

Order-preserving 1-string representations of planar graphs

Therese Biedl* and Martin Derka**

David R. Cheriton School of Computer Science, University of Waterloo
 {biedl, mderka}@uwaterloo.ca

Abstract. This paper considers 1-string representations of planar graphs that are *order-preserving* in the sense that the order of crossings along the curve representing vertex v is the same as the order of edges in the clockwise order around v in the planar embedding. We show that this does not exist for all planar graphs (not even for all planar 3-trees), but show existence for some subclasses of planar partial 3-trees. In particular, for outer-planar graphs it can be order-preserving and *outer-string* in the sense that all ends of strings are on the outside of the representation.

1 Introduction

String representations recently received a lot of attention, especially for planar graphs. Scheinerman [20] had asked in 1984 whether every planar graph can be represented as the intersection graph of segments in the plane. This was settled partially by Chalopin, Gonçalves and Ochem [6], who showed that every planar graph has a 1-string representation, i.e., a representation as an intersection graph of strings such that any two strings may cross at most once. Extending their result, in 2009 Chalopin and Gonçalves finally settled Scheinerman’s conjecture in the positive [5]. We later showed that 1-string representations of planar graphs can be achieved even with orthogonal curves with at most 2 bends [3]. A number of other papers gave string representations for subclasses of planar graphs that are simpler to build and/or have other useful properties, see for example [11,14,8,9,2]. Testing whether a graph has a string representation is NP-hard [15,18] and in NP [19]; the latter is not obvious because string representations may require exponentially many bends for non-planar graphs [16].

Our results: In this paper, we study the following question: Does every planar graph have a 1-string representation where the order of crossings along curves *preserves* the planar embedding in the sense that the order of crossings along the curve of v corresponds to the cyclic order of edges around v in some planar embedding? This is motivated by that we found string representations quite hard to read; during our work on [3] we struggled to verify correctness in some cases because the crossing of curves for an edge occurred at unexpected places.

* Research supported by NSERC.

** The author was supported by the Vanier CGS.

Furthermore, having an order-preserving string representation could make it easier to create such representations by using the typical incremental approach that adds one vertex on the outer-face at a time; for this it would be especially helpful if such representations were also *outer-string* in the sense that ends of strings are on the infinite region defined by the representation. We show the following:

- Not all planar graphs have order-preserving 1-string representations. In fact, we can construct a planar 3-tree that has no such representation.
- For some subclasses of planar partial 3-trees, we construct order-preserving 1-string representations. For outer-planar graphs, these are additionally outer-string (and use segments), while for the other graph classes we show that order-preserving outer-1-string representations do not always exist.

We are not aware of any previous results on order-preserving 1-string representations. (On the other hand, string-representations of planar graphs obtained from contact representations are usually order-preserving, but strings then intersect twice, at least for some edges.) The closest related results are on the *abstract graph realizability problem* [15,18], which asks to draw a graph such that only a given set of edge-pairs are allowed to cross.

2 Definitions

A *string representation* \mathcal{R} assigns a curve \mathbf{v} in the plane to every vertex v in a graph in such a way that (v,w) is an edge if and only if \mathbf{v} intersects \mathbf{w} . (Throughout the paper, bold-face \mathbf{x} always denotes the curve assigned to vertex x .) We demand that \mathbf{u} and \mathbf{v} intersect only if there is a proper crossing, i.e., any sufficiently small circle centered at an intersection-point crosses \mathbf{u} , \mathbf{v} , \mathbf{u} , \mathbf{v} in that order. (In particular no curve \mathbf{u} should end on another curve \mathbf{v} , though such a touching-point could always be resolved into a proper crossing by extending \mathbf{u} a bit.) We also do not allow three curves to share a point. A *1-string representation* is a string representation such that any two curves cross at most once. A *segment representation* uses straight-line segments in place of strings. A *B_k -VPG-representation* uses orthogonal curves with at most k bends as strings.

A string representation \mathcal{R} divides the plane into connected regions. The *contour* is the infinite region of $\mathbb{R}^2 - \mathcal{R}$. A string representation is called *weakly outer-string* if all vertex curves are incident to the contour. It is called *outer-string* if all vertex curves have an end incident to the contour.¹ A weakly outer-string representation can be made outer-string by “doubling back” along the curve of each vertex, but this does not work for an outer-1-string representation, because doubling back along the curve would make some curves cross twice. See [4,13] and the references therein for more on outer-string representations.

¹ One could distinguish this further by whether both ends must be on the contour or whether one end suffices. All our outer-string constructions have both ends on the contour, while all our impossibility-results hold even if only one end is required to be on the contour, so the distinction does not matter for the results in our paper.

