On the duration of human movement: from self-paced to slow/fast reaches up to Fitts's law

Frédéric Jean, Bastien Berret

On the duration of human movement: from self-paced to slow/fast reaches up to Fitts's law

Frédéric Jean[1] and Bastien Berret[2]

1. Unité de Mathématiques Appliquées, ENSTA ParisTech, Université Paris-Saclay, F-91120 Palaiseau, France

2. CIAMS, Univ. Paris-Sud, Université Paris-Saclay, F-91405 Orsay, France

Abstract

In this chapter, we present a mathematical theory of human movement vigor. At the core of the theory is the concept of the cost of time. According to it, natural movement cannot be too slow because the passage of time entails a cost which makes slow moves undesirable. Within this framework, an inverse methodology is available to reliably and robustly characterize how the brain penalizes time from experimental motion data. Yet, a general theory of human movement pace should not only account for the self-selected speed but should also include situations where slow or fast speed instructions are given by an experimenter or required by a task. In particular, the limit case of a "maximal speed" instruction is linked to Fitts's law, i.e. the speed/accuracy trade-off.

This chapter first summarizes the cost of time theory and the procedure used for its accurate identification. Then, the case of slow/fast movements is investigated but changing the duration of goal-directed movements can be done in various ways in this framework. Here we show that only one strategy seems plausible to account for both slow/fast and self-paced reaching movements. By relying upon a free-time optimal control formulation of the motor planning problem, this chapter provides a comprehensive treatment of the linear-quadratic

case for single degree of freedom arm movements but the principles are easily extendable to multijoint and/or artificial systems.

Acknowledgment. This work is supported by a public grant overseen by the French National Research Agency (ANR) as part of the « Investissement d'Avenir » program, through the "iCODE Institute project" funded by the IDEX Paris-Saclay, ANR-11-IDEX-0003-02.

Contents

1 Introduction

Everyday actions are usually performed at a pace that people would commonly qualify of "comfortable", which is neither too fast nor too slow. Movement duration or average speed are inherent characteristics of biological and artificial sensorimotor control, a process that takes place both in space and time. Understanding the underpinnings of movement pace formation is of crucial importance not only in motor neuroscience (as many disorders lead to bradykinesia, [2, 35]) but also in fields where humans are brought to interact with artificial systems, such as humanoid robotics, robot-assisted rehabilitation, neuroprosthetics or computer animation. The presence of temporal discrepancies may considerably affect the way humans perceive and collaborate with such entities. More generally, to improve the human-likeness of artificial sensorimotor systems, high-level computational principles leading to appropriate movement pace must be developed. In human motor control, most research efforts on the topic have been turned toward specific paradigms such as the speed/accuracy trade-off [22, 48, 23] where movements are assumed to be performed as fast as possible for a given level of accuracy [see 40, 14, for reviews]. This empirical observation has been formalized as Fitts's law [16] and successfully implemented in human-computer interaction to model movement time [33]. An interesting observation is that any system assuming an exponential decay of the distance left to the center of the target will trivially yield Fitts's law [9, 11]. Actually, robotic studies often exploit this property to drive reliably a robot to some desired spatial target in an adjustable amount of time (e.g. [34]). This is typically achieved by tuning a parameter that the modeler must set by hand. A similar tuning of parameters is required to vary movement time when using PID controllers and even more involved feedback schemes (e.g. [39]). Therefore, task duration is often hard coded by fixing a desired movement time at the planning stage or merely results from the application of a (possibly finely tuned) feedback gain at the execution stage. The approach undertaken in this chapter lies in-between.

A recent hypothesis advanced the idea that the duration of biological movement could be driven by a “cost of time” [43, 45, Chapter 11]. In this view, slow movements are undesirable because the passage of time incurs a cost: it is “better” to achieve a task soon than later. This would be a property of the neural controller for reasons that may relate to the functioning of the reward system (i.e. temporal discounting of reward [46, 43]) via the cortico-basal ganglia loop. Movement vigor may indeed originate from the basal ganglia [44, 53] and its interaction with cortical areas encoding movement speed [27, 10]. In [3], we developed an inverse approach allowing to automatically infer, from experimental data, what would be the cost of time for reaching movements. The time cost then proved to allow elaborating and predicting the duration of upcoming reaching movements of various amplitudes and directions performed at a self-selected speed: motion time was thus an emergent property of the motor preparation stage. Here, we further analyze how this framework can embrace task instructions such as “move slow” or “move fast”. We also give an account of Fitts’s law in this context. This work was conducted within the optimal control framework and, more precisely, the free-time optimal control formalism. Optimal control theory relies upon the choice of cost functions that define what is optimal behavior for a given system [49, 50]. One great feature of optimal control is the high-level of abstraction that it enables, allowing to easily port findings from biological to artificial systems and vice-versa. For our purpose, we shall distinguish between subjective and objective cost functions throughout the chapter. An objective cost function is specified or imposed by the task itself. Typical examples are the specification of a target location (e.g. endpoint error) or a reference trajectory to track (e.g. draw an ellipse). In contrast, a subjective cost function is specified by the sensorimotor system itself and crucially serves to resolve the remaining degrees of freedom that are left free by the (redundant) task. It may measure energy expenditure, effort, jerk or any other quantity such as the cost of time which is at the core of the present work.

This chapter is organized as follows. First, we briefly review how the cost of time can be characterized unequivocally from real data in the proposed framework. We then analyze quite extensively the linear quadratic case and explain how the theory can account for speed changes resulting from explicit constraints given by an experimenter such as Fitts's like instructions. Throughout the chapter, we give a theoretical treatment of the problem together with illustrations in the context of a single degree of freedom arm performing reaching movements in a horizontal plane. The concepts are however easily transferable to more complex systems and tasks.

2 Theory and results

2.1 Theory of the cost of time

The present theory is derived within the framework of optimal control (OC) theory, which assumes that the signature of human movement is optimality (with respect to a certain cost function) [49]. It implicitly supposes that the trajectories triggered by the central nervous system can be accounted for by a certain infinitesimal cost $h(\mathbf{x}, \mathbf{u}, t)$, which depends on the system state $\mathbf{x} \in \mathbb{R}^n$, the motor command $\mathbf{u} \in \mathbb{R}^m$ and the time $t \in I \subset \mathbb{R}$, respectively. In a sense, biological trajectories would adhere to a principle of least action where the "action" would be the time integral of h. In seminal studies assuming this framework [37, 17, 54], the time window of integration was set *a priori* by the modeler: movement time was simply fixed in accordance with experimental measurements. However, since movement time or average speed are motor decision variables, then a free-time formulation of the problem should rather be used [41, 29]. In this way, the duration of movement would emerge implicitly from the optimality of behavior, as already proposed by [24] who assumed to penalize the total motion duration itself. In the same vein, at the core of the present theory aiming to account for the vigor of movement is the idea of the "cost of time" (CoT) [43, 3]. The theory assumes that h can be separated into a term that penalizes time only, $g(t)$ (the infinitesimal CoT), plus a term that depends on the state/control variables, $l(\mathbf{x}, \mathbf{u})$, which allows to shape the trajectories followed by the system. Thus, if $h(\mathbf{x}, \mathbf{u}, t) = g(t) + l(\mathbf{x}, \mathbf{u})$, a mathematical analysis shows that it is actually possible to compute the value $g(t)$ by resolving an OC problem in fixed time t with known initial/final states (denoted by $\mathbf{x}^0$ and $\mathbf{x}^f$ respectively), given a system dynamics $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ and a trajectory cost $l(\mathbf{x}, \mathbf{u})$. We briefly recall how this is achieved but the reader is referred to [3] for more details.

Given an input $\mathbf{u}(\cdot)$ defined on an interval $[0, t_{\mathbf{u}}]$, we denote by $\mathbf{x}_{\mathbf{u}}(\cdot)$ the trajectory of $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ satisfying $\mathbf{x}_{\mathbf{u}}(t_{\mathbf{u}}) = \mathbf{x}^f$. As explained above, we consider the following cost function:

$$C(\mathbf{u}, t_{\mathbf{u}}) = \int_0^{t_{\mathbf{u}}} \big(g(t) + l(\mathbf{x}_{\mathbf{u}}(t), \mathbf{u}(t))\big)\ dt, \tag{1}$$

where the functions g and l are non-negative. The function l has been the subject of extensive investigations in motor control (e.g. [17, 54, 6]) and may capture both subjective (related to an individual's decision) and objective (task related) goals. The trajectory cost $l(\mathbf{x}, \mathbf{u})$ is assumed to be known or identifiable (in fixed time OC formulations). The function g is the infinitesimal (i.e. instantaneous) CoT we can identify and whose antiderivative is the actual CoT, $G(t) = \int_0^t g(s)ds$ (we assume $G(0) = 0$ for simplicity).

We consider the following *free-time* OC problems:

Given an initial state $\mathbf{x}^0$*, minimize the cost* $C(\mathbf{u}, t_{\mathbf{u}})$ *among all inputs* $\mathbf{u}(\cdot)$ *and all times* $t_{\mathbf{u}}$ *such that* $\mathbf{x}_{\mathbf{u}}(0) = \mathbf{x}^0$ *and* $\mathbf{x}_{\mathbf{u}}(t_{\mathbf{u}}) = \mathbf{x}^f$ *(by definition of* $\mathbf{x}_{\mathbf{u}}$*).*

We will assume the existence of minimal solutions $\mathbf{u}(\cdot)$ with a finite time $t_{\mathbf{u}}$, which may be guaranteed under some technical conditions on the dynamics and on the cost [30].

