# On chromatic number of colored mixed graphs 

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July 9, 2021


#### Abstract

An $(m, n)$-colored mixed graph $G$ is a graph with its arcs having one of the $m$ different colors and edges having one of the $n$ different colors. A homomorphism $f$ of an $(m, n)$ colored mixed graph $G$ to an $(m, n)$-colored mixed graph $H$ is a vertex mapping such that if $u v$ is an arc (edge) of color $c$ in $G$, then $f(u) f(v)$ is an arc (edge) of color $c$ in $H$. The $(m, n)$-colored mixed chromatic number $\chi_{(m, n)}(G)$ of an $(m, n)$-colored mixed graph $G$ is the order (number of vertices) of the smallest homomorphic image of $G$. This notion was introduced by Nešetřil and Raspaud (2000, J. Combin. Theory, Ser. B 80, 147-155). They showed that $\chi_{(m, n)}(G) \leq k(2 m+n)^{k-1}$ where $G$ is a $k$-acyclic colorable graph. We proved the tightness of this bound. We also showed that the acyclic chromatic number of a graph is bounded by $k^{2}+k^{2+\left\lceil\log _{(2 m+n)} \log _{(2 m+n)} k\right\rceil}$ if its $(m, n)$-colored mixed chromatic number is at most $k$. Furthermore, using probabilistic method, we showed that for graphs with maximum degree $\Delta$ its $(m, n)$-colored mixed chromatic number is at most $2(\Delta-1)^{2 m+n}(2 m+n)^{\Delta-1}$. In particular, the last result directly improves the upper bound $2 \Delta^{2} 2^{\Delta}$ of oriented chromatic number of graphs with maximum degree $\Delta$, obtained by Kostochka, Sopena and Zhu (1997, J. Graph Theory $24,331-340)$ to $2(\Delta-1)^{2} 2^{\Delta-1}$. We also show that there exists a graph with maximum degree $\Delta$ and $(m, n)$-colored mixed chromatic number at least $(2 m+n)^{\Delta / 2}$.


Keywords: colored mixed graphs, acyclic chromatic number, graphs with bounded maximum degree, arboricity, chromatic number.

## 1 Introduction

An ( $m, n$ )-colored mixed graph $G=(V, A \cup E)$ is a graph $G$ with set of vertices $V$, set of arcs $A$ and set of edges $E$ where each arc is colored by one of the $m$ colors $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}$ and each edge is colored by one of the $n$ colors $\beta_{1}, \beta_{2}, \ldots, \beta_{n}$. We denote the number of vertices and the number of edges of the underlying graph of $G$ by $v_{G}$ and $e_{G}$, respectively. Also, we will consider only those ( $m, n$ )-colored mixed graphs for which the underlying undirected graph is simple. Nešetřil and Raspaud [5] generalized the notion of vertex coloring and chromatic number for ( $m, n$ )-colored mixed graphs by definining colored homomorphism.

Let $G=\left(V_{1}, A_{1} \cup E_{1}\right)$ and $H=\left(V_{2}, A_{2} \cup E_{2}\right)$ be two ( $m, n$ )-colored mixed graphs. A colored homomorphism of $G$ to $H$ is a function $f: V_{1} \rightarrow V_{2}$ satisfying

$$
u v \in A_{1} \Rightarrow f(u) f(v) \in A_{2},
$$

[^0]$$
u v \in E_{1} \Rightarrow f(u) f(v) \in E_{2}
$$
and the color of the arc or edge linking $f(u)$ and $f(v)$ is the same as the color of the arc or the edge linking $u$ and $v[5]$. We write $G \rightarrow H$ whenever there exists a homomorphism of $G$ to $H$.

Given an $(m, n)$-colored mixed graph $G$ let $H$ be an $(m, n)$-colored mixed graph with minimum order (number of vertices) such that $G \rightarrow H$. Then the order of $H$ is the $(m, n)$-colored mixed chromatic number $\chi_{(m, n)}(G)$ of $G$. For an undirected simple graph $G$, the maximum $(m, n)$-colored mixed chromatic number taken over all $(m, n)$-colored mixed graphs having underlying undirected simple graph $G$ is denoted by $\chi_{(m, n)}(G)$. Let $\mathcal{F}$ be a family of undirected simple graphs. Then $\chi_{(m, n)}(\mathcal{F})$ is the maximum of $\chi_{(m, n)}(G)$ taken over all $G \in \mathcal{F}$.

Note that a $(0,1)$-colored mixed graph $G$ is nothing but an undirected simple graph while $\chi_{(0,1)}(G)$ is the ordinary chromatic number. Similarly, the study of $\chi_{(1,0)}(G)$ is the study of oriented chromatic number which is considered by several researchers in the last two decades (for details please check the recent updated survey [8]). Alon and Marshall [1] studied the homomorphism of $(0, n)$-colored mixed graphs with a particular focus on $n=2$.

A simple graph $G$ is $k$-acyclic colorable if we can color its vertices with $k$ colors such that each color class induces an independent set and any two color class induces a forest. The acyclic chromatic number $\chi_{a}(G)$ of a simple graph $G$ is the minimum $k$ such that $G$ is $k$-acyclic colorable. Nešetřil and Raspaud [5] showed that $\chi_{(m, n)}(G) \leq k(2 m+n)^{k-1}$ where $G$ is a $k$ acyclic colorable graph. As planar graphs are 5-acyclic colorable due to Borodin [2], the same authors implied $\chi_{(m, n)}(\mathcal{P}) \leq 5(2 m+n)^{4}$ for the family $\mathcal{P}$ of planar graphs as a corollary. This result, in particular, implies $\chi_{(1,0)}(\mathcal{P}) \leq 80$ and $\chi_{(0,2)}(\mathcal{P}) \leq 80$ (independently proved before in [7] and [1], respectively).