In this paper, we only consider connected graphs. A graph is called *planar* if it can be drawn in the plane without crossing. Such a planar drawing Γ defines, by enumerating edges around vertices in clockwise order, a *rotation scheme*, i.e., an assignment of a cyclic order of edges at each vertex. From the rotation scheme, one can read the *faces*, i.e., the vertices and edges that are incident to each connected piece of $\mathbb{R}^2 - \Gamma$. A *plane graph* is a planar graph with a fixed rotation scheme. An *outer-planar graph* is a planar graph that has a rotation scheme such that all vertices are incident to one face. An *outer-plane graph* is a plane graph with the rotation system that describes such an embedding. A *k-tree* (used here only for $k = 2, 3$) is a graph that has a vertex order v_1, \dots, v_n such that v_1, \dots, v_k is a clique, and each v_i for $i > k$ has exactly k neighbours in v_1, \dots, v_{i-1} , and they form a clique. A *partial k-tree* is a subgraph of a k -tree. Every outer-planar graph is a partial 2-tree.

Fix a rotation scheme of a graph. We say that a 1-string representation is *order-preserving* with respect to the rotation scheme if for any vertex v , we can walk along curve \mathbf{v} from one end to the other and encounter the crossings with $\mathbf{w}_1, \dots, \mathbf{w}_k$ in the same order in which the neighbours w_1, \dots, w_k of v appear in the cyclic order of edges around v . This leaves open the choice which neighbour of v should be w_1 , since the order at v is cyclic while the order along \mathbf{v} is not.²

3 Graphs with no order-preserving representations

In this section, we show that there exist planar graphs that have no 1-string representation that preserves the order of any planar embedding. To define them, we need the following graph operation: Given a plane graph G , the *stellation* of G is obtained by inserting a new vertex into every face of G , and making it adjacent to all vertices incident to that face. The *triple-stellation* of G is obtained by stellating G to get G' , stellating G' to get G'' , and finally stellating G'' .

Lemma 1. *Let G be a plane graph with minimum degree 3 and at least $|V(G)| + 1$ faces that are triangles. Then the triple-stellation G''' of G has no order-preserving 1-string representation with respect to this rotation scheme.*

Proof. Assume for contradiction we had such a 1-string representation \mathcal{R} , and let \mathcal{R}_G be the induced 1-string representation of G , which is also order-preserving. The following notation will be helpful: If a, c are neighbours of b , then let $\mathbf{b}[a, c]$ be the stretch of \mathbf{b} between the intersection with \mathbf{a} and \mathbf{c} .

Consider a face-vertex-incidence in G , which can be described by giving a vertex b and two neighbours a, c of b that are consecutive in the clockwise order at b . We call such a face-vertex-incidence *unbroken* if (in \mathcal{R}_G) $\mathbf{b}[a, c]$ contains no other crossing, else we call it *broken*. Since \mathcal{R}_G is order-preserving, for every vertex b in G only one face-vertex-incidence at b is broken. Since G has at least

² Once we fix how to break up the cyclic order at all vertices, an order-preserving 1-string representation can be described abstractly as a graph H and can be realized if and only if H is planar. Hence the problem is interesting only if we keep this choice.

$|V(G)| + 1$ triangular faces, there exists a face $T = \{u, v, w\}$ of G such that all face-vertex-incidences at T are unbroken. We will find a contradiction at the stellation vertices that were placed in T . See also Fig. 1.

Let x be the vertex that (during the stellation of G to get G') was placed in face T . We claim that \mathbf{x} must intersect \mathbf{u} in $\mathbf{u}[v, w]$. To see this, recall that $\deg_G(u) \geq 3$, hence u has at least one other neighbour u' in G . Since the face-incidence at u is unbroken, $\mathbf{u}[v, w]$ contains no other crossing of \mathcal{R}_G , so \mathbf{u}' intersects \mathbf{u} outside this stretch. Since T is a face in G , the (clockwise or counter-clockwise) order of neighbours at u in G' contains u', v, x, w . To maintain this order in the string representation, the intersection between \mathbf{x} and \mathbf{u} (in \mathcal{R}) must be on $\mathbf{u}[v, w]$. Similarly one argues that \mathbf{x} intersects $\mathbf{v}[u, w]$ and $\mathbf{w}[u, v]$.

Let C be the region bounded by $\mathbf{u}[v, w] \cup \mathbf{w}[u, v] \cup \mathbf{v}[w, u]$. Curve \mathbf{x} intersects δC three times, and no more since curves intersect at most once in a 1-string representation. So \mathbf{x} starts (say) inside C , crosses δC to go outside, crosses δC to go inside, and then crosses δC again to end outside. Between the second and third crossing, \mathbf{x} contains a stretch that is inside C ; after possible renaming of $\{u, v, w\}$ we assume that this is $\mathbf{x}[v, w]$. This stretch splits C into two parts, say C' (incident to parts of \mathbf{u}) and C'' (incident to the crossing of \mathbf{v} and \mathbf{w}).

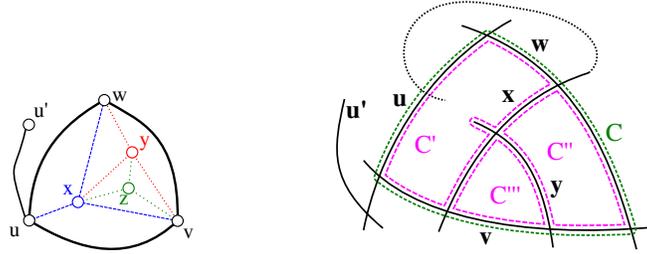


Fig. 1. For the proof of Lemma 1.