Next, let $V_{\mathbf{x}^f}(t, \mathbf{x}^0)$ be the value function[1] of the OC problem joining $\mathbf{x}^0$ to $\mathbf{x}^f$ in fixed-time t, that is

$$V_{\mathbf{x}^f}(t, \mathbf{x}^0) = \inf \int_0^t l(\mathbf{x}_{\mathbf{u}}(s), \mathbf{u}(s))ds, \tag{2}$$

where the infimum is taken among all inputs $\mathbf{u}(\cdot)$ such that $\mathbf{x}_{\mathbf{u}}(0) = \mathbf{x}^0$ and $\mathbf{x}_{\mathbf{u}}(t) = \mathbf{x}^f$. It is the optimal cost of a motion in time t between $\mathbf{x}^0$ and $\mathbf{x}^f$.

[1] Note that we did not use the standard way to define the value function: for a movement duration equal to t, this is usually $\tilde{V}_{\mathbf{x}^f}(w, \mathbf{x}^0(w)) = \inf \int_w^t l\big(\mathbf{x}_{\mathbf{u}}(s), \mathbf{u}(s)\big)ds$. Here we set $V_{\mathbf{x}^f}(t - w, \mathbf{x}^0(w)) = \tilde{V}_{\mathbf{x}^f}(w, \mathbf{x}^0(w))$, hence $\frac{\partial V_{\mathbf{x}^f}}{\partial t} = -\frac{\partial \tilde{V}_{\mathbf{x}^f}}{\partial t}$.

Then the movement time τ, that is the time $t_{\mathbf{u}}$ of an optimal solution $\mathbf{u}(\cdot)$ of the free-time OC problem, satisfies

$$\tau \in \operatorname{argmin}_{t\geq 0}\left(\int_0^t g(s)ds + V_{\mathbf{x}^f}(t,\mathbf{x}^0)\right), \tag{3}$$

and, assuming that $V_{\mathbf{x}^f}$ is differentiable with respect to t, we get:

$$g(\tau) = -\frac{\partial V_{\mathbf{x}^f}}{\partial t}(\tau,\mathbf{x}^0). \tag{4}$$

It is well-known from the Hamilton-Jacobi-Bellman theory that $\frac{\partial V_{\mathbf{x}^f}}{\partial t}(\tau,\mathbf{x}^0) = \mathcal{H}_0^\star(\mathbf{x}(\tau),\mathbf{p}(\tau))$, with $\mathcal{H}_0^\star(\mathbf{x},\mathbf{p}) = \max_{\mathbf{v}} \mathcal{H}_0(\mathbf{x},\mathbf{p},\mathbf{v})$ where $\mathcal{H}_0 = \mathbf{p}^\top \mathbf{f}(\mathbf{x},\mathbf{v}) + l(\mathbf{x},\mathbf{v})$ is the Hamiltonian associated with the fixed-time OC problem[2], $\mathbf{x}(t)$ is an optimal solution, and $\mathbf{p}(t) \in \mathbb{R}^n$ is the *co-state vector* [41]. Since it is obvious that the corresponding optimal control $\mathbf{u}(\cdot)$ is also a minimal solution of the OC problem in *fixed time* τ we then have $\mathcal{H}_0^\star(\mathbf{x}(\tau),\mathbf{p}(\tau)) = \mathcal{H}_0(\mathbf{x}(\tau),\mathbf{p}(\tau),\mathbf{u}(\tau))$, we get in this way $g(\tau) = -\mathcal{H}_0(\mathbf{x}(\tau),\mathbf{p}(\tau),\mathbf{u}(\tau))$.

Interestingly, the above analysis shows that the derivation extends to stochastic settings [47, 50]. In particular in the linear quadratic Gaussian (LQG) case, the infinitesimal CoT can be easily computed because the value function has a parametric form whose parameters can be evaluated via the resolution of decoupled ordinary differential equations [28].

In summary, it suffices to solve a stochastic or deterministic OC problem in fixed time t to recover the value of $g(t)$. This will be exemplified in the linear quadratic (LQ) case in the next section, before the problem of tuning movement time (around the optimal one) will be addressed.

To test the above methodology, we asked subjects to perform 1-dof arm movements in the horizontal plane. These reaching movements were of different amplitudes and, for each amplitude, the duration was estimated from motion capture data. In Figure 1, we depict the main results. Overall, an affine relationship between movement extent and time can be drawn from the experimental data. When identifying $g(t)$ for several movement times t, one can charac-

[2]We assume here that there are no abnormal extremals (an hypothesis which is satisfied in particular by controllable linear systems). As a consequence, it is not necessary to put a Lagrange multiplier in front of l in $\mathcal{H}_0$.

terize the shape of g on the interval of actual movement durations. For the depicted subjects, movement times varied between about 600 ms (for an amplitude of 5 degrees) to about 1400 ms (for an amplitude of 95 degrees). Therefore, we were able to identify the CoT in a robust and reliable manner on the interval 600-1400 ms. Outside of this interval, extrapolation was required. However, it must be noticed that the shape of g on the range of empirical movement times was sufficient to conclude that the CoT was neither linear nor purely convex or concave. Actually, its shape tended to be sigmoidal. The present shapes were obtained when assuming the torque change [54] as trajectory cost l. Assuming the angle jerk [17] as trajectory cost would not change the sigmoidal shape. For a more thorough analysis with additional assessments, the reader is referred to [3]. The biomechanical model of the arm is described in Sect. A.2. It must also be noted that the free-time optimal control model predicts smooth and bell-shaped velocity profiles, which agrees with classical observations for such planar arm movements.

2.2 Linear quadratic models

2.2.1 General settings and solutions

Let us focus on deterministic LQ models for 1 degree-of-freedom (dof) motions. This framework is relevant to model simple arm reaching movements. The state of such systems can be described by $\mathbf{x} = (\theta, \dots, \theta^{(n-1)}) \in \mathbb{R}^n$ and then the dynamics has the form

$$\theta^{(n)} + c_{n-1}\theta^{(n-1)} + \cdots + c_0\theta = u, \tag{5}$$

which is a single-input linear system $\dot{\mathbf{x}} = A\mathbf{x} + Bu$, $u \in \mathbb{R}$. Typically $n = 2$ or 3 for dynamical models of the arm (see below). The single-input LQ case is also interesting from a theoretical point of view as strong results of well-posedness of the inverse problem exist. In particular (see [3]), the uniqueness and robustness to perturbations of experimental data can be proven for $g(t)$.

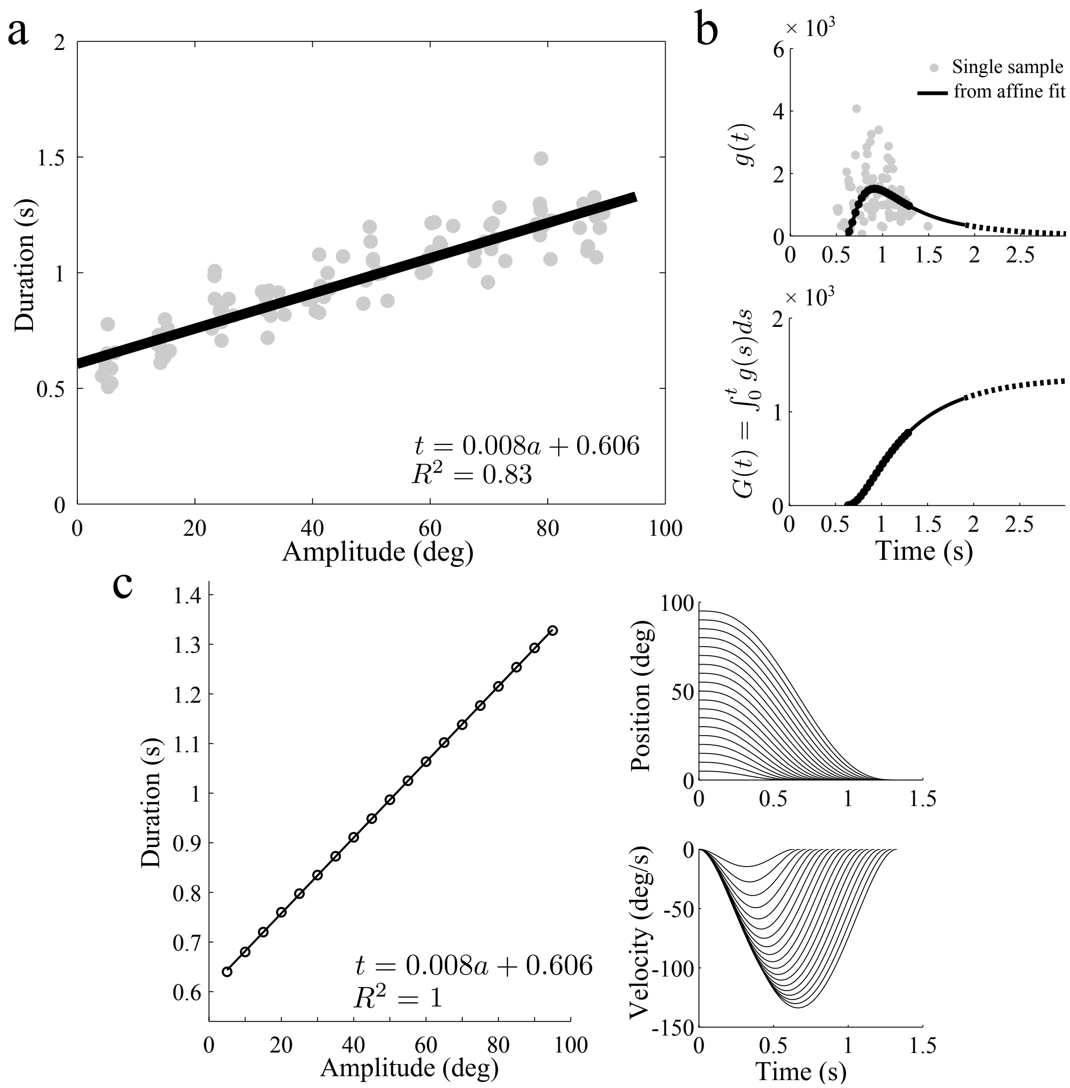

Figure 1. Movements at spontaneous speed. **a**. Experimental amplitude-duration relationship for one individual. **b**. Infinitesimal and integral time costs. Values of the infinitesimal CoT were recovered either in single trials (gray dots) or by making use of the affine fit of the amplitude-duration relationship presented in the first panel. Dotted lines are extrapolated values. **c**. OC simulations in free-time using the CoT $G(t)$. The duration movement is exactly recovered for every amplitude, and the corresponding trajectories are displayed (standard bell-shaped velocity profiles for such 1-dof movements in the horizontal plane).