Let $\mathcal{A}_{k}$ be the family of graphs with acyclic chromatic number at most $k$. Ochem [6] showed that the upper bound $\chi_{(1,0)}\left(\mathcal{A}_{k}\right) \leq 80$ is tight. We generalize it for all $(m, n) \neq(0,1)$ to show that the upper bound $\chi_{(m, n)}\left(\mathcal{A}_{k}\right) \leq k(2 m+n)^{k-1}$ obtained by Nešetřil and Raspaud [5] is tight. This implies that the upper bound $\chi_{(m, n)}(\mathcal{P}) \leq 5(2 m+n)^{4}$ cannot be improved using the upper bound of $\chi_{(m, n)}\left(\mathcal{A}_{5}\right)$.

The arboricity $\operatorname{arb}(G)$ of a graph $G$ is the minimum $k$ such that the edges of $G$ can be decomposed into $k$ forests. Kostochka, Sopena and Zhu [3] showed that given a simple graph $G$, the acyclic chromatic number $\chi_{a}(G)$ of $G$ is also bounded by a function of $\chi_{(1,0)}(G)$. We generalize this result for all $(m, n) \neq(0,1)$ by showing that for a graph $G$ with $\chi_{(m, n)}(G) \leq k$ we have $\chi_{a}(G) \leq k^{2}+k^{2+\left\lceil\log _{2} \log _{p} k\right\rceil}$ where $p=2 m+n$. Our bound slightly improves the bound obtained by Kostochka, Sopena and Zhu [3] for $(m, n)=(1,0)$. For achieving this result we first establish some relations among arboricity of a graph, $(m, n)$-colored mixed chromatic number and acyclic chromatic number.

Let $\mathcal{G}_{\Delta}$ be the family of graphs with maximum degree $\Delta$. Kostochka, Sopena and Zhu [3] proved that $2^{\Delta / 2} \chi_{(1,0)}\left(\mathcal{G}_{\Delta}\right) \leq 2 \Delta^{2} 2^{\Delta}$. We improve this result in a generalized setting by proving $p^{\Delta / 2} \leq \chi_{(m, n)}\left(\mathcal{G}_{\Delta}\right) \leq 2(\Delta-1)^{p} p^{\Delta-1}$ for all $(m, n) \neq(0,1)$ where $p=2 m+n$.

## 2 Preliminaries

A special 2-path uvw of an $(m, n)$-colored mixed graph $G$ is a 2-path satisfying one of the following conditions:
(i) $u v$ and $v w$ are edges of different colors,
(ii) $u v$ and $v w$ are arcs (possibly of the same color),
(iii) $u v$ and $w v$ are arcs of different colors,
(iv) $v u$ and $v w$ are arcs of different colors,
(v) exactly one of $u v$ and $v w$ is an edge and the other is an arc.

Observation 1. The endpoints of a special 2-path must have different image under any homomorphism of $G$.

Proof. Let $u v w$ be a special 2-path in an $(m, n)$-colored mixed graph $G$. Let $f: G \rightarrow H$ be a colored homomorphism of $G$ to an $(m, n)$-colored mixed graph $H$ such that $f(u)=f(w)$. Then $f(u) f(v)$ and $f(w) f(v)$ will induce parallel edges in the underlying graph of $H$. But as we are dealing with $(m, n)$-colored mixed graphs with underlying simple graphs, this is not possible.

Let $G=(V, A \cup E)$ be an $(m, n)$-colored mixed graph. Let $u v$ be an arc of $G$ with color $\alpha_{i}$ for some $i \in\{1,2, \ldots, m\}$. Then $u$ is a $-\alpha_{i}$-neighbor of $v$ and $v$ is a $+\alpha_{i}$-neighbor of $u$. The set of all $+\alpha_{i}$-neighbors and $-\alpha_{i}$-neighbors of $v$ is denoted by $N^{+\alpha_{i}}(v)$ and $N^{-\alpha_{i}}(v)$, respectively. Similarly, let $u v$ be an edge of $G$ with color $\beta_{i}$ for some $i \in\{1,2, \ldots, n\}$. Then $u$ is a $\beta_{i}$ neighbor of $v$ and the set of all $\beta_{i}$-neighbors of $v$ is denoted by $N^{\beta_{i}}(v)$. Let $\vec{a}=\left(a_{1}, a_{2}, \ldots, a_{j}\right)$ be a $j$-vector such that $a_{i} \in\left\{ \pm \alpha_{1}, \pm \alpha_{2}, \ldots, \pm \alpha_{m}, \beta_{1}, \beta_{2}, \ldots, \beta_{n}\right\}$ where $i \in\{1,2, \ldots, j\}$. Let $J=\left(v_{1}, v_{2}, \ldots, v_{j}\right)$ be a $j$-tuple (without repetition) of vertices from $G$. Then we define the set $N^{\vec{a}}(J)=\left\{v \in V \mid v \in N^{a_{i}}\left(v_{i}\right)\right.$ for all $\left.1 \leq i \leq j\right\}$. Finally, we say that $G$ has property $Q_{g(j)}^{t, j}$ if for each $j$-vector $\vec{a}$ and each $j$-tuple $J$ we have $\left|N^{\vec{a}}(J)\right| \geq g(j)$ where $j \in\{0,1, \ldots, t\}$ and $g:\{0,1, \ldots, t\} \rightarrow\{0,1, \ldots \infty\}$ is an integral function.