Let y be the vertex that (during the stellation of G' to get G'') was placed in the face $\{v, w, x\}$ of G' . Since v, w, x all have degree 3 or more in G' , as before one argues that \mathbf{y} must intersect $\mathbf{x}[v, w]$, $\mathbf{w}[x, v]$ and $\mathbf{v}[w, x]$. Curve \mathbf{y} intersects $\delta C'$ (in $\mathbf{x}[v, w]$), but cannot intersect $\delta C'$ a second time, else it would cross \mathbf{u} (but $(u, y) \notin E$) or would cross one of $\mathbf{x}, \mathbf{v}, \mathbf{w}$ twice (which is not allowed). Hence \mathbf{y} starts inside C' , then crosses \mathbf{x} , and then crosses one of \mathbf{v} and \mathbf{w} . Up to renaming of $\{v, w\}$ we may assume that \mathbf{y} crosses \mathbf{v} first. Hence $\mathbf{y}[x, v]$ splits C'' into two parts, say C''' (incident to parts of \mathbf{w}) and C'''' (incident to the crossing of \mathbf{v} and \mathbf{x}).

Now finally consider the vertex z that was placed in $\{x, y, v\}$ when stellating G'' to obtain G''' . As before one argues that \mathbf{z} has an end inside C' , because it crosses \mathbf{x} in stretch $\mathbf{x}[v, y] \subset \mathbf{x}[v, w]$, and it cannot cross C' again. But we can also see that \mathbf{z} has an end inside C''' , since it crosses $\mathbf{y}[x, v]$ and crosses no other curve on the boundary of C''' . But this means that \mathbf{z} has both ends outside

C''' , contradicting that it must intersect the boundary of C''' three times to respect the edge-orders at x, y, v . Contradiction, so G''' does not have an order-preserving 1-string representation. \square

Theorem 1. *There exists a planar 3-tree that has no order-preserving 1-string representation.*

Proof. Start with an arbitrary planar 3-tree G with $n \geq 6$ vertices; this has minimum degree 3 and $2n - 4 \geq n + 2$ triangular faces in its (unique) rotation scheme. Stellating a 3-tree gives again a 3-tree, so by Lemma 1 the triple-stellation of G is a 3-tree that has no order-preserving 1-string representation. \square

4 Order-preserving outer-1-string representations

Now we turn towards positive results and show that every outer-plane graph has an order-preserving outer-1-string representation. We first discuss one existing result that does not quite achieve this. It is easy to show that every outer-planar graph can be represented as touching-graph of line segments (see e.g. [14] for much broader results). The standard way to do this (see also Fig. 2) results, after extending the segments a bit, in a segment-representation that is order-preserving and weakly outer-string. However, this does not quite achieve our goal, because the ends of segments are not necessarily on the outer-face.

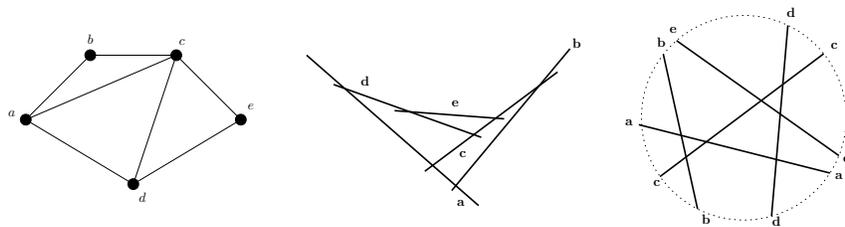


Fig. 2. An outer-planar graph, a weakly segment-representation that is not outer-segment at **e**, and a representation as circle graph that is not order-preserving at **c**.

We instead give two other constructions. The first one uses that any outer-planar graph is a *circle graph*, i.e., the intersection graph of chords of a circle [21]. This obviously gives an end-outer-segment representation, but it need not be order-preserving (see Fig. 2). Our first construction hence re-proves this result and maintains invariants to ensure that the representation is indeed order-preserving.

The resolution in this representation could be very bad, and we therefore give a second construction where the curves are orthogonal instead. We use one bend for each vertex curve here, and so obtain a B_1 -VPG-representation. Since there are n vertices and at most n bends, the representation can be embedded into a grid of size $O(n) \times O(n)$.

In our proofs, we use that any 2-connected outer-planar graph G can be built up as follows [12, Lemma 3]: Fix an edge (u, v) . Now repeatedly add an *ear*, i.e., a path $P = u_0, u_1, \dots, u_k, u_{k+1}$ with $k \geq 1$ where (u_0, u_{k+1}) is an edge on the outer-face of the current graph G' , and u_1, \dots, u_k are new vertices that induce a path and have no edges to G' other than (u_0, u_1) and (u_k, u_{k+1}) .

A crucial requirement of the constructed representation \mathcal{R} of such a subgraph is the following *order-condition*: If w and w' are the counterclockwise and clockwise neighbours of v on the outer-face, then we encounter the neighbours of v in order, starting with w and ending with w' , while walking along \mathbf{v} . Put differently, the broken face-vertex-incidence is the one with the outer-face. We consider \mathbf{v} to be directed so that it intersects first \mathbf{w} and last \mathbf{w}' .

