On chromatic number of colored mixed graphs

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Abstract

An (m, n)-colored mixed graph G is a graph with its arcs having one of the m different colors and edges having one of the n different colors. A homomorphism f of an (m, n)colored mixed graph G to an (m, n)-colored mixed graph H is a vertex mapping such that if uv is an arc (edge) of color c in G, then f(u)f(v) is an arc (edge) of color c in H. The (m,n)-colored mixed chromatic number $\chi_{(m,n)}(G)$ of an (m,n)-colored mixed graph G is the order (number of vertices) of the smallest homomorphic image of G. This notion was introduced by Nešetřil and Raspaud (2000, J. Combin. Theory, Ser. B 80, 147–155). They showed that $\chi_{(m,n)}(G) \leq k(2m+n)^{k-1}$ where G is a k-acyclic colorable graph. We proved the tightness of this bound. We also showed that the acyclic chromatic number of a graph is bounded by $k^2 + k^{2 + \lceil \log_{(2m+n)} \log_{(2m+n)} k \rceil}$ if its (m, n)-colored mixed chromatic number is at most k. Furthermore, using probabilistic method, we showed that for graphs with maximum degree Δ its (m, n)-colored mixed chromatic number is at most $2(\Delta - 1)^{2m+n}(2m+n)^{\Delta-1}$. In particular, the last result directly improves the upper bound $2\Delta^2 2^{\Delta}$ of oriented chromatic number of graphs with maximum degree Δ , obtained by Kostochka, Sopena and Zhu (1997, J. Graph Theory 24, 331–340) to $2(\Delta - 1)^2 2^{\Delta - 1}$. We also show that there exists a graph with maximum degree Δ and (m, n)-colored mixed chromatic number at least $(2m + n)^{\Delta/2}$.

Keywords: colored mixed graphs, acyclic chromatic number, graphs with bounded maximum degree, arboricity, chromatic number.

1 Introduction

An (m, n)-colored mixed graph $G = (V, A \cup E)$ is a graph G with set of vertices V, set of arcs A and set of edges E where each arc is colored by one of the m colors $\alpha_1, \alpha_2, ..., \alpha_m$ and each edge is colored by one of the n colors $\beta_1, \beta_2, ..., \beta_n$. We denote the number of vertices and the number of edges of the underlying graph of G by v_G and e_G , respectively. Also, we will consider only those (m, n)-colored mixed graphs for which the underlying undirected graph is simple. Nešetřil and Raspaud [5] generalized the notion of vertex coloring and chromatic number for (m, n)-colored mixed graphs by definining colored homomorphism.

Let $G = (V_1, A_1 \cup E_1)$ and $H = (V_2, A_2 \cup E_2)$ be two (m, n)-colored mixed graphs. A colored homomorphism of G to H is a function $f : V_1 \to V_2$ satisfying

 $uv \in A_1 \Rightarrow f(u)f(v) \in A_2,$

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$$uv \in E_1 \Rightarrow f(u)f(v) \in E_2,$$

and the color of the arc or edge linking f(u) and f(v) is the same as the color of the arc or the edge linking u and v [5]. We write $G \to H$ whenever there exists a homomorphism of G to H.

Given an (m, n)-colored mixed graph G let H be an (m, n)-colored mixed graph with minimum order (number of vertices) such that $G \to H$. Then the order of H is the (m, n)-colored mixed chromatic number $\chi_{(m,n)}(G)$ of G. For an undirected simple graph G, the maximum (m, n)-colored mixed chromatic number taken over all (m, n)-colored mixed graphs having underlying undirected simple graph G is denoted by $\chi_{(m,n)}(G)$. Let \mathcal{F} be a family of undirected simple graphs. Then $\chi_{(m,n)}(\mathcal{F})$ is the maximum of $\chi_{(m,n)}(G)$ taken over all $G \in \mathcal{F}$.

Note that a (0, 1)-colored mixed graph G is nothing but an undirected simple graph while $\chi_{(0,1)}(G)$ is the ordinary chromatic number. Similarly, the study of $\chi_{(1,0)}(G)$ is the study of oriented chromatic number which is considered by several researchers in the last two decades (for details please check the recent updated survey [8]). Alon and Marshall [1] studied the homomorphism of (0, n)-colored mixed graphs with a particular focus on n = 2.