## 3 On graphs with bounded acyclic chromatic number

First we will construct examples of $(m, n)$-colored mixed graphs $H_{k}^{(m, n)}$ with acyclic chromatic number at most $k$ and $\chi_{(m, n)}\left(H_{k}^{(m, n)}\right)=k(2 m+n)^{k-1}$ for all $k \geq 3$ and for all $(m, n) \neq(0,1)$. This, along with the upper bound established by Nešetřil and Raspaud [5, will imply the following result:

Theorem 3.1. Let $\mathcal{A}_{k}$ be the family of graphs with acyclic chromatic number at most $k$. Then $\chi_{(m, n)}\left(\mathcal{A}_{k}\right)=k(2 m+n)^{k-1}$ for all $k \geq 3$ and for all $(m, n) \neq(0,1)$.

Proof. First we will construct an $(m, n)$-colored mixed graph $H_{k}^{(m, n)}$, where $p=2 m+n \geq 2$, as follows. Let $A_{k-1}$ be the set of all $(k-1)$-vectors. Thus, $\left|A_{k-1}\right|=p^{k-1}$.

Define $B_{i}$ as a set of $(k-1)$ vertices $B_{i}=\left\{b_{1}^{i}, b_{2}^{i}, \ldots, b_{k-1}^{i}\right\}$ for all $i \in\{1,2, \ldots, k\}$ such that $B_{r} \cap B_{s}=\emptyset$ when $r \neq s$. The vertices of $B_{i}$ 's are called bottom vertices for each $i \in\{1,2, \ldots, k\}$. Furthermore, let $T B_{i}=\left(b_{1}^{i}, b_{2}^{i}, \ldots, b_{k-1}^{i}\right)$ be a $(k-1)$-tuple.

After that define the set of vertices $T_{i}=\left\{t_{\vec{a}}^{i} \mid t_{\vec{a}}^{i} \in N^{\vec{a}}\left(T B_{i}\right)\right.$ for all $\left.\vec{a} \in A_{k-1}\right\}$ for all $i \in$ $\{1,2, \ldots, k\}$. The vertices of $T_{i}$ 's are called top vertices for each $i \in\{1,2, \ldots, k\}$. Observe that there are $p^{k-1}$ vertices in $T_{i}$ for each $i \in\{1,2, \ldots, k\}$.

Note that the definition of $T_{i}$ already implies some colored arcs and edges between the set of vertices $B_{i}$ and $T_{i}$ for all $i \in\{1,2, \ldots, k\}$.

As $p \geq 2$ it is possible to construct a special 2-path. Now for each pair of vertices $u \in T_{i}$ and $v \in T_{j}(i \neq j)$, construct a special 2-path $u w_{u v} v$ and call these new vertices $w_{u v}$ as internal vertices for all $i, j \in\{1,2, \ldots, k\}$. This so obtained graph is $H_{k}^{(m, n)}$.

Now we will show that $\chi_{(m, n)}\left(H_{k}^{(m, n)}\right) \geq k(2 m+n)^{k-1}$. Let $\vec{a} \neq \overrightarrow{a^{\prime}}$ be two distinct $(k-1)$ vectors. Assume that the $j^{t} h$ co-ordinate of $\vec{a}$ and $\overrightarrow{a^{\prime}}$ is different. Then note that $t_{\vec{a}}^{i} b_{j}^{i} t_{a^{\prime}}^{i}$ is a special 2-path. Therefore, $t_{\vec{a}}^{i}$ and $t_{\vec{a}}^{i}$ must have different homomorphic image under any homomorphism. Thus, all the vertices in $T_{i}$ must have distinct homomorphic image under any homomorphism. Moreover, as a vertex of $T_{i}$ is connected by a special 2-path with a vertex of $T_{j}$ for all $i \neq j$, all the top vertices must have distinct homomorphic image under any homomorphism. It is easy to see that $\left|T_{i}\right|=p^{k-1}$ for all $i \in\{1,2, \ldots, k\}$. Hence $\chi_{(m, n)}\left(H_{k}^{(m, n)}\right) \geq$ $\sum_{i=1}^{k}\left|T_{i}\right|=k(2 m+n)^{k-1}$.

Then we will show that $\chi_{a}\left(H_{k}^{(m, n)}\right) \leq k$. From now on, by $H_{k}^{(m, n)}$, we mean the underlying undirected simple graph of the $(m, n)$-colored mixed graph $H_{k}^{(m, n)}$. We will provide an acyclic coloring of this graph with $\{1,2, \ldots, k\}$. Color all the vertices of $T_{i}$ with $i$ for all $i \in\{1,2, \ldots, k\}$. Then color all the vertices of $B_{i}$ with distinct $(k-1)$ colors from the set $\{1,2, \ldots, k\} \backslash\{i\}$ of colors for all $i \in\{1,2, \ldots, k\}$. Note that each internal vertex have exactly two neighbors. Color each internal vertex with a color different from its neighbors. It is easy to check that this is an acyclic coloring.