The second crucial ingredient for both proofs is to reserve for edges (somewhat similar as was done for faces in [3,2,6,9]) a region that can be used to attach subgraphs. Thus define a *private region* $S_{\mathbf{u}\mathbf{v}}$ of edge (u, v) to be a region that contains an end of \mathbf{u} and an end of \mathbf{v} and does not intersect any other curve or private regions of \mathcal{R} . Both constructions maintain such a private region $S_{\mathbf{u}\mathbf{v}}$ for every outer-face edge (u, v) . Moreover, if v is the clockwise neighbour of u , then $S_{\mathbf{u}\mathbf{v}}$ contains the tail of \mathbf{u} and the head of \mathbf{v} .

4.1 Circle-chord representation

We now re-prove that outer-planar graphs are circle graphs, and show that furthermore the order can be preserved.

Theorem 2. *Every outer-plane graph has an order-preserving representation as intersection graph of chords of a circle C .*

Proof. It suffices to prove the claim for a 2-connected outer-planar graph G since every outer-planar graph G' is an induced subgraph of a 2-connected outer-planar graph G , and therefore a string representation for G also yields one for G' by deleting curves of vertices in $G - G'$.

We create a representation \mathcal{R} while building up the graph via adding ears, and maintain curve directions and private regions as explained before. Each private region $S_{\mathbf{u}\mathbf{v}}$ is bounded by parts of circle C and a chord of C and does not contain the crossing of \mathbf{u} and \mathbf{v} . Further, the tail of \mathbf{u} and the head of \mathbf{v} are in the interior of the circular arc that bounds $S_{\mathbf{u}\mathbf{v}}$.

In the base case, G is an edge (u, v) which can be represented by two chords through the center of C . See Fig. 3. We reserve two private regions for (u, v) , because the outer-face of a single-edge graph should be viewed as containing this edge twice (we can add ears twice at it). All conditions are easily verified.

For the induction step, let us assume that G was obtained by adding an ear $P = u, x_1, \dots, x_k, v$ at some edge (u, v) , with u the counter-clockwise neighbour of v on the outer-face. Let $C[u, v]$ be the arc of C between the tail of \mathbf{u} and the head of \mathbf{v} that lies inside $S_{\mathbf{u}\mathbf{v}}$. Let u' and v' be two points on C just outside $C[u, v]$ but still within $S_{\mathbf{u}\mathbf{v}}$. If $k = 1$, then we add x_1 by using chord $\overline{u'v'}$ for \mathbf{x}_1 . If $k > 1$, then we insert $2k - 2$ points on the interior of $C[u, v]$ and create

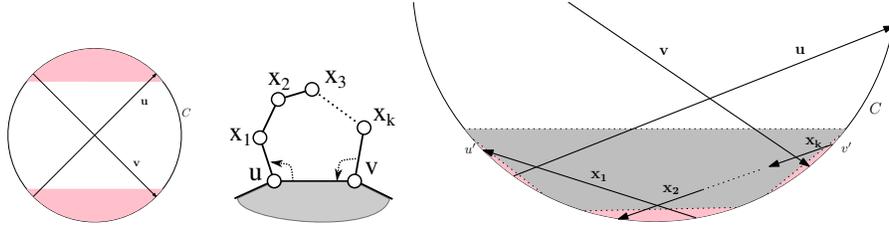


Fig. 3. The base case, and adding chords for an ear.

chords for $\mathbf{x}_1, \dots, \mathbf{x}_k$ so that everyone intersects as required. See Fig. 3, which also shows the private regions that we define for the new outer-face edges.

Since $S_{\mathbf{u}\mathbf{v}}$ was convex, all new curves are inside it and do not intersect any other curves. The orientation of these new curves is determined by the order-condition: \mathbf{x}_i should be oriented so that it intersects first \mathbf{x}_{i+1} (where $x_0 := u$) and then \mathbf{x}_{i-1} (where $x_{k+1} := v$). In particular this means that the private region $S_{\mathbf{x}_i\mathbf{x}_{i+1}}$ contains the tail of \mathbf{x}_i and the head of \mathbf{x}_{i+1} , and hence satisfies the condition on private regions.

It remains to check that the order-condition is satisfied for \mathbf{u} . Since $S_{\mathbf{u}\mathbf{v}}$ contained the tail of \mathbf{u} , this means that \mathbf{x}_1 becomes the first curve to be intersected by \mathbf{u} , which is correct since x_1 is the clockwise neighbour of u on the outer-face. Likewise one argues that the order-condition holds for \mathbf{v} . Hence all conditions hold, and after repeating for all ears we obtain an order-preserving representation as intersection graph of chords of a circle. \square

4.2 B_1 -VPG representation

Now we create, for any outer-planar graph, a B_1 -VPG representation that is order-preserving and outer-string. However, the ends will not be on a circle; instead they will lie on a closed curve S that we maintain throughout the construction and that surrounds the entire representation \mathcal{R} without truly intersecting any curve. All vertices are 1-bend poly-lines with slopes ± 1 (after rotating by 45° this gives the B_1 -VPG representation); this allows us to use an orthogonal curve for S . Fig. 4 illustrates types of private regions that we will use for this construction: $S_{\mathbf{u}\mathbf{v}}$ contains no bend of \mathbf{u} or \mathbf{v} , and it is an isosceles right triangle whose hypotenuse lies on S .