Indeed, the underlying quadratic cost can be identified unequivocally [38] from the empirical and presumably optimal trajectories and in a continuous way (roughly speaking, the mapping between the optimal trajectories and the quadratic cost is continuous, so that small changes in experimental trajectories result in small changes in the quadratic cost). This remarkable theoretical result motivates a deeper investigation of the LQ scenario.

A quadratic cost for the system given in Eq. (5) is a function $\alpha u^2 + \mathbf{x}^T Q \mathbf{x} + 2\mathbf{x}^T S u$, with $\alpha > 0$, which is a positive semidefinite quadratic form in $(\mathbf{x}, u)$. Up to a normalization we can assume $\alpha = 1$ and then consider a cost of the form $l(\mathbf{x}, u) = u^2 + \mathbf{x}^T Q \mathbf{x} + 2\mathbf{x}^T S u$.

The associated OC problem in fixed time $\tau > 0$ is the following: *given terminal conditions* $\mathbf{x}^0, \mathbf{x}^f \in \mathbb{R}^n$*, minimize the cost*

$$C_\tau(u) = \int_0^\tau (u(t)^2 + \mathbf{x}_u(t)^T Q \mathbf{x}_u(t) + 2\mathbf{x}_u(t)^T S u(t)) dt,$$

among all controls u *such that the solution* $\mathbf{x}_u$ *of* $\dot{\mathbf{x}} = A\mathbf{x} + Bu$, $\mathbf{x}_u(0) = \mathbf{x}^0$*, satisfies* $\mathbf{x}_u(\tau) = \mathbf{x}^f$.

Initial and final points $\mathbf{x}^0$ and $\mathbf{x}^f$ are always chosen as equilibrium states of the system, that is, $\mathbf{x}^0 = (\theta^0, 0, \ldots, 0)$ and $\mathbf{x}^f = (\theta^f, 0, \ldots, 0)$. We also make the technical assumption that the pair $(A, Q^{1/2})$ is observable (this assumption is necessary for Eq. (6) below to hold). Ferrante et al. [15] showed that the optimal trajectory $\mathbf{x}_u$ of this problem is given by

$$\mathbf{x}_u(t) = e^{tA_+}\mathbf{p}_1 + e^{tA_-}\mathbf{p}_2, \tag{6}$$

where the vectors $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^n$ are the unique solution of

$$\begin{cases} \mathbf{x}^0 & = \mathbf{p}_1 + \mathbf{p}_2, \\ \mathbf{x}^f & = e^{\tau A_+}\mathbf{p}_1 + e^{\tau A_-}\mathbf{p}_2. \end{cases} \tag{7}$$

The matrices A_-, A_+ are respectively anti-stable and stable (the eigenvalues of A_+ are actually the opposite of the ones of A_-). These matrices are determined through a Riccati equation and do not depend on $\mathbf{x}^0, \mathbf{x}^f$, and τ, but only on the parameters (A, B) of the dynamic and (Q, S) of the cost.

Remark 1. When $\mathbf{x}^f = 0$, the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ depend linearly on $\mathbf{x}^0$ and so do $\mathbf{x}_u(t)$ for every t. This is also true for $u(t)$ since its expression has a form similar to Eq. 6 (see [15]).

Remark 2. Note that the corresponding OC problem in infinite time is the following: *given an initial condition* $\mathbf{x}^0 \in \mathbb{R}^n$, *minimize the cost*

$$C_\infty(u) = \int_0^\infty (u(t)^2 + \mathbf{x}_u(t)^T Q \mathbf{x}_u(t) + 2\mathbf{x}_u(t)^T S u(t))dt,$$

among all controls u, *where* $\mathbf{x}_u$ *is the solution of* $\dot{\mathbf{x}} = A\mathbf{x} + Bu$, $\mathbf{x}_u(0) = \mathbf{x}^0$. The solution of this problem is given again by Eq. (6), with the same matrices A_-, A_+, but with parameters $\mathbf{p}_1 = \mathbf{x}^0$ and $\mathbf{p}_2 = 0$.

2.2.2 Computation of the infinitesimal CoT

Up to a translation in θ (position variable), we can always assume $\mathbf{x}^f = 0$. We then choose a family of initial conditions $\mathbf{x}^0(a) = (a, 0, \ldots, 0)$, parameterized by the movement extent $a > 0$ (i.e. the amplitude of the motion). For every amplitude $a > 0$ we denote by $t^*(a)$ the duration (which can be estimated experimentally) of the motion between $\mathbf{x}^0(a)$ and $\mathbf{x}^f$, and by $u^a(\cdot)$ a control minimizing the integral cost $C_{t^*(a)}(u)$ in fixed time $t^*(a)$ between $\mathbf{x}_u(0) = \mathbf{x}^0(a)$ and $\mathbf{x}_u(t^*(a)) = \mathbf{x}^f = 0$. By standard computations (see [3]) we obtain $\frac{\partial V_{\mathbf{x}^f}}{\partial t}(t^*(a), \mathbf{x}^0(a)) = -u^a(t^*(a))^2$, and so from Eq. 4,

$$g(t^*(a)) = u^a(t^*(a))^2. \tag{8}$$

Moreover, the value $u^a(t^*(a))$ can be seen to depend linearly on $\mathbf{x}_u(0)$ in the LQ case (see Remark 1), and so it depends linearly on a since $\mathbf{x}_u(0) = \mathbf{x}^0(a) = a\mathbf{x}^0(1)$. In other words, $u^a(t^*(a)) = a\varphi(t^*(a))$, where the function $\varphi(\cdot)$ is defined as follows: *for every* $\tau > 0$, $\varphi(\tau)$ *is the value* $u^1(\tau)$ *of the control minimizing the integral cost* $C_\tau(u) = \int_0^\tau (u^2 + \mathbf{x}^T Q \mathbf{x} + 2\mathbf{x}^T S u)dt$

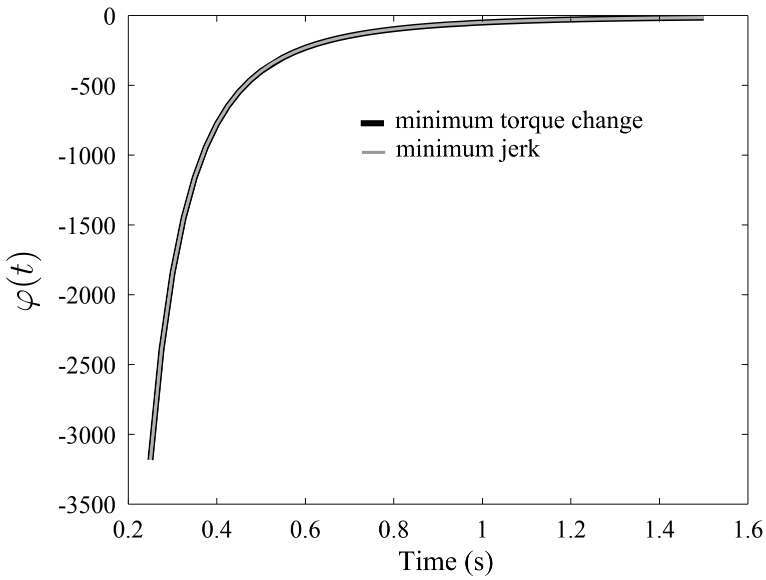

Figure 2. The universal function φ for the minimum torque change and minimum jerk optimality criteria, for a 1-dof arm moving in the horizontal plane. The function was found to be quasi identical for these two costs. From this function, the infinitesimal CoT can be recovered as $g(t) = \varphi(t)^2 a(t)^2$ where $a(t)$ is the amplitude corresponding to a movement in time t (which can be determined experimentally).

in fixed time τ between $\mathbf{x}_u(0) = \mathbf{x}^0(1)$ and $\mathbf{x}_u(\tau) = \mathbf{0}$. Note that $\varphi(\cdot)$ is a *universal function of time* that depends only on the system dynamics (A, B) and the trajectory cost and not on the specific behavior of an individual. This universal function of time can be computed explicitly thanks to the equations given in [15]. We finally obtain $g(t^*(a)) = \varphi\big(t^*(a)\big)^2 a^2$.