A simple graph G is k-acyclic colorable if we can color its vertices with k colors such that each color class induces an independent set and any two color class induces a forest. The acyclic chromatic number $\chi_a(G)$ of a simple graph G is the minimum k such that G is k-acyclic colorable. Nešetřil and Raspaud [5] showed that $\chi_{(m,n)}(G) \leq k(2m+n)^{k-1}$ where G is a kacyclic colorable graph. As planar graphs are 5-acyclic colorable due to Borodin [2], the same authors implied $\chi_{(m,n)}(\mathcal{P}) \leq 5(2m+n)^4$ for the family \mathcal{P} of planar graphs as a corollary. This result, in particular, implies $\chi_{(1,0)}(\mathcal{P}) \leq 80$ and $\chi_{(0,2)}(\mathcal{P}) \leq 80$ (independently proved before in [7] and [1], respectively).

Let \mathcal{A}_k be the family of graphs with acyclic chromatic number at most k. Ochem [6] showed that the upper bound $\chi_{(1,0)}(\mathcal{A}_k) \leq 80$ is tight. We generalize it for all $(m,n) \neq (0,1)$ to show that the upper bound $\chi_{(m,n)}(\mathcal{A}_k) \leq k(2m+n)^{k-1}$ obtained by Nešetřil and Raspaud [5] is tight. This implies that the upper bound $\chi_{(m,n)}(\mathcal{P}) \leq 5(2m+n)^4$ cannot be improved using the upper bound of $\chi_{(m,n)}(\mathcal{A}_5)$.

The arboricity arb(G) of a graph G is the minimum k such that the edges of G can be decomposed into k forests. Kostochka, Sopena and Zhu [3] showed that given a simple graph G, the acyclic chromatic number $\chi_a(G)$ of G is also bounded by a function of $\chi_{(1,0)}(G)$. We generalize this result for all $(m, n) \neq (0, 1)$ by showing that for a graph G with $\chi_{(m,n)}(G) \leq k$ we have $\chi_a(G) \leq k^2 + k^{2+\lceil \log_2 \log_p k \rceil}$ where p = 2m + n. Our bound slightly improves the bound obtained by Kostochka, Sopena and Zhu [3] for (m, n) = (1, 0). For achieving this result we first establish some relations among arboricity of a graph, (m, n)-colored mixed chromatic number and acyclic chromatic number.

Let \mathcal{G}_{Δ} be the family of graphs with maximum degree Δ . Kostochka, Sopena and Zhu [3] proved that $2^{\Delta/2}\chi_{(1,0)}(\mathcal{G}_{\Delta}) \leq 2\Delta^2 2^{\Delta}$. We improve this result in a generalized setting by proving $p^{\Delta/2} \leq \chi_{(m,n)}(\mathcal{G}_{\Delta}) \leq 2(\Delta-1)^p p^{\Delta-1}$ for all $(m,n) \neq (0,1)$ where p = 2m + n.

2 Preliminaries

A special 2-path uvw of an (m, n)-colored mixed graph G is a 2-path satisfying one of the following conditions:

- (i) uv and vw are edges of different colors,
- (ii) uv and vw are arcs (possibly of the same color),

- (iii) uv and wv are arcs of different colors,
- (iv) vu and vw are arcs of different colors,
- (v) exactly one of uv and vw is an edge and the other is an arc.

Observation 1. The endpoints of a special 2-path must have different image under any homomorphism of G.

Proof. Let uvw be a special 2-path in an (m, n)-colored mixed graph G. Let $f : G \to H$ be a colored homomorphism of G to an (m, n)-colored mixed graph H such that f(u) = f(w). Then f(u)f(v) and f(w)f(v) will induce parallel edges in the underlying graph of H. But as we are dealing with (m, n)-colored mixed graphs with underlying simple graphs, this is not possible. \Box