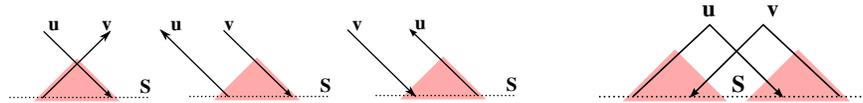


Fig. 4. Three types of private regions (three more can be obtained by flipping horizontally), and the base case.

Theorem 3. *Every outer-planar graph G has an order-preserving outer-1-string B_1 -VPG-representation \mathcal{R} .*

Proof. As before it suffices to prove the claim for 2-connected outer-planar graphs G . We proceed by induction on the number of vertices, building \mathcal{R} while adding ears. In the base case, G is an edge (u, v) which can be represented by two 1-bend curves positioned and oriented as shown in Fig. 4, which also shows the private region. We use a horizontal segment for S (this can be expanded into a closed curve surrounding \mathcal{R} arbitrarily).

For the induction step, let us assume that G was obtained by adding an ear $P = u, x_1, \dots, x_k, v$ at some edge (u, v) , with u the counter-clockwise neighbour of v on the outer-face. After possible rotation the hypotenuse of the private region S_{uv} is horizontal with S_{uv} above it. We distinguish cases:

1. \mathbf{u} and \mathbf{v} have different slopes in S_{uv} and $k = 1$ (i.e. we add one vertex x).

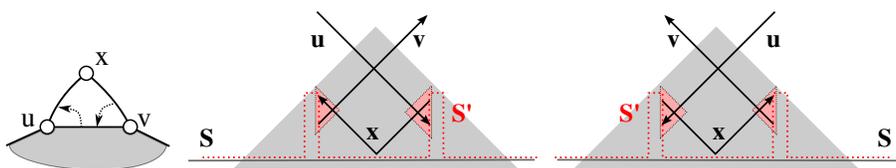


Fig. 5. Adding a single node if \mathbf{u} and \mathbf{v} have different slopes.

We add a 1-bend curve \mathbf{x} with the bend pointing downwards. See Fig. 5, which also shows the private regions that we define for (u, x) and (x, v) . Curve \mathbf{x} fits entirely inside S_{uv} by placing the bend in the interior of S_{uv} and shortening \mathbf{u} and \mathbf{v} appropriately so that the ends of \mathbf{x} are vertically aligned with those of \mathbf{u} and \mathbf{v} . We can now easily find a new curve S' by adding “detours” to S that reach the hypotenuses of the new private regions. These detours are inside S_{uv} and hence intersect no other curves (since we shortened \mathbf{u} and \mathbf{v}). So the new curve S' is a closed curve that surround the new representation as desired.

The orientation of \mathbf{x} is again determined by the order-condition, and exactly as in Theorem 2 one argues that this respects the order-condition at \mathbf{u} and \mathbf{v} , since our choice of curve for \mathbf{x} ensures that it crosses \mathbf{u} *after* the crossing of \mathbf{u} with \mathbf{v} .

2. \mathbf{u} and \mathbf{v} have different slopes in S_{uv} and $k > 1$ (i.e. we add at least two vertices x_1, \dots, x_k .)

We add a path of 1-bend curves $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ with their bends at the top, and define private regions as illustrated in Fig. 6. Each curve \mathbf{x}_i is oriented as required by the order-condition, and again one verifies the order-condition for \mathbf{u} and \mathbf{v} . We can re-use the same S .

3. \mathbf{u} and \mathbf{v} have the same slope inside S_{uv} .

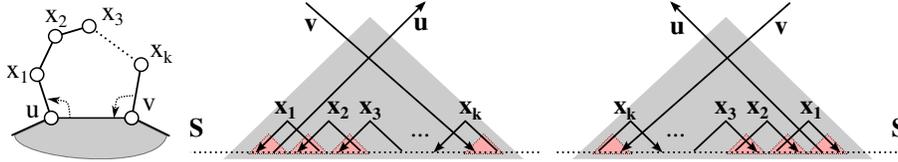


Fig. 6. Adding 2 or more nodes if u and v have different slopes.

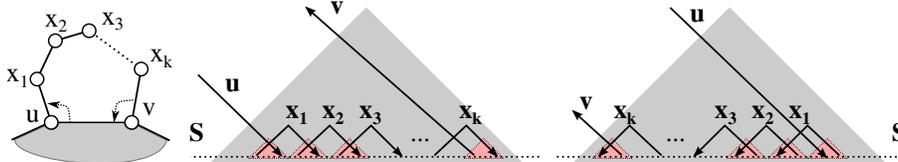


Fig. 7. Adding one or more vertices if u and v have the same slope. We only show two of the four possible configurations.

We add a path of 1-bend curves x_1, x_2, \dots, x_k (possibly $k = 1$) with their bends at the top, and define private regions as illustrated in Fig. 7. Each curve x_i is oriented as required by the order-condition, and one verifies all conditions using the same S .