Empirical observations show that the time $t^*(a)$ is typically an increasing function of the amplitude, so that its inverse $a^*(t)$ exists. We can then determine the function $g(\cdot)$ by $g(t) = \varphi(t)^2 a^*(t)^2$. In particular, if it appears from experiments that the function t^* is approximately affine of the form $t^*(a) = \alpha a + \beta$, then the infinitesimal CoT can be written $g(t) = \varphi(t)^2 (\frac{1}{\alpha} t - \frac{\beta}{\alpha})^2$. Hence, it suffices to compute $\varphi(t)$, which can be done explicitly, to recover the actual infinitesimal CoT from the experimental duration/amplitude mapping. For illustration, the function φ is plotted in the Figure 2 for the two main trajectory costs considered here, namely the angle jerk and torque change optimality criteria.

3 Accounting for other motion speeds

A general theory of human movement vigor should also be able to account for movement times departing from the self-chosen ones. It is clear that motion duration can vary in function of the task, in particular verbal instructions given by an experimenter such as go quickly/slowly to the target. How can the CoT theory take account of this variation? Remind that, from Eq. (3), the duration of a motion satisfies

$$\tau \in \mathrm{argmin}_{t \geq 0} \left(\int_0^t g(s)ds + \inf_{\mathbf{u}} \int_0^t l(\mathbf{x_u}(s), \mathbf{u}(s))ds \right). \tag{9}$$

Hence variations of the motion duration can be explained by changes either of $g(t)$ or of $l(\mathbf{x}, \mathbf{u})$. The first question is: is it possible to explain changes of motion duration by playing on the CoT $g(t)$?

3.1 The sole modification of the cost of time cannot explain slower/faster movements

Let us assume first that the cost of the trajectory is independent of the task and hence that changes in motion duration only result from changes of the CoT. Typically, it is clear from Eq. (9) that any increase of the values of $G(t) = \int_0^t g(s)ds$ implies a decrease of the duration τ. The effect of instructions such as "go quickly to the target" could then simply correspond to an increase of the CoT $G(t)$. Conversely, instructions such as "go slowly to the target" could produce a decrease of the CoT, which implies in turn an increase of the motion duration. Let us examine the consequence of this hypothesis for the model described in Sect. 2.2, that is, in the context of linear quadratic (LQ) models. In this case, motions are always solutions of a LQ optimal control problem in fixed finite time, with always the same cost $l(\mathbf{x}, u) = u^2 + \mathbf{x}^T Q \mathbf{x} + 2\mathbf{x}^T S u$

but with a time τ that depends on the term $G(t)$.

Rescaling of g. An intuitive idea would be to rescale g by multiplying it by some positive parameter κ. A simple investigation however proves that such an approach is falsified by experimental findings (see for instance [8, 57]). Indeed, such a rescaling would induce a new amplitude $\sqrt{\kappa}a^*(t)$ for a movement in time t (we use the notations of Sect. 2.2.2, i.e. $\kappa g(t) = \varphi(t)^2 (\sqrt{\kappa}a^*(t))^2$). Therefore, the rescaled CoT would yield the affine amplitude-duration relationship $t^*(a) = \alpha \frac{a}{\sqrt{\kappa}} + \beta$. It can be concluded that just rescaling the CoT does not allow to change both the slope and the intercept of the amplitude/duration relationship. Hence, since both the intercept and slope are found to change experimentally when the instructed speed is varied, this observation cannot be attributed to a global rescaling of the CoT g. Typically, when a subject is asked to move faster, not only α is reduced significantly but also β.

Arbitrary change of g. We now consider that g can be changed both in shape and magnitude. Consider first the case of an overall decrease of the CoT, which produces a longer duration τ, i.e. slower movements. The asymptotic analysis of Sect. A.1.1 shows that, for a large duration τ, the solution in time τ of the LQ problem associated with the cost $l(\mathbf{x}, u)$ looks alike the solution of the same LQ problem in infinite time (see Lemma 4). The latter solutions have an exponential decay to the final state and, moreover, a single peak of velocity whose magnitude is independent of the time τ. These characteristics are not compatible with what is known of slow reaching movements where velocity traces are gradually more multipeaked [26, 55], which moreover seems to be a preplanned property not simply due to sensory feedback processing [13].

Consider now the case of an increase of the CoT, which induces a shorter duration τ and hence faster movements. The asymptotic analysis when $\tau \to 0$ shows that, for small enough durations, the solutions of the LQ problem are almost identical, up to a change of time-parameterization, and are of polynomial form (see Lemma 5). More precisely, the theory would predict that for faster and faster movements the velocity profiles are dilatations of each other and have a symmetric shape (see Remark 6). Such a strict scaling law of symmetric speed profiles is falsified by experimental observations. Indeed, it was shown that movements become more asymmetric as speed increases, where the relative duration of the deceleration phase increases during extremely fast reaches (from ~50% of total motion duration for rapid reaches to ~70% for maximally fast reaches) [32, 31]. This is moreover incompatible with the exponential decay of the distance left to the target observed in Fitts's like studies.

In summary, the sole modification of the CoT cannot explain slower/faster movements in the LQ framework. Of course we could consider different models than the linear-quadratic ones. Indeed, in the latter models we make several hypotheses: first, the evolution of the state is given by linear differential equations; second, the state and the control are unbounded; third, we restrict ourselves to the class of costs function which are quadratic function of both state and control. The first assumption is not questionable as soon as we do not finely model the dynamics of muscles or do not consider multijoint systems, which is consistent at the present level of investigation. The third one seems to be reasonable since the class of quadratic costs is sufficient to reproduce accurately simple arm motions at least [37, 25, 17]. Moreover, the conclusions above should be very similar for a slightly larger class of costs functions (for instance a class including the absolute work as in [4]), even if the asymptotic study would be much more difficult in that case. The most critical hypothesis actually is the second one. Indeed it is evident that the state and the control, being physical quantities, are bounded. In a LQ model, this fact is taken into account implicitly since high values are penalized in the cost. This approach is valid as long as the values of the state and the control in the optimal solutions do not exceed the bounds. This condition is not easy to check since most of the bounds are not really known, it is however clearly satisfied when the duration of the motion is not too small. Another approach would be to take into account explicitly these bounds. In that case, it exists a minimum time to go from one given state to another one. When the CoT $G(t)$ increases (for instance because of instructions such as "go quickly to the target"), the time τ of the motion converges to the minimum time and, under standard convexity hypotheses on the cost [18], the optimal solutions converge to the minimum time solutions. This scenario is not really plausible for different reasons. First, minimum time solutions present, in general time-intervals, saturations of the control's bounds. Since such characteristic saturations have never been observed in fast motions for quantities such as velocity, acceleration and jerk, the control would necessarily be a higher-order quantity. Even if they existed, such saturations would hardly be compatible with trajectories satisfying Fitts's law (exponential decay of the end-effector position to the goal). Conceivably, saturation may occur at the level of motoneurons activity but experimental data of surface electromyography (EMG), the main non-invasive approach to estimate the overall activity of motor units, indicate that EMG activity is relatively far from maximal during rapid reaching [1]. Moreover, no plateau is visible on any sensible time window and the so-called triphasic pattern, with well-distinguished EMG bursts, is known to govern ballistic movements [21, 7]. Secondly, the hypothesis of minimum time trajectories has already been studied in [48] and contradicted in [57]. Intuitively, the reason is that humans do not always move as fast as possible for a given level of accuracy: in most daily activities, we could move faster without degrading task performance. At last, one may mention that even when instructed to move as fast as possible, the actual maximal speed of a subject is not attained. It has been proven that subjects can move faster without altering accuracy when explicitly asked to co-contract muscles, an energy consuming strategy [36].

In summary, we presented strong arguments supporting that the sole modification of the CoT cannot be put forward to explain neither slower nor faster movements. Then, it seems necessary to assume that task-induced changes of motion duration are due to changes of the cost of the trajectory, i.e. $l(\mathbf{x}, u)$.

3.2 From self-paced motion to slower/faster movements

How do changes of the cost $l(\mathbf{x}, u)$ affect the duration? Consider the example of the linear quadratic models. For our purpose, we propose to interpret the total infinitesimal cost

$$g(t) + l(x, u) = g(t) + u^2 + \mathbf{x}^T Q\mathbf{x} \quad [+ \text{eventually mixed terms } \mathbf{x}^T Su].$$

by distinguishing three types of terms:

- $g(t)$ is the cost of the time, it penalizes slow motion by accumulating infinitesimal values during the passage of time;
- u^2 is a subjective cost that evaluates the "effort" associated with a movement. It can reflect mechanical energy expenditure, amount of joint torques, smoothness etc., depending on the modeling; we can also include the mixed terms in the subjective cost, and possibly some part $\mathbf{x}^T Q'\mathbf{x}$ of the quadratic terms in $\mathbf{x}$. In essence, the subjective part of the trajectory cost reflects an individual's motor decision (often useful to resolve all residual task redundancy).
- $\mathbf{x}^T Q\mathbf{x} = (\mathbf{x} - \mathbf{x}^f)^T Q(\mathbf{x} - \mathbf{x}^f)$ is an objective cost, also part of the trajectory cost. Here, it penalizes the fact of being away from the goal $\mathbf{x}^f$ (recall that $\mathbf{x}^f = 0$ here without loss of generality) and can be modulated by the requirements of the task. It is objective in the sense that it is directly related to the task's demand.