Therefore, we showed that $\chi_{(m, n)}\left(\mathcal{A}_{k}\right) \geq k(2 m+n)^{k-1}$ while, on the other hand, Nešetřil and Raspaud [5] showed that $\chi_{(m, n)}\left(\mathcal{A}_{k}\right) \leq k(2 m+n)^{k-1}$ for all $k \geq 3$ and for all $(m, n) \neq(0,1)$.

Consider a complete graph $K_{t}$. Replace all its edges by a 2-path to obtain the graph $S$. For all $(m, n) \neq(0,1)$, it is possible to assign colored edges/arcs to the edges of $S$ such that it becomes an $(m, n)$-colored mixed graph with $t$ vertices that are pairwise connected by a special 2-path. Therefore, by Observation 1 we know that $\chi_{(m, n)}(S) \geq t$ whereas, it is easy to note that $S$ has arboricity 2. Thus, the ( $m, n$ )-colored mixed chromatic number is not bounded by any function of arboricity. Though the reverse type of bound exists. Kostochka, Sopena and Zhu [3] proved such a bound for $(m, n)=(1,0)$. We generalize their result for all $(m, n) \neq(0,1)$.

Theorem 3.2. Let $G$ be an $(m, n)$-colored mixed graph with $\chi_{(m, n)}(G)=k$ where $p=2 m+n \geq$ 2. Then $\operatorname{arb}(G) \leq\left\lceil\log _{p} k+k / 2\right\rceil$.

Proof. Let $G^{\prime}$ be an arbitrary labeled subgraph of $G$ consisting $v_{G^{\prime}}$ vertices and $e_{G^{\prime}}$ edges. We know from Nash-Williams' Theorem [4] that the arboricity $\operatorname{arb}(G)$ of any graph $G$ is equal to the maximum of $\left\lceil e_{G^{\prime}} /\left(v_{G^{\prime}}-1\right)\right\rceil$ over all subgraphs $G^{\prime}$ of $G$. So it is sufficient to prove that for any subgraph $G^{\prime}$ of $G, e_{G^{\prime}} /\left(v_{G^{\prime}}-1\right) \leq \log _{p} k+k / 2$. As $G^{\prime}$ is a labeled graph, so there are $p^{e} G^{\prime}$ different ( $m, n$ )-colored mixed graphs with underlying graph $G^{\prime}$. As $\chi_{(m, n)}(G)=k$, there exits a homomorphism from $G^{\prime}$ to a $(m, n)$-colored mixed graph $G_{k}$ which has the complete graph on $k$ vertices as its underlying graph. Note that the number of possible homomorphisms of $G^{\prime}$ to $G_{k}$ is at most $k^{v} G_{G^{\prime}}$. For each such homomorphism of $G^{\prime}$ to $G_{k}$ there are at most $p^{\binom{k}{2}}$ different ( $m, n$ )-colored mixed graphs with underlying labeled graph $G^{\prime}$ as there are $p^{\binom{k}{2}}$ choices of $G_{k}$. Therefore,

$$
\begin{equation*}
p^{\binom{k}{2}} \cdot k^{v_{G^{\prime}}} \geq p^{e} G_{G^{\prime}} \tag{1}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\log _{p} k \geq\left(e_{G^{\prime}} / v_{G^{\prime}}\right)-\binom{k}{2} / v_{G^{\prime}} \tag{2}
\end{equation*}
$$

If $v_{G^{\prime}} \leq k$, then $e_{G^{\prime}} /\left(v_{G^{\prime}}-1\right) \leq v_{G^{\prime}} / 2 \leq k / 2$. Now let $v_{G^{\prime}}>k$. We know that $\chi_{(m, n)}\left(G^{\prime}\right) \leq$ $\chi_{(m, n)}(G)=k$. So

$$
\begin{aligned}
\log _{p} k & \geq \frac{e_{G^{\prime}}}{v_{G^{\prime}}}-\frac{k(k-1)}{2 v_{G^{\prime}}} \\
& \geq \frac{e_{G^{\prime}}}{\left(v_{G^{\prime}}-1\right)}-\frac{e_{G^{\prime}}}{v_{G^{\prime}}\left(v_{G^{\prime}}-1\right)}-\frac{k-1}{2} \\
& \geq \frac{e_{G^{\prime}}}{\left(v_{G^{\prime}}-1\right)}-1 / 2-k / 2+1 / 2 \\
& \geq \frac{e_{G^{\prime}}}{\left(v_{G^{\prime}}-1\right)}-k / 2
\end{aligned}
$$

Therefore, $\frac{e_{G^{\prime}}}{\left(v_{G^{\prime}}-1\right)} \leq \log _{p} k+k / 2$.
We have seen that the $(m, n)$-colored mixed chromatic number of a graph $G$ is bounded by a function of the acyclic chromatic number of $G$. Here we show that it is possible to bound the acyclic chromatic number of a graph in terms of its $(m, n)$-colored mixed chromatic number and arboricity. Our result is a generalization of a similar result proved for $(m, n)=(1,0)$ by Kostochka, Sopena and Zhu [3].

Theorem 3.3. Let $G$ be an $(m, n)$-colored mixed graph with $\operatorname{arb}(G)=r$ and $\chi_{(m, n)}(G)=k$ where $p=2 m+n \geq 2$. Then $\chi_{a}(G) \leq k^{\left\lceil\log _{p} r\right\rceil+1}$.