After having represented the entire graph in this way, we are order-preserving due to the order-condition, outer-string due to poly-line S , and B_1 -VPG (after a 45° -rotation) since every curve has one bend. \square

In our B_1 -VPG-representation, every vertex-curve is an Lin one of the four possible rotations $\sqsubset, \sqsupset, \sqcap, \sqsupset$. (All four may be used, since private regions get rotated in Case 1.) We would have preferred a representation that uses L (or the two shapes \sqsubset and \sqsupset), because then the stretching-techniques by Middendorf and Pfeiffer [17] could have been applied to obtain another segment-representation. It is easy to create a representations with L only if we need not be order-preserving (use \wedge in Case 1) or need not be outer-string (see also Lemma 2), but finding an outer-string order-preserving representation using only L s remains open.

5 Beyond outer-planar graphs?

One wonders what other graph classes might have order-preserving 1-string representations, preferably outer-string ones. We study this here for some graph classes. We start with the *series-parallel graphs*, which are the same as the partial 2-trees, and hence generalize outer-planar graphs.

Lemma 2. *Every series-parallel graph G has a 1-string representation with L s that is order-preserving for some planar embedding of G .*

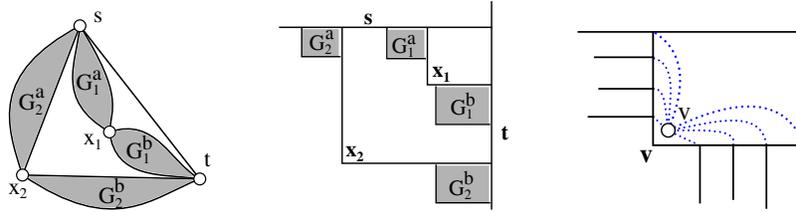


Fig. 8. Representing series-parallel graphs by touching Ls, and converting this into a planar drawing with the same order.

Proof. It is easy to show that every 2-tree has a representation by touching true Ls, i.e., each vertex is assigned an L (not rotated and not degenerated into a line segment), curves are disjoint except at ends, and (u, v) is an edge if and only if the end of \mathbf{u} lies on the interior of \mathbf{v} or vice versa.³ See also Fig. 8. Extending the Ls slightly gives a 1-string representation, and it is order-preserving for a planar embedding easily derived from the touching L representation. Details are provided in Appendix A. \square

It would be interesting to know whether this result can be extended to the so-called *planar Laman-graphs*, which have a representation by touching Ls [14], but not all Ls are necessarily in the same rotation and so it is not clear whether this is order-preserving. Of particular interest would be planar bipartite graphs, which can even be represented by horizontal and vertical touching line segments [11], but again it is not clear how to make this order-preserving.

As for having strings additionally end at the contour for series-parallel graphs: this is not always possible. Let H be the graph obtained by subdividing every edge in a $K_{2,3}$; one verifies that H is series-parallel. It is easy to see (see also [4]) that H cannot be outer-string, since $K_{2,3}$ is not outer-planar. So H has no outer-string representation, much less one that is 1-string and order-preserving.

Now we turn to partial 3-trees. We showed in Theorem 1 that there exist planar 3-trees (hence partial 3-trees) that do not have an order-preserving 1-string representation. We now study some subclasses of partial 3-trees that are superclasses of outer-planar graphs.

An *IO-graph* is a planar graph G that has an independent set I such that $G - I$ is a 2-connected outer-planar graph O for which all vertices in I are inside inner faces of O . A *Halin-graph* is a graph that consists of a tree T and a cycle C that connects all leaves of T . Both types of graphs are well-known to be partial 3-trees. In [2], we gave 1-string representations for both Halin graphs and IO-graphs; the latter uses only unrotated Ls. Independently, Francis and Lahiri also constructed 1-string representations of Halin-graphs, using only unrotated

³ We have not been able to find a direct reference for this, but it follows for example from the works of Chaplick et al. [7] or with an iterative approach similar to the 6-sided contact representations in [1].

Es [10]. Inspection of both constructions shows that these respect the standard planar embedding (where O respectively C is one face). We hence have:

Theorem 4 (based on [2,10]). *Every IO-graph and every Halin-graph has an order-preserving 1-string representation in which every vertex is an L .*

In these constructions, the ends of the strings are not on the outer-face, and we now show that this is unavoidable. This is obvious for Halin-graphs, since the subdivided $K_{2,3}$ is an induced subgraph of a Halin-graph. As for IO-graphs, define the *wheel* W_n be the graph that consists of a cycle $C = \{v_1, \dots, v_n\}$ with n vertices and one universal vertex c connected to all of them. Let the *extended wheel-graph* W_n^+ be the wheel-graph W_n with additionally a vertex w_i incident to v_i and v_{i+1} for $i = 1, \dots, n$ (and $w_{n+1} := w_1$). Notice that W_n^+ is an IO-graph. The proof of the following is presented in Appendix A.