Hence we postulate that a change in the description of the task (e.g. go quickly/slowly to the target) will affect only the objective cost, not the two other ones. Let us explain how it could work. To simplify, we assume that the matrix Q is diagonal (i.e. cost function with separate variables), $\mathbf{x}^T Q\mathbf{x} = r\theta^2 + s_1\dot{\theta}^2 + \cdots + s_{n-1}(\theta^{(n-1)})^2$. Since the term $r\theta^2 = r(\theta - \theta^f)^2$ penalizes the fact of being away from the goal, the instruction "go quickly to the goal" translates as "increase r". In the same way, since the term $s_1\dot{\theta}^2$ penalizes high velocities, the instruction "go slower" translates as “increase s_1” (it can also increase the other parameters s_i). And it appears actually that the duration of a trajectory minimizing the cost

$$\int_0^{t_u} \Big(g(t) + u^2 + r\theta^2 + s_1\dot{\theta}^2 + \cdots + s_{n-1}(\theta^{(n-1)})^2\Big)dt, \tag{10}$$

decreases with r and increases with s_1, as expected. This fact was checked through numerical simulations and the dependence of motion duration on r and s_1 is reported in Figure 3. Tuning the objective cost provides a means to modify the motion duration around the reference value corresponding to a self-selected movement pace. Verbal instructions such as “produce a quick movement” or “produce a slow movement” can thus be accounted for in this way. It should be noted that increasing r or s_1 breaks the affine relationship between movement amplitude and time. If linearity is preserved for relatively small enough values of r and s_1, the correlation coefficients nevertheless go down as these weights increase. The next section actually shows that there is a gradual distortion of the amplitude/time relationship such that Fitts’s law is actually recovered for very large values of r.

3.3 Towards Fitts’s law and the speed/accuracy trade-off

To take into account accuracy constraints, we propose to consider goal-directed movements such as arm pointing as the superposition of an open-loop motion (the planned trajectory) and of a feedback process whose role is to provide on-line corrections and in particular to stabilize the hand around the target [12, 51]. We then distinguish two different motion times:

- the *planning time*, denoted by τ_p, which is the duration of the planned trajectory and can be determined by solving a free time OC problem involving the CoT as described previously;

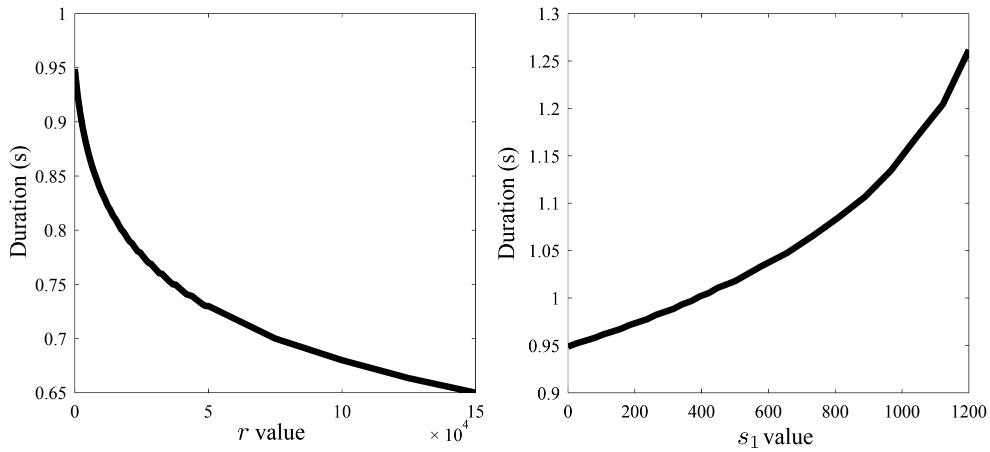

Figure 3. Slower and faster movements for the 1-dof planar arm movement under consideration. Effects of varying r and s_1 on the motion duration. These graphs were drawn from the CoT of the individual presented in Fig. 1 and amplitude was set to $a = 45°$. Increasing the positional weight r induces an decrease of movement duration, as predicted by the free-time optimal solutions. The opposite effect can be observed when increasing the velocity weight s_1. Modifying the objective trajectory cost is a sensible way to tune movement duration around the individual's self-selected one.

- the *execution time*, denoted by τ_s, which is the actual duration of the motion; it may differ from the planning time because of the feedback process.

In general planning and execution times may differ for two reasons: the presence of perturbations and the fact that the point aimed at differs from the actual stopping point. The former situation occurs because of the presence of sensorimotor noise in the nervous system and the latter may occur in the case of accuracy requirements. For example, if the target has a width w, and if the instruction is "as fast and as accurate as possible", the subject will conceivably aim at a point inside or near the center of the target to ensure target achievement (see [52]), whereas the motion can actually be stopped once the trajectory meets the target via the activation of terminal feedback processes.

Again, we will consider a 1-dof LQ model as in Sect. 2.2. Let $\mathbf{x}^0$ be the starting point, $\mathbf{x}^f = 0$ be the center of a target of width w and $\mathbf{x}(\cdot)$ be the planned trajectory between these points. On the one hand, the planning time τ_p satisfies $\mathbf{x}(\tau_p) = 0$. In other words, the end-effector attains the center of the target exactly in time τ_p (no perturbations are assumed here). On the other hand, the movement will be stopped as soon as $\theta(t) \leq w/2$, i.e. the stopping time τ_s satisfies approximately $\theta(\tau_s) = w/2$.

Consider for instance the case of as fast as possible and as accurate as possible movements, which is the standard scenario behind Fitts's law [16]. As explained in the previous section, the instruction "move fast" corresponds to a cost with a very large coefficient r (see Eq. (10)). It can be shown (see Sect. A.1.2) that in that case the planning time τ_p is rather small and the planned trajectory is of the form

$$\theta(t) \approx c\theta^0 e^{-\alpha_r t}, \qquad \text{for } t/\tau_p \text{ large enough,}$$

where c, α_r are positive constants with $\alpha_r \tau_p$ large (i.e. $\alpha_r \tau_p \to \infty$ as $r \to \infty$).

The stopping time is determined by the constraint $\theta(\tau_s) = w/2$. Therefore it satisfies

$$\tau_s \approx \frac{1}{\alpha_r} \log c + \frac{1}{\alpha_r} \log(2\theta^0/w),$$

which is of the same form than the original formulation of Fitts's law, that is, $t = \tilde{\alpha} \log_2(2a/w) + \tilde{\beta}$ with $t = \tau_s$ and $a = \theta^0$ [16]. Hence Fitts's law can be accounted for by our theory, although developed in a deterministic context, without explicitly assuming a linear feedback control law that would lead to an exponential decay of the distance left to the target as done in [11] or [42].

Remark 3. Note that the distinction between planning and stopping movement times allows one to recover Fitts's law as soon as the planned trajectory decreases exponentially. It is in particular the case in all models with infinite horizon and quadratic costs (either deterministic, i.e. LQR, or stochastic, i.e. LQG), and more generally in all linear models with a proportional feedback $u = K\mathbf{x}$, even though in those cases there would be no planned movement time. Hence, this is mainly the shape of the trajectory which explains Fitts's law in such models: one does not increase motion duration specifically because of a higher accuracy demand but rather motion duration increases as a consequence of the exponential decrease of the distance left to the target during maximally fast reachings. It is likely that the planned motion duration could also be increased on purpose if modeling signal-dependent noise and adding a terminal error term in a stochastic context [22] but we did not consider stochastic formulations of the present deterministic free-time OC problems.

To illustrate the convergence to Fitts's law, we performed simulations for the same 1-dof arm model with $r = 10^8$ and $s_1 = 0.05r$ in Eq. (10). The results are depicted in Figure 4 where the switch from affine to logarithmic relationships between amplitude and duration is illustrated. In accordance with experimental findings, velocity profiles also become more asymmetrical in the sense that the relative duration of deceleration drastically increases for maximally fast reaches [32, 31]. These graphs also explain why Fitts's law does not hold for self-paced movements but is mainly a limit case, which agrees with experimental observations (see [57]).

4 Conclusion

In this chapter, we have presented a theoretical view of the computational principles that may underlie the control of movement vigor within the central nervous system. We tackled the issue of how reach duration can be adjusted to speed instructions in this framework. At the core of the theory is the hypothesis of the existence of a "cost of time" [43]. It assumes that the passage of time has a cost *per se*, which explains why our movements are not slower. Using an inverse optimal control approach, we showed that this hypothetical time cost can be reliably identified from experimental data of movement extent and duration and without resorting to any parametric adjustment [3]. Yet, the cost of time aims at explaining the spontaneous/natural movement vigor, i.e. self-chosen motion pace. When explicitly asked to move slower or faster, we argued that humans do not seem to modify the cost of time itself but rather an objective trajectory