Proof. First we rename the following symbols: $\alpha_{1}=a_{0},-\alpha_{1}=a_{1}, \alpha_{2}=a_{2},-\alpha_{2}=a_{3}, \ldots, \alpha_{m}=$ $a_{2 m-2},-\alpha_{m}=a_{2 m-1}, \beta_{1}=a_{2 m}, \beta_{2}=a_{2 m+1}, \ldots, \beta_{n}=a_{2 m+n-1}$.

Let $G$ be a graph with $\chi_{(m, n)}(G)=k$ where $2 m+n=p$. Let $v_{1}, v_{2}, \ldots, v_{t}$ be some ordering of the vertices of $G$. Now consider the $(m, n)$-colored mixed graph $G_{0}$ with underlying graph $G$ such that for any $i<j$ we have $v_{j} \in N^{a_{0}}\left(v_{i}\right)$ whenever $v_{i} v_{j}$ is an edge of $G$.

Note that the edges of $G$ can be covered by $r$ edge disjoint forests $F_{1}, F_{2}, \ldots, F_{r}$ as $\operatorname{arb}(G)=r$. Let $s_{i}$ be the number $i$ expressed with base $p$ for all $i \in\{1,2, \ldots, r\}$. Note that $s_{i}$ can have at most $s=\left\lceil\log _{p} r\right\rceil$ digits.

Now we will construct a sequence of $(m, n)$-colored mixed graphs $G_{1}, G_{2}, \ldots, G_{s}$ each having underlying graph $G$. For a fixed $l \in\{1,2, \ldots, s\}$ we will describe the construction of $G_{l}$. Let $i<j$ and $v_{i} v_{j}$ is an edge of $G$. Suppose $v_{i} v_{j}$ is an edge of the forest $F_{l^{\prime}}$ for some $l^{\prime} \in\{1,2, \ldots, r\}$. Let the $l^{t h}$ digit of $s_{l^{\prime}}$ be $s_{l^{\prime}}(l)$. Then $G_{l}$ is constructed in a way such that we have $v_{j} \in N^{a_{l_{l^{\prime}}(l)}}\left(v_{i}\right)$ in $G_{l}$.

Note that there is a homomorphism $f_{l}: G_{l} \rightarrow H_{l}$ for each $l \in\{1,2, \ldots, s\}$ such that $H_{l}$ is an $(m, n)$-colored mixed graph on $k$ vertices. Now we claim that $f(v)=\left(f_{0}(v), f_{1}(v), \ldots, f_{s}(v)\right)$ for each $v \in V(G)$ is an acyclic coloring of $G$.

For adjacent vertices $u, v$ in $G$ clearly we have $f(v) \neq f(u)$ as $f_{0}(v) \neq f_{0}(u)$. Let $C$ be a cycle in $G$. We have to show that at least 3 colors have been used to color this cycle with respect to the coloring given by $f$. Note that in $C$ there must be two incident edges $u v$ and $v w$ such that they belong to different forests, say, $F_{i}$ and $F_{i^{\prime}}$, respectively. Now suppose that $C$ received two colors with respect to $f$. Then we must have $f(u)=f(w) \neq f(v)$. In particular we must have $f_{0}(u)=f_{0}(w) \neq f_{0}(v)$. To have that we must also have $u, w \in N^{a_{i}}(v)$ for some $i \in\{0,1, \ldots, p-1\}$ in $G_{0}$. Let $s_{i}$ and $s_{i^{\prime}}$ differ in their $j^{t h}$ digit. Then in $G_{j}$ we have $u \in N^{a_{i}^{\prime}}(v)$ and $w \in N^{a_{i}^{\prime \prime}}(v)$ for some $i^{\prime} \neq i^{\prime \prime}$. Then we must have $f_{j}(u) \neq f_{j}(w)$. Therefore, we also have $f(u) \neq f(w)$. Thus, the cycle $C$ cannot be colored with two colors under the coloring $f$. So $f$ is indeed an acyclic coloring of $G$.

Thus, combining Theorem 3.2 and 3.3 we have $\chi_{a}(G) \leq k^{\left\lceil\log _{p}\left\lceil\log _{p} k+k / 2\right\rceil\right\rceil+1}$ for $\chi_{(m, n)}(G)=k$ where $p=2 m+n \geq 2$. However, we managed to obtain the following better bound.

Theorem 3.4. Let $G$ be an ( $m, n$ )-colored mixed graph with $\chi_{(m, n)}(G)=k \geq 4$ where $p=$ $2 m+n \geq 2$. Then $\chi_{a}(G) \leq k^{2}+k^{2+\left\lceil\log _{2} \log _{p} k\right\rceil}$.

Proof. Let $t$ be the maximum real number such that there exists a subgraph $G^{\prime}$ of $G$ with $v_{G^{\prime}} \geq k^{2}$ and $e_{G^{\prime}} \geq t . v_{G^{\prime}}$. Let $G^{\prime \prime}$ be the biggest subgraph of $G$ with $e_{G^{\prime \prime}}>t . v_{G^{\prime \prime}}$. Thus, by maximality of $t, v_{G^{\prime \prime}}<k^{2}$.

Let $G_{0}=G-G^{\prime \prime}$. Hence $\chi_{a}(G) \leq \chi_{a}\left(G_{0}\right)+k^{2}$. By maximality of $G^{\prime \prime}$, for each subgraph $H$ of $G_{0}$, we have $e_{H} \leq t . v_{H}$.