Let $G = (V, A \cup E)$ be an (m, n)-colored mixed graph. Let uv be an arc of G with color α_i for some $i \in \{1, 2, ..., m\}$. Then u is a $-\alpha_i$ -neighbor of v and v is a $+\alpha_i$ -neighbor of u. The set of all $+\alpha_i$ -neighbors and $-\alpha_i$ -neighbors of v is denoted by $N^{+\alpha_i}(v)$ and $N^{-\alpha_i}(v)$, respectively. Similarly, let uv be an edge of G with color β_i for some $i \in \{1, 2, ..., n\}$. Then u is a β_i neighbor of v and the set of all β_i -neighbors of v is denoted by $N^{\beta_i}(v)$. Let $\vec{a} = (a_1, a_2, ..., a_j)$ be a *j*-vector such that $a_i \in \{\pm\alpha_1, \pm\alpha_2, ..., \pm\alpha_m, \beta_1, \beta_2, ..., \beta_n\}$ where $i \in \{1, 2, ..., j\}$. Let $J = (v_1, v_2, ..., v_j)$ be a *j*-tuple (without repetition) of vertices from G. Then we define the set $N^{\vec{a}}(J) = \{v \in V | v \in N^{a_i}(v_i) \text{ for all } 1 \leq i \leq j\}$. Finally, we say that G has property $Q_{g(j)}^{t,j}$ if for each *j*-vector \vec{a} and each *j*-tuple J we have $|N^{\vec{a}}(J)| \geq g(j)$ where $j \in \{0, 1, ..., t\}$ and $g: \{0, 1, ..., t\} \to \{0, 1, ..., \infty\}$ is an integral function.

3 On graphs with bounded acyclic chromatic number

First we will construct examples of (m, n)-colored mixed graphs $H_k^{(m,n)}$ with acyclic chromatic number at most k and $\chi_{(m,n)}(H_k^{(m,n)}) = k(2m+n)^{k-1}$ for all $k \ge 3$ and for all $(m,n) \ne (0,1)$. This, along with the upper bound established by Nešetřil and Raspaud [5], will imply the following result:

Theorem 3.1. Let \mathcal{A}_k be the family of graphs with acyclic chromatic number at most k. Then $\chi_{(m,n)}(\mathcal{A}_k) = k(2m+n)^{k-1}$ for all $k \geq 3$ and for all $(m,n) \neq (0,1)$.

Proof. First we will construct an (m, n)-colored mixed graph $H_k^{(m,n)}$, where $p = 2m + n \ge 2$, as follows. Let A_{k-1} be the set of all (k-1)-vectors. Thus, $|A_{k-1}| = p^{k-1}$.

Define B_i as a set of (k-1) vertices $B_i = \{b_1^i, b_2^i, ..., b_{k-1}^i\}$ for all $i \in \{1, 2, ..., k\}$ such that $B_r \cap B_s = \emptyset$ when $r \neq s$. The vertices of B_i 's are called *bottom* vertices for each $i \in \{1, 2, ..., k\}$. Furthermore, let $TB_i = (b_1^i, b_2^i, ..., b_{k-1}^i)$ be a (k-1)-tuple.

After that define the set of vertices $T_i = \{t^i_{\vec{a}} | t^i_{\vec{a}} \in N^{\vec{a}}(TB_i) \text{ for all } \vec{a} \in A_{k-1}\}$ for all $i \in \{1, 2, ..., k\}$. The vertices of T_i 's are called *top* vertices for each $i \in \{1, 2, ..., k\}$. Observe that there are p^{k-1} vertices in T_i for each $i \in \{1, 2, ..., k\}$.

Note that the definition of T_i already implies some colored arcs and edges between the set of vertices B_i and T_i for all $i \in \{1, 2, ..., k\}$.