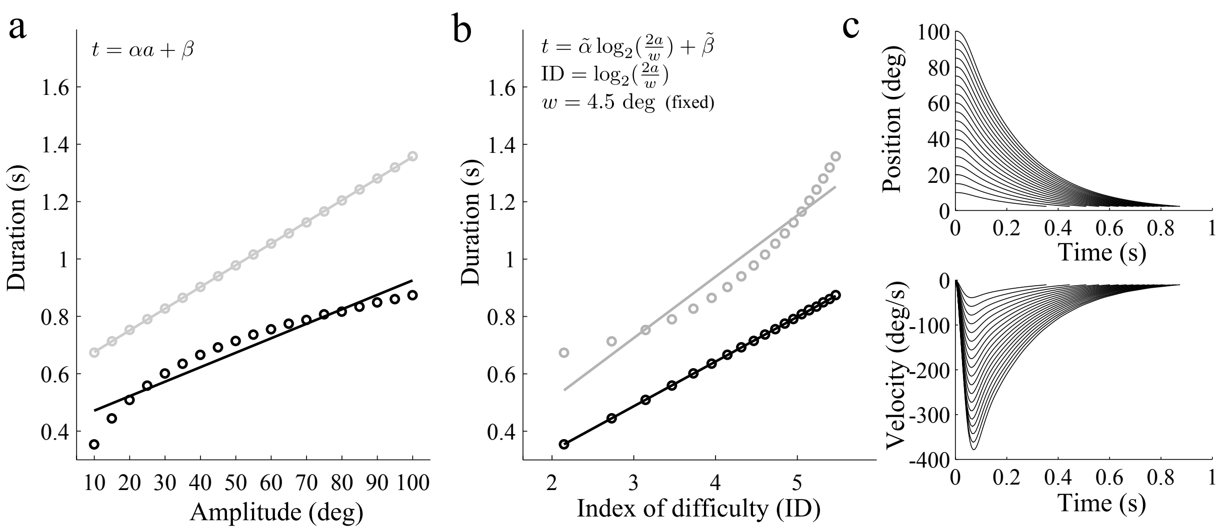

Figure 4. The case of Fitts's law for 1-dof planar arm pointing movements. **a**. Relationship between amplitude and duration. In gray, the original amplitude/duration of the example individual is recalled (i.e. $r = s_1 = 0$ for self-paced movements). When r becomes very large (here $r = 10^8$ and $s_1 = 0.05r$), movements become faster and the amplitude/duration relationship departs from its linear shape (black traces, where a logarithmic profile is visible). **b**. Relationship between index of difficulty ($\mathrm{ID} = \log_2(\frac{2a}{w})$) and duration. In gray, for self-paced movements ($r = s_1 = 0$). In black, for Fitts's instructions ($r = 10^8$ and $s_1 = 0.05r$). Fitts's law is recovered very accurately in the latter case in contrast to self-paced motions where a convex, instead of linear, trend is observed. **c**. Position and velocity profiles corresponding to Fitts's law simulations in the free time OC formalism. The exponential decrease of the distance left to the target is visible for the position variable and the asymmetry of speed profiles can be compared to those of Fig. 1c.

cost reflecting the specific task constraints. Whereas the time cost and the subjective trajectory cost seem to be relatively invariant (at least on a short time scale), we provided evidence that the introduction of an objective trajectory cost is crucial to capture speed instructions given by an experimenter. In particular, Fitts's law is recovered in our framework as a limit case.

A Appendix: Technical details

A.1 Asymptotic study

We describe in this asymptotic studies the behavior of the solutions of the linear quadratic model introduced in Sect.. 2.2 when some parameters of the problem go to zero or infinity.

A.1.1 Asymptotic study for small/large time and fixed cost

Let us study the behavior of the optimal solutions when the final time τ varies, the quadratic cost $l(\mathbf{x}, u) = u^2 + \mathbf{x}^T Q\mathbf{x} + 2\mathbf{x}^T Su$ and the terminal conditions $\mathbf{x}^0 = (\theta^0, 0, \dots, 0)$, $\mathbf{x}^f = 0$ being fixed. For every $\tau > 0$ we denote by $\mathbf{x}_\tau(t) = \left(\theta_\tau(t), \dots, \theta_\tau^{(n-1)}(t)\right)$, $t \in [0, \tau]$, the solution of the free-time OC problem in fixed time τ whose expression is given by Eq. (6). Consider first the case of large times, that is the case where $\tau \to \infty$. Remind that in this case $e^{\tau A_+}$ and $e^{-\tau A_-}$ tend to zero.

Lemma 4. *When $\tau \to \infty$, there holds*

$$\mathbf{x}_\tau(t) = e^{tA_+}\mathbf{x}^0 + O(\|e^{\tau A_+}\mathbf{x}^0\|).$$

As a consequence, there exists constants $c, \alpha > 0$, and $\epsilon \in (0, 1)$, such that

$$\theta_\tau(t) = c\theta^0 e^{-\alpha t} + O(e^{-\alpha\tau}), \qquad \text{for any } t \in [\epsilon\tau, \tau].$$

We thus recover a somewhat intuitive result: the solution of a LQ problem in fixed time converges to the solution of the same LQ problem with infinite horizon when the time goes to infinity (see Remark 2).

Proof. We deduce directly from the conditions of Eq. (7) the values of $\mathbf{p}_1 = \mathbf{p}_1(\tau)$ and $\mathbf{p}_2 = \mathbf{p}_2(\tau)$ in function of $\mathbf{x}^0$. By putting these values into Eq. (6), we obtain

$$\mathbf{x}_\tau(t) = e^{tA_+}\left(I - e^{-\tau A_-}e^{\tau A_+}\right)^{-1}\mathbf{x}^{\mathbf{0}} - e^{(t-\tau)A_-}e^{\tau A_+}\mathbf{x}^{\mathbf{0}},$$

which is of the form $e^{tA_+}\mathbf{x}^0 + O(\|e^{\tau A_+}\mathbf{x}^0\|)$ since A_+ is stable and A_- is anti-stable. Now, $e^{tA_+}\mathbf{x}^0$ is a function of t which can be written as a sum of decreasing exponential terms. Denoting by $e^{-\alpha t}$ the less decreasing term in this sum, it appears that all other exponential terms in $e^{tA_+}\mathbf{x}^0$ are negligible in front of $e^{-\alpha\tau}$ for t/τ not too small and we obtain the formula for θ_τ (note that in general $\alpha = \min\{-\Re(\lambda) \ : \ \lambda \textit{ eigenvalue of } A_+\}$). □

Consider now the case of small times, i.e. the case where $\tau \to 0$. In that case we can prove the following result.

Lemma 5. *Let $p(s)$ be the polynomial function of degree $2n-1$ defined by $\left(p(0), p'(0), \ldots, p^{(n-1)}(0)\right) = \mathbf{x}^0$ and $\left(p(1), p'(1), \ldots, p^{(n-1)}(1)\right) = \mathbf{x}^f$. Then*

$$\theta_\tau(t) = p(\frac{t}{\tau}) + O(\tau).$$

As a consequence, $\theta_\tau(t) \approx p(\frac{t}{\tau})$ for small times τ: a change of the final time induces approximately a temporal rescaling of the solutions.

Remark 6. Note that since the terminal conditions are equilibriums, the polynomial $p(\cdot)$ satisfies $\dot{p}(t) = \dot{p}(1-t)$, which implies that the velocity profiles of θ_τ have an almost symmetric shape for small times τ. Indeed, the polynomial function $\widetilde{p}(t) = \theta^0 - p(1-t)$ satisfies the same conditions at $t = 0$ and $t = 1$ as $p(t)$, which implies by unicity of the solution that $\widetilde{p}(t) = p(t)$, and so the conclusion.

Proof. Let us start with a preliminary remark on the optimal solution θ_τ. On one hand, it follows from Eq. (6) that $\theta_\tau(t)$ is an analytic function (i.e. it is equal to its Taylor series) which depends linearly on the vectors $\mathbf{p}_1 = \mathbf{p}_1(\tau)$ and $\mathbf{p}_2 = \mathbf{p}_2(\tau)$. Hence, all derivatives of θ_τ at 0 depend linearly on the pair $(\mathbf{p}_1, \mathbf{p}_2)$. On the other hand, due to the particular properties of the matrices A_-, A_+ (see [15, Lemma 1]), there is a one-to-one correspondence between $(\mathbf{p}_1, \mathbf{p}_2)$ and the $2n$ first derivatives of θ_τ at 0, i.e. by $\theta_\tau^{(k)}(0),\ 0 \leq k \leq 2n-1$. As a consequence, all derivatives of θ_τ at 0 depend linearly on the $2n$ first ones: for every integer k there exists a constant C_k such that, for any τ, $|\theta_\tau^{(k)}(0)| \leq C_k \Theta_\tau$, where

$$\Theta_\tau = \max\left\{|\theta_\tau^{(k)}(0)|,\ 0 \leq k \leq 2n-1\right\}.$$

Set $\phi_\tau(t) = \theta_\tau(t) - p(\frac{t}{\tau})$. We have to prove that $\phi_\tau(t) = O(\tau)$. The above remark and the fact that $\left(p(0), \ldots, p^{(n-1)}(0)\right) = \left(\theta_\tau(0), \ldots, \theta_\tau^{(n-1)}(0)\right) = (\theta^0, 0, \ldots, 0)$ imply that the Taylor expansion of ϕ_τ has the form,

$$\phi_\tau(t) = \sum_{k=n}^{2n-1} \frac{t^k}{k!}\left(\theta_\tau^{(k)}(0) - \frac{p^{(k)}(0)}{\tau^k}\right) + \Theta_\tau O(t^{2n}), \tag{11}$$

where all $O(\cdot)$ are uniform with respect to τ. By definition of $p(\cdot)$ we have also $\phi_\tau^{(j)}(\tau) = 0$ for $j = 0, \ldots, n-1$, and from Eq. (11) we get

$$\sum_{k=n}^{2n-1} \frac{1}{(k-j)!}\left(\tau^k \theta_\tau^{(k)}(0) - p^{(k)}(0)\right) = \Theta_\tau O(\tau^{2n}), \qquad j = 0, \ldots, n-1.$$