Theorem 5. *For $n \geq 7$, the IO-graph W_n^+ has no order-preserving outer-1-string representation.*

6 Final remarks

In this paper, we studied 1-string representations that respect a planar embedding. As for open problems, what other graph classes have order-preserving 1-string representations? A natural candidate to investigate would be the 2-outer-planar graphs, for which Lemma 1 cannot be applied since a triple-stellation is never 2-outer-planar. Other interesting candidates would be planar bipartite graphs (or more generally planar Laman-graphs), or planar 4-connected graphs.

Secondly, what is the complexity of testing whether an order-preserving 1-string representation exists? Given the NP-hardness of the abstract graph realization problem [15,18], this is very likely NP-hard if we are allowed to prescribe an arbitrary rotation scheme (not from a planar drawing). But is it NP-hard for plane graphs?

One unsatisfactory aspect of our definition of “order-preserving” is that graphs with an end-contact representation (i.e., with disjoint strings where for every edge one string ends on the other string) do not automatically have an order-preserving 1-string representation: We can obtain a 1-string representation by extending the strings slightly, but it does not need to be order-preserving. A reviewer hence suggested to us the following alternate model: Thicken each string slightly, and consider the cyclic order of intersections while walking around the thickened string. Let now “order-preserving” mean that the cyclic order of neighbours around a vertex forms a subsequence of the intersections encountered while walking “around” its string. With this, any end-contact representation becomes an order-preserving 1-string representation after extending the curves a bit. This includes for example planar bipartite graphs and Laman graphs. Since this model’s restriction is weaker, all our positive results transfer, but the proofs of the negative results no longer hold. Are there plane graphs that do not have an order-preserving 1-string representation in this new model?

References

1. Alam, M.J., Biedl, T., Felsner, S., Gerasch, A., Kaufmann, M., Kobourov, S.G.: Linear-time algorithms for hole-free rectilinear proportional contact graph representations. *Algorithmica* 67(1), 3–22 (2013)
2. Biedl, T., Derka, M.: 1-string B_1 -VPG-representations of planar partial 3-trees and some subclasses. In: *Cnd. Conf. Comp. Geometry (CCCG 2015)*. pp. 37–42 (2015)
3. Biedl, T., Derka, M.: 1-string B_2 -VPG representations of planar graphs. *Journal of Computational Geometry* (2016), to appear. Preliminary version at *SoCG 2015*.
4. Cabello, S., Jejčić, M.: Refining the hierarchies of classes of geometric intersection graphs. *CoRR* abs/1603.08974 (2016)
5. Chalopin, J., Gonçalves, D.: Every planar graph is the intersection graph of segments in the plane: extended abstract. In: *ACM Symposium on Theory of Computing, (STOC 2009)*. pp. 631–638. ACM (2009)
6. Chalopin, J., Gonçalves, D., Ochem, P.: Planar graphs have 1-string representations. *Discrete & Computational Geometry* 43(3), 626–647 (2010)
7. Chaplick, S., Kobourov, S.G., Ueckerdt, T.: Equilateral L-contact graphs. In: *Graph-Theoretic Concepts in Computer Science (WG 2013)*. Lecture Notes in Computer Science, vol. 8165, pp. 139–151. Springer (2013)
8. Chaplick, S., Ueckerdt, T.: Planar graphs as VPG-graphs. *J. Graph Algorithms Appl.* 17(4), 475–494 (2013)
9. Felsner, S., Knauer, K.B., Mertziós, G.B., Ueckerdt, T.: Intersection graphs of L-shapes and segments in the plane. *Discrete Applied Mathematics* 206, 48–55 (2016)
10. Francis, M.C., Lahiri, A.: VPG and EPG bend-numbers of Halin graphs. *CoRR* abs/1505.06036 (2015)
11. de Fraysseix, H., de Mendez, P.O., Pach, J.: Representation of planar graphs by segments. *Intuitive Geometry* 63, 109–117 (1991)
12. Govindan, R., Langston, M.A., Yan, X.: Approximating the pathwidth of outerplanar graphs. *Inf. Process. Lett.* 68(1), 17–23 (1998)
13. Keil, J.M., Mitchell, J.S., Pradhan, D., Vatshelle, M.: An algorithm for the maximum weight independent set problem on outerstring graphs. *Computational Geometry* pp. – (2016)
14. Kobourov, S.G., Ueckerdt, T., Verbeek, K.: Combinatorial and geometric properties of planar Laman graphs. In: *SIAM Symposium on Discrete Algorithms (SODA 2013)*. pp. 1668–1678. SIAM (2013)
15. Kratochvíl, J.: String graphs II. Recognizing string graphs is NP-hard. *J. Comb. Theory Ser. B* 52(1), 67–78 (1991)
16. Kratochvíl, J., Matoušek, J.: Intersection graphs of segments. *J. Comb. Theory, Ser. B* 62(2), 289–315 (1994)
17. Middendorf, M., Pfeiffer, F.: The max clique problem in classes of string-graphs. *Discrete Mathematics* 108, 365–372 (1992)
18. Middendorf, M., Pfeiffer, F.: Weakly transitive orientations, Hasse diagrams and string graphs. *Discrete Mathematics* 111, 393–400 (1993)
19. Schaefer, M., Sedgwick, E., Štefankovič, D.: Recognizing string graphs is in NP. *Journal of Computer and System Sciences* 67(2), 365380 (2003)
20. Scheinerman, E.R.: Intersection Classes and Multiple Intersection Parameters of Graphs. Ph.D. thesis, Princeton University (1984)
21. Wessel, W., Pöschel, R.: On circle graphs. In: Sachs, H. (ed.) *Graphs, Hypergraphs and Applications*, pp. 207–210. Teubner (1985)