If $t \leq \frac{v_{H}-1}{2}$, then $e_{H} \leq(t+1 / 2)\left(v_{H}-1\right)$. If $t>\frac{v_{H}-1}{2}$, then $\frac{v_{H}}{2}<t+1 / 2$. So $e_{H} \leq$ $\frac{\left(v_{H}-1\right) \cdot v_{H}}{2} \leq(t+1 / 2)\left(v_{H}-1\right)$. Therefore, $e_{H} \leq(t+1 / 2)\left(v_{H}-1\right)$ for each subgraph $H$ of $G_{0}$.

By Nash-Williams' Theorem [4], there exists $r=\lceil t+1 / 2\rceil$ forests $F_{1}, F_{2}, \cdots, F_{r}$ which covers all the edges of $G_{0}$. We know from Theorem $3.3 \chi_{a}\left(G_{0}\right) \leq k^{s+1}$ where $s=\left\lceil\log _{p} r\right\rceil$.

Using inequality (2) we get $\log _{p} k \geq t-1 / 2$. Therfore

$$
s=\left\lceil\log _{p}(\lceil t+1 / 2\rceil)\right\rceil \leq\left\lceil\log _{p}\left(1+\left\lceil\log _{p} k\right\rceil\right)\right\rceil \leq 1+\left\lceil\log _{p} \log _{p} k\right\rceil .
$$

Hence $\chi_{a}(G) \leq k^{2}+k^{2+\left\lceil\log _{p} \log _{p} k\right\rceil}$.
Our bound, when restricted to the case of $(m, n)=(1,0)$, slightly improves the existing bound [3].

## 4 On graphs with bounded maximum degree

Recall that $\mathcal{G}_{\Delta}$ is the family of graphs with maximum degree $\Delta$. It is known that $\chi_{(1,0)}\left(\mathcal{G}_{\Delta}\right) \leq$ $2 \Delta^{2} 2^{\Delta}$ [3]. Here we prove that $\chi_{(m, n)}\left(\mathcal{G}_{\Delta}\right) \leq 2(\Delta-1)^{p} . p^{(\Delta-1)}+2$ for all $p=2 m+n \geq 2$ and $\Delta \geq 5$. Our result, restricted to the case $(m, n)=(1,0)$, slightly improves the upper bound of Kostochka, Sopena and Zhu 3].

Theorem 4.1. For the family $\mathcal{G}_{\Delta}$ of graphs with maximum degree $\Delta$ we have $p^{\Delta / 2} \leq \chi_{(m, n)}\left(\mathcal{G}_{\Delta}\right) \leq$ $2(\Delta-1)^{p} . p^{(\Delta-1)}+2$ for all $p=2 m+n \geq 2$ and for all $\Delta \geq 5$.

If every subgraph of a graph $G$ have at least one vertex with degree at most $d$, then $G$ is $d$-degenerated. Minimum such $d$ is the degeneracy of $G$. To prove the above theorem we need the following result.

Theorem 4.2. Let $\mathcal{G}_{\Delta}^{\prime}$ be the family of graphs with maximum degree $\Delta$ and degeneracy $(\Delta-1)$. Then $\chi_{(m, n)}\left(\mathcal{G}_{\Delta}^{\prime}\right) \leq 2(\Delta-1)^{p} . p^{(\Delta-1)}$ for all $p=2 m+n \geq 2$ and for all $\Delta \geq 5$.

To prove the above theorem we need the following lemma.
Lemma 4.3. There exists an ( $m, n$ )-colored complete mixed graph with property $Q_{1+(t-j)(t-2)}^{t-1, j}$ on $c=2(t-1)^{p} . p^{(t-1)}$ vertices where $p=2 m+n \geq 2$ and $t \geq 5$.

Proof. Let $C$ be a random $(m, n)$-colored mixed graph with underlying complete graph. Let $u, v$ be two vertices of $C$ and the events $u \in N^{a}(v)$ for $a \in\left\{ \pm \alpha_{1}, \pm \alpha_{2}, \ldots, \pm \alpha_{m}, \beta_{1}, \beta_{2}, \ldots, \beta_{n}\right\}$ are equiprobable and independent with probability $\frac{1}{2 m+n}=\frac{1}{p}$. We will show that the probability of $C$ not having property $Q_{1+(t-j)(t-2)}^{t-1, j}$ is strictly less than 1 when $|C|=c=2(t-1)^{p} . p^{(t-1)}$. Let $P(J, \vec{a})$ denote the probability of the event $\left|N^{\vec{a}}(J)\right|<1+(t-j)(t-2)$ where $J$ is a $j$-tuple of $C$ and $\vec{a}$ is a $j$-vector for some $j \in\{0,1, \ldots, t-1\}$. Call such an event a bad event. Thus,

$$
\begin{align*}
P(J, \vec{a}) & =\sum_{i=0}^{(t-j)(t-2)}\binom{c-j}{i} p^{-i j}\left(1-p^{-j}\right)^{c-i-j} \\
& <\left(1-p^{-j}\right)^{c} \sum_{i=0}^{(t-j)(t-2)} \frac{c^{i}}{i!}\left(1-p^{-j}\right)^{-i-j} p^{-i j}  \tag{3}\\
& <2 e^{-c p^{-j}} \sum_{i=0}^{(t-j)(t-2)} c^{i} \\
& <e^{-c p^{-j}} c^{(t-j)(t-2)+1} .
\end{align*}
$$