It follows that, for $k = n, \ldots, 2n-1$ there holds $\tau^k \theta_\tau^{(k)}(0) - p^{(k)}(0) = \Theta_\tau O(\tau^{2n})$, and thus from the definition of Θ_τ we obtain that $\Theta_\tau \tau^{2n} = O(\tau)$. This and Eq. (11) give $\phi_\tau(t) = O(\tau)$, which proves the lemma. $\square$

A.1.2 Asymptotic study for fixed time

Let us try to understand now how the optimal solutions behave when some coefficients in the cost function are modified. We fix an initial state $\mathbf{x}^0 = (\theta^0, 0, \ldots, 0)$, a final one $\mathbf{x}^f = 0$, and an infinitesimal CoT $g(t)$. We consider a family of costs $l_r(\mathbf{x}, u)$ depending on a parameter r of the form

$$l_r(\mathbf{x}, u) = u^2 + r\theta^2 + \mathbf{x}^T Q_0 \mathbf{x} + 2\mathbf{x}^T S u,$$

that is, with a matrix $Q(r) = Q_0 + re_1e_1^T$ ($e_1 = (1, 0, \ldots, 0)$ denotes the first vector of the canonical basis of $\mathbb{R}^n$). We want to study the behavior when r tends to ∞ of the optimal solutions of the following free-time OC problem: *minimize the cost*

$$C_r(u, t_u) = \int_0^{t_u} \left(g(t) + l_r(\mathbf{x}_u(t), u(t))\right) dt,$$

among all inputs $u(\cdot)$ and all times t_u such that $\mathbf{x}_u(0) = \mathbf{x}^0$ and $\mathbf{x}_u(t_u) = \mathbf{x}^f$. As we have seen previously, the time $\tau = \tau(r)$ is determined by Eq. (9) and the optimal solutions are the one of the OC problem $\min \int_0^\tau l_r(\mathbf{x}, u)$ in fixed time τ.

Lemma 7. *For every $r > 0$ we denote by $\mathbf{x}_r(t) = \left(\theta_r(t), \ldots, \theta_r{}^{(n-1)}(t)\right)$, $t \in [0, \tau(r)]$, the solution of the free-time OC problem associated with C_r. Assume that the infinitesimal cost of time $g(\cdot)$ is a bounded function. Then there exists constants $c, \alpha > 0$, and $\epsilon \in (0, 1)$, such that, when $r \to \infty$, we have $r^{1/2n}\tau(r) \to \infty$ and*

$$\theta_r(t) = c\theta^0 e^{-\alpha r^{1/2n} t} + O\left(e^{-\alpha r^{1/2n}\tau(r)}\right), \qquad \textit{for any } t \in [\epsilon\tau(r), \tau(r)].$$

Note that the boundedness assumption on g is very natural and seems to be verified experimentally since we obtain functions $g(t)$ that are decreasing for large t.

Proof. To simplify the study, we give only the proof in the case where the matrices Q_0 and S are zero, and the dynamics (Eq. (5)) is of the form $\theta^{(n)} = u$. The proof of the complete result can be obtained by showing that this case actually gives the highest order terms with respect to r. With the preceding hypothesis, $\theta_r(t)$ is the solution of the OC problem in fixed time $\tau = \tau(r)$ associated with the infinitesimal cost $u^2 + r\theta^2$, or equivalently with $\frac{1}{r}u^2 + \theta^2$. Set $\widetilde{\theta_r}(t) = \theta_r(tr^{-1/2n})$. Then $\widetilde{\theta_r}(t)$ is the solution of the OC problem in fixed time $r^{1/2n}\tau$ associated with the infinitesimal cost $u^2 + \theta^2$. In the latter problem, nothing depends on r except the duration $r^{1/2n}\tau$. It results from the analysis of Sect. 2.2.2 that there exists a universal function of time $\varphi(\cdot)$ such that $\widetilde{u_r}(r^{1/2n}\tau) = \theta^0\varphi(r^{1/2n}\tau)$. Since we have $u_r(t) = r^{1/2}\widetilde{u_r}(r^{1/2n}t)$, we obtain

$$u_r(\tau) = r^{1/2}\theta^0\varphi(r^{1/2n}\tau).$$

Now remember (see Eq. (8)) that the time τ must satisfy $g(\tau) = (u_r(\tau))^2$, which gives $g(\tau) = r\left(\theta^0\varphi(r^{1/2n}\tau)\right)^2$. Assume by contradiction that the quantity $r^{1/2n}\tau(r)$ is bounded as $r \to \infty$. Then $\varphi(r^{1/2n}\tau)$ is bounded away from zero (φ is positive and continuous on $(0, +\infty)$, and converges to $+\infty$ as $t \to 0$, see Fig. 2), and therefore $g(\tau(r)) \to \infty$ as $r \to \infty$, which contradicts the boundedness of g. Thus we get $r^{1/2n}\tau(r) \to \infty$.

Since $\widetilde{\theta_r}(t)$ is the solution of an OC problem in fixed time with a very large time $r^{1/2n}\tau(r)$, it results from Lemma 4 that $\widetilde{\theta_r}(t) = c\theta^0e^{-\alpha t} + O\left(e^{-\alpha r^{1/2n}\tau(r)}\right)$ for t larger than $\epsilon r^{1/2n}\tau(r)$ for some $\epsilon \in (0, 1)$. The conclusion follows from $\theta_r(t) = \widetilde{\theta_r}(tr^{1/2n})$. $\square$

A.2 Model for arm reaching movements

Single degree-of-freedom (dof) limb. For a 1-dof arm moving in the horizontal plane, the basic model used throughout the study was already described in numerous other studies (e.g. [25, 20, 48, 19, 3]) and is as follows:

$$\begin{cases} I\ddot{\theta} & = \tau - b\dot{\theta} \\ \dot{\tau} & = u \end{cases} \tag{12}$$

where is θ the shoulder joint angle, τ is the muscle torque, b is the friction coefficient ($b = 0.87$ here), I is the moment of inertia of the arm with respect to the shoulder joint (value estimated based upon Winter's table for each participant; [56]) and u is the single control variable.

For the trajectory cost we typically considered canonical quadratic costs of the form $l(\mathbf{x}, u) = u^2 + \mathbf{x}^T Q\mathbf{x} + 2\mathbf{x}^T Su$, where $\mathbf{x} = (\theta, \dot{\theta}, \ddot{\theta}) \in \mathbb{R}^3$ denotes the system state. The two most famous examples are the minimum torque change corresponding to $l(\mathbf{x}, u) = u^2$ [54] and the minimum jerk corresponding to $l(\mathbf{x}, u) = \dddot{\theta}^2$ [17]. Other costs, possibly composite, may account for such planar movements in fixed time but such an investigation is out of the scope of the present chapter (but see [4, 5, 18, 6, 3] for studies related to the trajectory cost identification).

References

1. Bagesteiro LB, Sainburg RL (2002) Handedness: dominant arm advantages in control of limb dynamics. J Neurophysiol 88(5):2408–2421, DOI 10.1152/jn.00901.2001, URL http://dx.doi.org/10.1152/jn.00901.2001

2. Berardelli A, Rothwell JC, Thompson PD, Hallett M (2001) Pathophysiology of bradykinesia in parkinson's disease. Brain 124(Pt 11):2131–2146

3. Berret B, Jean F (2016) Why don't we move slower? the value of time in the neural control of action. J Neurosci 36(4):1056–1070, DOI 10.1523/JNEUROSCI.1921-15.2016, URL http://dx.doi.org/10.1523/JNEUROSCI.1921-15.2016

4. Berret B, Darlot C, Jean F, Pozzo T, Papaxanthis C, Gauthier JP (2008) The inactivation principle: mathematical solutions minimizing the absolute work and biological implications for the planning of arm movements. PLoS Comput Biol 4(10):e1000,194, DOI 10.1371/journal.pcbi.1000194, URL http://dx.doi.org/10.1371/journal.pcbi.1000194

5. Berret B, Gauthier JP, Papaxanthis C (2008) How humans control arm movements. Proceedings of the Steklov Institute of Mathematics 261:44–58

6. Berret B, Chiovetto E, Nori F, Pozzo T (2011) Evidence for composite cost functions in arm movement planning: an inverse optimal control approach. PLoS Comput Biol 7(10):e1002,183, DOI 10.1371/journal.pcbi.1002183, URL http://dx.doi.org/10.1371/journal.pcbi.1002183

7. Brown JMM, Gilleard W (1991) Transition from slow to ballistic movement: development of triphasic electromyogram patterns. European Journal of Applied Physiology and Occupational Physiology 63(5):381–386, DOI 10.1007/BF00364466, URL http://dx.doi.org/10.1007/BF00364466

8. Brown SH, Hefter H, Mertens M, Freund HJ (1990) Disturbances in human arm movement

trajectory due to mild cerebellar dysfunction. J Neurol Neurosurg Psychiatry 53(4):306–313

9. Card S, Moran T, Newell A (1983) The psychology of human-computer interaction. L. Erlbaum Associates

10. Churchland MM, Santhanam G, Shenoy KV (2006) Preparatory activity in premotor and motor cortex reflects the speed of the upcoming reach. J Neurophysiol 96(6):3130–3146, DOI 10.1152/jn.00307.2006, URL http://dx.doi.org/10.1152/jn.00307.2006

11. Connelly EM (1984) A control model: An alternative interpretation of fitts' law. Proceedings of the Human Factors and Ergonomics Society Annual Meeting 28(7):625–628, DOI 10.1177/154193128402800722