A Appendix

Proof (of Lemma 2). Start with the representation by touching true Ls explained in the main part of the paper. For any contact-representation with true Ls (neither rotated nor degenerated into a horizontal or vertical line segment), we can create a planar drawing that matches the order of touching-points along each L. Namely, draw a point for v slightly above and to the right of the corner of the bend in \mathbf{v} . Connect v to all touching-points on \mathbf{v} , and to the two ends of \mathbf{v} . Because every curve is an L, the curves whose ends touch \mathbf{v} all come from the left at the vertical segment of \mathbf{v} or from the bottom at the horizontal segment of \mathbf{v} . Therefore the added lines do not cross any curves and so give a planar drawing of G that is clearly respected by the representation. Extending the Ls slightly hence gives the desired 1-string representation. \square

Proof (of Theorem 5). Assume for contradiction that it did, and consider the induced representation \mathcal{R}_W of W_n . Let the naming of cycle C be such that \mathbf{c} intersects $\mathbf{v}_1, \dots, \mathbf{v}_n$ in this order. Define as before $\mathbf{u}[v, w]$ (for any 2-path v, u, w) to be the stretch of \mathbf{u} between the intersection with \mathbf{v} and \mathbf{w} . Now define R to be the region bounded by $\mathbf{c}[v_1, v_n]$ (which is almost the entire curve \mathbf{c}), as well as $\mathbf{v}_n[c, v_1]$ and $\mathbf{v}_1[v_n, c]$ (which exist since (v_1, v_n) is an edge). See also Fig. 9.

Consider v_i for $i = 3, 4, 5$, which is adjacent to neither v_1 nor v_n . \mathbf{v}_i intersects the boundary of R (because it intersects $\mathbf{c}[v_1, v_n]$ by assumption), but does not intersect it twice, else it would intersect \mathbf{c} twice or intersect \mathbf{v}_1 or \mathbf{v}_n . Hence one end of \mathbf{v}_i is inside R while the other one is outside, and so not both ends of \mathbf{v}_i can be on the contour for $i = 3, 4, 5$.

This shows that W_n is not outer-1-string in the sense that for some vertex not both ends of the curves are on the contour. Now consider W_n^+ , and the vertices w_3 and w_4 that were added at v_4 when creating W_n^+ . Since w_3 and w_4 are adjacent to none of c, v_1, v_n , and since the drawing is outer-string, both \mathbf{w}_3 and \mathbf{w}_4 (and therefore their intersections with \mathbf{v}_4) must be outside R .

So walking along \mathbf{v}_4 starting at the end inside R , we encounter \mathbf{c} and then one of $\{\mathbf{w}_3, \mathbf{w}_4\}$. We assume that we encounter \mathbf{w}_3 before \mathbf{w}_4 ; the other case is symmetric (and results in \mathbf{v}_5 having no end on the contour). Consider the region R' enclosed by $\mathbf{v}_4[c, w_4]$, $\mathbf{w}_4[v_4, v_5]$, $\mathbf{v}_5[w_4, c]$ and $\mathbf{c}[v_5, v_4]$. Since \mathbf{w}_4 is outside R , so is R' . Curve \mathbf{v}_3 intersects $\delta R'$, because it intersects \mathbf{v}_4 , and this intersection must be on $\mathbf{v}_4[c, w_3]$ to preserve the order of edges around v_4 (and since we know that $\mathbf{c}, \mathbf{w}_3, \mathbf{w}_4$ intersect \mathbf{v}_4 in this order). Curve \mathbf{v}_3 cannot intersect $\delta R'$ again, else it would intersect \mathbf{c} or \mathbf{v}_4 twice or would intersect \mathbf{w}_4 or \mathbf{v}_5 , which it shouldn't. Therefore one end of \mathbf{v}_3 is inside R' , which is outside R . The other end of \mathbf{v}_3 is inside R . So neither end of \mathbf{v}_3 is on the contour. Contradiction. \square

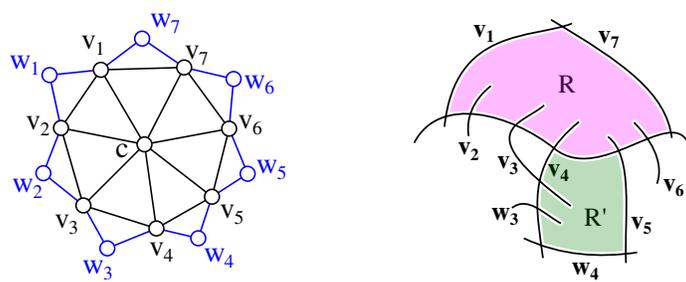


Fig. 9. For the proof of Theorem 5.