Let $P(B)$ denote the probability of the occurrence of at least one bad event. To prove this lemma it is enough to show that $P(B)<1$. Let $T^{j}$ denote the set of all $j$-tuples and $W^{j}$ denote the set of all $j$-vectors. Then

$$
\begin{align*}
P(B)=\sum_{j=0}^{t-1} \sum_{J \in T^{j}} \sum_{\vec{a} \in W^{j}} P(J, \vec{a}) & <\sum_{j=0}^{t-1}\binom{c}{j} p^{j} e^{-c p^{-j}} c^{(t-j)(t-2)+1} \\
& <\sum_{j=0}^{t-1} \frac{c^{j}}{j!} p^{j} e^{-c p^{-j}} c^{(t-j)(t-2)+1}  \tag{4}\\
& =2 \sum_{j=0}^{t-1} \frac{p^{j}}{2^{j}} \frac{2^{j-1}}{j!} c^{j} e^{-c p^{-j}} c^{(t-j)(t-2)+1} \\
& <2 \sum_{j=0}^{t-1} \frac{p^{j}}{2^{j}} e^{-c p^{-j}} c^{(t-j)(t-2)+1+j} .
\end{align*}
$$

Consider the function $f(j)=2(p / 2)^{j} e^{-c p^{-j}} c^{(t-j)(t-2)+1+j}$. Observe that $f(j)$ is the $j^{\text {th }}$ summand of the last sum from equation (4). Now

$$
\begin{align*}
\frac{f(j+1)}{f(j)} & =\frac{p}{2} \frac{e^{(p-1) c p^{-j-1}}}{c^{t-3}} \\
& >\frac{p}{2} \frac{e^{(p-1) c p^{-(t-1)}}}{c^{t-3}}  \tag{5}\\
& >\frac{p}{2}\left(\frac{e^{2(p-1)(t-1)^{p-1}}}{c}\right)^{t-3}
\end{align*}
$$

As $\frac{p-1}{p}>\frac{1}{2}$,

$$
\frac{(k-1)^{p-1}}{2}>\ln (k-1) \Longrightarrow(p-1)(k-1)^{p-1}>\ln (k-1)^{p} .
$$

Furthermore,

$$
\frac{(p-1)}{\ln p}(k-1)^{p-1}>\frac{\ln 2}{\ln p}+(k-1) \Longrightarrow(p-1)(k-1)^{p-1}>\ln \left(2 p^{k-1}\right)
$$

Adding the above two inequalities we get

$$
e^{2(p-1)(t-1)^{p-1}}>2(t-1)^{p} p^{t-1}=c .
$$

Hence $\frac{f(j+1)}{f(j)}>\frac{p}{2}$. Thus, using inequality (44) we get $P(B)<\sum_{j=0}^{t-1} f(j)$. This implies

$$
P(B)< \begin{cases}\frac{(p / 2)^{t}-1}{(p / 2)-1} f(0), & \text { if } p>2 \\ t f(0), & \text { if } p=2\end{cases}
$$

Case.1: $p>2$.

$$
\begin{align*}
P(B) & <2 \cdot \frac{(p / 2)^{t}-1}{(p / 2)-1} \cdot \frac{c^{(t-1)^{2}}}{e^{2(t-1)^{p} p^{t-1}}} \\
& <4 \cdot \frac{(p / 2)^{t}-1}{p-2} \cdot\left(\frac{c}{e^{2 p^{t-1}}}\right)^{(t-1)^{p}}  \tag{6}\\
& <4 \cdot(p / 2)^{t} \cdot\left(\frac{c}{e^{2 p^{t-1}}}\right)^{(t-1)^{p}} \\
& <\left(\frac{p c}{e^{2 p^{t-1}}}\right)^{(t-1)^{p}}
\end{align*}
$$

Now, we observe that

$$
\begin{aligned}
\ln (p c) & <\ln p+\ln 2+p \ln (t-1)+(t-1) \ln p \\
& =t \ln p+p \ln (t-1)+\ln 2 \\
& <t p+p(t-1)+2 \\
& <2 t p<2 p^{t-1}
\end{aligned}
$$

So from the inequality (6), we can say that $P(B)<1$ for $p>2$.
Case.2: $p=2$.

$$
\begin{align*}
P(B) & <2 t \cdot \frac{c^{(t-1)^{2}}}{e^{(t-1)^{2} 2^{t}}} \\
& =2 t \cdot\left(\frac{c}{e^{2^{t}}}\right)^{(t-1)^{2}}  \tag{7}\\
& <\left(\frac{2 t c}{e^{2^{t}}}\right)^{(t-1)^{2}}
\end{align*}
$$

Observe that, $\ln c=2 \ln (t-1)+t \ln 2<2(t-1)+2 t=4 t-2$.
Now, we see that

$$
\ln (2 t c)<4 t-2+2 t<6 t<2^{t} \Longrightarrow 2 t c<e^{2^{t}} \Longrightarrow \frac{2 t c}{e^{2^{t}}}<1
$$

So from the inequality (7), we can say that $P(B)<1$ for $p=2$.

Now we are ready to prove Theorem 4.2.
Proof of Theorem 4.2. Suppose that $G$ is an $(m, n)$-colored mixed graph with maximum degree $\Delta$ and degeneracy $(\Delta-1)$. By Lemma 4.3 we know that there exists an $(m, n)$-colored mixed graph $C$ with property $Q_{1+(\Delta-j)(\Delta-2)}^{\Delta-1, j}$ on $2(\Delta-1)^{p} . p^{(\Delta-1)}$ vertices where $p=2 m+n \geq 2$ and $\Delta \geq 5$. We will show that $G$ admits a homomorphism to $C$.