12. Desmurget M, Grafton S (2000) Forward modeling allows feedback control for fast reaching movements. Trends Cogn Sci 4(11):423–431

13. Doeringer JA, Hogan N (1998) Intermittency in preplanned elbow movements persists in the absence of visual feedback. J Neurophysiol 80(4):1787–1799

14. Elliott D, Helsen WF, Chua R (2001) A century later: Woodworth's (1899) two-component model of goal-directed aiming. Psychol Bull 127(3):342–357

15. Ferrante A, Marro G, Ntogramatzidis L (2005) A parametrization of the solutions of the finite-horizon lq problem with general cost and boundary conditions. Automatica 41:1359–1366

16. Fitts PM (1954) The information capacity of the human motor system in controlling the amplitude of movement. J Exp Psychol 47(6):381–391

17. Flash T, Hogan N (1985) The coordination of arm movements: an experimentally confirmed mathematical model. J Neurosci 5(7):1688–1703

18. Gauthier JP, Berret B, Jean F (2010) A biomechanical inactivation principle. Proceedings of the Steklov Institute of Mathematics 268:93–116

19. Gaveau J, Berret B, Demougeot L, Fadiga L, Pozzo T, Papaxanthis C (2014) Energy-related optimal control accounts for gravitational load: comparing shoulder, elbow, and wrist rotations. J Neurophysiol 111(1):4–16, DOI 10.1152/jn.01029.2012, URL http://dx.doi.org/10.1152/jn.01029.2012

20. Gentili R, Cahouet V, Papaxanthis C (2007) Motor planning of arm movements is direction-dependent in the gravity field. Neuroscience 145(1):20–32, DOI 10.1016/j.neuroscience.2006.11.035, URL http://dx.doi.org/10.1016/j.neuroscience.2006.11.035

21. Hallett M, Marsden CD (1979) Ballistic flexion movements of the human thumb. J Physiol 294:33–50

22. Harris CM, Wolpert DM (1998) Signal-dependent noise determines motor planning. Nature 394(6695):780–784, DOI 10.1038/29528, URL http://dx.doi.org/10.1038/29528

23. Harris CM, Wolpert DM (2006) The main sequence of saccades optimizes speed-accuracy trade-off. Biol Cybern 95(1):21–29, DOI 10.1007/s00422-006-0064-x, URL http://dx.doi.org/10.1007/s00422-006-0064-x

24. Hoff B (1994) A model of duration in normal and perturbed reaching movement. Biological Cybernetics pp 481–488

25. Hogan N (1984) An organizing principle for a class of voluntary movements. J Neurosci 4(11):2745–2754

26. Isenberg C, Conrad B (1994) Kinematic properties of slow arm movements in parkinson's disease. J Neurol 241(5):323–330

27. Johnson MT, Coltz JD, Ebner TJ (1999) Encoding of target direction and speed during visual instruction and arm tracking in dorsal premotor and primary motor cortical neurons. Eur J Neurosci 11(12):4433–4445

28. Kappen HJ (2011) Optimal control theory and the linear bellman equation. In: Barber D, Cemgil AT, Chiappa S (eds) Bayesian Time Series Models, Cambridge University Press, pp 363–387, URL http://dx.doi.org/10.1017/CBO9780511984679.018, cambridge Books Online

29. Kirk DE (1970) Optimal control theory: An Introduction. Prentice-Hall, New Jersey

30. Lee EB, Markus L (1967) Foundations of Optimal Control Theory. John Wiley

31. MacKenzie C, Iberall T (1994) The Grasping Hand. Advances in psychology, North-Holland

32. MacKenzie CL, Marteniuk RG, Dugas C, Liske D, Eickmeier B (1987) Three-dimensional movement trajectories in fitts' task: Implications for control. The Quarterly Journal of Experimental Psychology Section A 39(4):629–647, DOI 10.1080/14640748708401806, URL http://dx.doi.org/10.1080/14640748708401806, http://dx.doi.org/10.1080/14640748708401806

33. MacKenzie IS (1992) Fitts' law as a research and design tool in human-computer interaction. Hum-Comput Interact 7(1):91–139

34. Mansard N, Stasse O, Evrard P, Kheddar A (2009) A versatile generalized inverted kinematics implementation for collaborative working humanoid robots: the Stack of Tasks. In: ICAR'09: International Conference on Advanced Robotics, Munich, Germany, pp 1–6, URL http://hal-lirmm.ccsd.cnrs.fr/lirmm-00796736

35. Mazzoni P, Hristova A, Krakauer JW (2007) Why don't we move faster? parkinson's disease, movement vigor, and implicit motivation. J Neurosci 27(27):7105–7116, DOI

10.1523/JNEUROSCI.0264-07.2007, URL http://dx.doi.org/10.1523/JNEUROSCI.0264-07.2007

36. Missenard O, Fernandez L (2011) Moving faster while preserving accuracy. Neuroscience 197:233–241, DOI 10.1016/j.neuroscience.2011.09.020, URL http://dx.doi.org/10.1016/j.neuroscience.2011.09.020

37. Nelson WL (1983) Physical principles for economies of skilled movements. Biol Cybern 46(2):135–147

38. Nori F, Frezza R (2004) Linear optimal control problems and quadratic cost functions estimation. In: 12th Mediterranean Conference on Control and Automation, MED'04. Kusadasi, Aydin, Turkey

39. Pattacini U, Nori F, Natale L, Metta G, Sandini G (2010) An experimental evaluation of a novel minimum-jerk cartesian controller for humanoid robots. In: Intelligent Robots and Systems (IROS), 2010 IEEE/RSJ International Conference on, pp 1668–1674, DOI 10.1109/IROS.2010.5650851

40. Plamondon R, Alimi AM (1997) Speed/accuracy trade-offs in target-directed movements. Behavioral and Brain Sciences 20:279–303, DOI null

41. Pontryagin LS, Boltyanskii VG, Gamkrelidze RV, Mishchenko EF (1964) The Mathematical Theory of Optimal Processes. Pergamon Press

42. Qian N, Jiang Y, Jiang ZP, Mazzoni P (2013) Movement duration, fitts's law, and an infinite-horizon optimal feedback control model for biological motor systems. Neural Comput 25(3):697–724

43. Shadmehr R (2010) Control of movements and temporal discounting of reward. Curr Opin Neurobiol 20(6):726–730, DOI 10.1016/j.conb.2010.08.017, URL http://dx.doi.org/10.1016/j.conb.2010.08.017

44. Shadmehr R, Krakauer JW (2008) A computational neuroanatomy for motor control. Exp Brain Res 185(3):359–381, DOI 10.1007/s00221-008-1280-5, URL http://dx.doi.org/10.1007/s00221-008-1280-5

45. Shadmehr R, Mussa-Ivaldi S (2012) Biological Learning and Control. MIT Press

46. Shadmehr R, Orban de Xivry JJ, Xu-Wilson M, Shih TY (2010) Temporal discounting of reward and the cost of time in motor control. J Neurosci 30(31):10,507–10,516, DOI 10.1523/JNEUROSCI.1343-10.2010, URL http://dx.doi.org/10.1523/JNEUROSCI.1343-10.2010

47. Stengel R (1986) Optimal Control and Estimation. Dover books on advanced mathematics, Dover Publications

48. Tanaka H, Krakauer JW, Qian N (2006) An optimization principle for determining movement duration. J Neurophysiol 95(6):3875–3886, DOI 10.1152/jn.00751.2005, URL http://dx.doi.org/10.1152/jn.00751.2005

49. Todorov E (2004) Optimality principles in sensorimotor control. Nat Neurosci 7(9):907–915, DOI 10.1038/nn1309, URL http://dx.doi.org/10.1038/nn1309

50. Todorov E (2006) Optimal control theory, Bayesian Brain: Probabilistic Approaches to Neural Coding, Doya K (ed), chap 12, pp 269–298

51. Todorov E, Jordan MI (2002) Optimal feedback control as a theory of motor coordination. Nat Neurosci 5(11):1226–1235, DOI 10.1038/nn963, URL http://dx.doi.org/10.1038/nn963

52. Trommershï¿œuser J, Maloney LT, Landy MS (2008) Decision making, movement planning and statistical decision theory. Trends Cogn Sci 12(8):291–297, DOI 10.1016/j.tics.2008.04.010, URL http://dx.doi.org/10.1016/j.tics.2008.04.010

53. Turner RS, Desmurget M (2010) Basal ganglia contributions to motor control: a vigorous tutor. Curr Opin Neurobiol 20(6):704–716, DOI 10.1016/j.conb.2010.08.022, URL http://dx.doi.org/10.1016/j.conb.2010.08.022

54. Uno Y, Kawato M, Suzuki R (1989) Formation and control of optimal trajectory in human multijoint arm movement. minimum torque-change model. Biol Cybern 61(2):89–101

55. van der Wel RPRD, Sternad D, Rosenbaum DA (2010) Moving the arm at different rates: slow movements are avoided. J Mot Behav 42(1):29–36, DOI 10.1080/00222890903267116, URL http://dx.doi.org/10.1080/00222890903267116

56. Winter D (1990) Biomechanics and Motor Control of Human Movement. New York: John Wiley & Sons

57. Young SJ, Pratt J, Chau T (2009) Target-directed movements at a comfortable pace: movement duration and fitts's law. J Mot Behav 41(4):339–346, DOI 10.3200/JMBR.41.4.339-346, URL http://dx.doi.org/10.3200/JMBR.41.4.339-346