As $G$ has degeneracy $(\Delta-1)$, we can provide an ordering $v_{1}, v_{2}, \ldots, v_{k}$ of the vertices of $G$ in such a way that each vertex $v_{j}$ has at most $(\Delta-1)$ neighbors with lower indices. Let $G_{l}$ be the $(m, n)$-colored mixed graph induced by the vertices $v_{1}, v_{2}, \ldots, v_{l}$ from $G$ for $l \in\{1,2, \ldots, k\}$. Now we will recursively construct a homomorphism $f: G \rightarrow C$ with the following properties:
(i) The partial mapping $f\left(v_{1}\right), f\left(v_{2}\right), \ldots, f\left(v_{l}\right)$ is a homomorphism of $G_{l}$ to $C$ for all $l \in$ $\{1,2, \ldots, k\}$.
(ii) For each $i>l$, all the neighbors of $v_{i}$ with indices less than or equal to $l$ has different images with respect to the mapping $f$.

Note that the base case is trivial, that is, any partial mapping $f\left(v_{1}\right)$ is enough. Suppose that the function $f$ satisfies the above properties for all $j \leq t$ where $t \in\{1,2, \ldots, k-1\}$ is fixed. Now assume that $v_{t+1}$ has $s$ neighbors with indices greater than $t+1$. Then $v_{t+1}$ has at most $(\Delta-s)$ neighbors with indices less than $t+1$. Let $A$ be the set of neighbors of $v_{t+1}$ with indices greater than $t+1$. Let $B$ be the set of vertices with indices at most $t$ and with at least one neighbor in $A$. Note that as each vertex of $A$ is a neighbor of $v_{t+1}$ and has at most $\Delta-1$ neighbors with lesser indices, $|B|=(\Delta-2)|A|=s(\Delta-2)$. Let $D$ be the set of possible options for $f\left(v_{t+1}\right)$ such that the partial mapping is a homomorphism of $G_{t+1}$ to $C$. As $C$ has property $Q_{1+(\Delta-j)(\Delta-2)}^{\Delta-1, j}$ we have $|C| \geq 1+s(\Delta-1)$. So the set $D \backslash B$ is non-empty. Thus, choose any vertex from $D \backslash B$ as the image $f\left(v_{t+1}\right)$. Note that this partial mapping satisfies the required conditions.

Finally, we are ready to prove Theorem 4.1.
Proof of Theorem 4.1. First we will prove the lower bound. Let $G_{t}$ be a $\Delta$ regular graph on $t$ vertices. Thus, $G_{t}$ has $\frac{t \Delta}{2}$ edges. Then we have

$$
k_{t}=\chi_{(m, n)}\left(G_{t}\right) \geq \frac{p^{\Delta / 2}}{p^{\binom{k_{t}}{2} / t}}
$$

using inequality (1) (see Section 3). If $\chi_{(m, n)}\left(G_{t}\right) \geq p^{\Delta / 2}$ for some $t$, then we are done. Otherwise, $\chi_{(m, n)}\left(G_{t}\right)=k_{t}$ is bounded. In that case, if $t$ is sufficiently large, then $\chi_{(m, n)}\left(G_{t}\right) \geq p^{\Delta / 2}$ as $\chi_{(m, n)}\left(G_{t}\right)$ is a positive integer.

Let $G=(V, A \cup E)$ be a connected $(m, n)$-colored mixed graph with maximum degree $\Delta \geq 5$ and $p=2 m+n \geq 2$. If $G$ has a vertex of degree at most $(\Delta-1)$ then it has degeneracy at most $(\Delta-1)$. In that case by Theorem 4.1 we are done.

Otherwise, $G$ is $\Delta$ regular. In that case, remove an edge $u v$ of $G$ to obtain the graph $G^{\prime}$. Note that $G^{\prime}$ has maximum degree at most $\Delta$ and has degeneracy at most $(\Delta-1)$. Therefore, by Theorem 4.1 there exists an $(m, n)$-colored complete mixed graph $C$ on $2(\Delta-1)^{p} \cdot p^{(\Delta-1)}$ vertices to which $G^{\prime}$ admits a $f$ homomorphism to. Let $G^{\prime \prime}$ be the graph obtained by deleting the vertices $u$ and $v$ of $G^{\prime}$. Note that the homomorphism $f$ restricted to $G^{\prime \prime}$ is a homomorphism $f_{\text {res }}$ of $G^{\prime \prime}$ to $C$. Now include two new vertices $u^{\prime}$ and $v^{\prime}$ to $C$ and obtain a new graph $C^{\prime}$. Color the edges or arcs between the vertices of $C$ and $\left\{u^{\prime}, v^{\prime}\right\}$ in such a way so that we can extend the homomorphism $f_{\text {res }}$ to a homomorphism $f_{\text {ext }}$ of $G$ to $C^{\prime}$ where $f_{\text {ext }}(u)=u^{\prime}, f_{\text {ext }}(v)=v^{\prime}$ and
$f_{\text {ext }}(x)=f_{\text {res }}(x)$ for all $x \in V(G) \backslash\{u, v\}$. It is easy to note that the above mentioned process is possible.

Thus, every connected ( $m, n$ )-colored mixed graph with maximum degree $\Delta$ admits a homomorphism to $C^{\prime}$.

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