As $p \geq 2$ it is possible to construct a special 2-path. Now for each pair of vertices $u \in T_i$ and $v \in T_j$ $(i \neq j)$, construct a special 2-path $uw_{uv}v$ and call these new vertices w_{uv} as *internal* vertices for all $i, j \in \{1, 2, ..., k\}$. This so obtained graph is $H_k^{(m,n)}$. Now we will show that $\chi_{(m,n)}(H_k^{(m,n)}) \geq k(2m+n)^{k-1}$. Let $\vec{a} \neq \vec{a'}$ be two distinct (k-1)-vectors. Assume that the $j^t h$ co-ordinate of \vec{a} and $\vec{a'}$ is different. Then note that $t_{\vec{a}}^i b_j^i t_{\vec{a'}}^i$ is a special 2-path. Therefore, $t_{\vec{a}}^i$ and $t_{\vec{a}}^i$ must have different homomorphic image under any homomorphism. Thus, all the vertices in T_i must have distinct homomorphic image under any homomorphism. Moreover, as a vertex of T_i is connected by a special 2-path with a vertex of T_j for all $i \neq j$, all the top vertices must have distinct homomorphic image under any homomorphism. It is easy to see that $|T_i| = p^{k-1}$ for all $i \in \{1, 2, ..., k\}$. Hence $\chi_{(m,n)}(H_k^{(m,n)}) \geq \sum_{i=1}^k |T_i| = k(2m+n)^{k-1}$.

Then we will show that $\chi_a(H_k^{(m,n)}) \leq k$. From now on, by $H_k^{(m,n)}$, we mean the underlying undirected simple graph of the (m, n)-colored mixed graph $H_k^{(m,n)}$. We will provide an acyclic coloring of this graph with $\{1, 2, ..., k\}$. Color all the vertices of T_i with *i* for all $i \in \{1, 2, ..., k\}$. Then color all the vertices of B_i with distinct (k - 1) colors from the set $\{1, 2, ..., k\} \setminus \{i\}$ of colors for all $i \in \{1, 2, ..., k\}$. Note that each internal vertex have exactly two neighbors. Color each internal vertex with a color different from its neighbors. It is easy to check that this is an acyclic coloring.

Therefore, we showed that $\chi_{(m,n)}(\mathcal{A}_k) \geq k(2m+n)^{k-1}$ while, on the other hand, Nešetřil and Raspaud [5] showed that $\chi_{(m,n)}(\mathcal{A}_k) \leq k(2m+n)^{k-1}$ for all $k \geq 3$ and for all $(m,n) \neq (0,1)$. \Box

Consider a complete graph K_t . Replace all its edges by a 2-path to obtain the graph S. For all $(m, n) \neq (0, 1)$, it is possible to assign colored edges/arcs to the edges of S such that it becomes an (m, n)-colored mixed graph with t vertices that are pairwise connected by a special 2-path. Therefore, by Observation 1 we know that $\chi_{(m,n)}(S) \geq t$ whereas, it is easy to note that S has arboricity 2. Thus, the (m, n)-colored mixed chromatic number is not bounded by any function of arboricity. Though the reverse type of bound exists. Kostochka, Sopena and Zhu [3] proved such a bound for (m, n) = (1, 0). We generalize their result for all $(m, n) \neq (0, 1)$.

Theorem 3.2. Let G be an (m, n)-colored mixed graph with $\chi_{(m,n)}(G) = k$ where $p = 2m + n \ge 2$. 2. Then $arb(G) \le \lceil log_p k + k/2 \rceil$.

Proof. Let G' be an arbitrary labeled subgraph of G consisting $v_{G'}$ vertices and $e_{G'}$ edges. We know from Nash-Williams' Theorem [4] that the arboricity arb(G) of any graph G is equal to the maximum of $[e_{G'}/(v_{G'}-1)]$ over all subgraphs G' of G. So it is sufficient to prove that for any subgraph G' of G, $e_{G'}/(v_{G'}-1) \leq \log_p k + k/2$. As G' is a labeled graph, so there are $p^{e_{G'}}$ different (m, n)-colored mixed graphs with underlying graph G'. As $\chi_{(m,n)}(G) = k$, there exits a homomorphism from G' to a (m, n)-colored mixed graph G by which has the complete graph on k vertices as its underlying graph. Note that the number of possible homomorphisms of G' to G_k is at most $k^{v_{G'}}$. For each such homomorphism of G' to G_k there are $p^{\binom{k}{2}}$ different (m, n)-colored mixed graphs with underlying labeled graph G' as there are $p^{\binom{k}{2}}$ choices of G_k . Therefore,

$$p^{\binom{k}{2}} k^{v_{G'}} \ge p^{e_{G'}} \tag{1}$$

which implies

$$log_p k \ge (e_{G'}/v_{G'}) - \binom{k}{2}/v_{G'}.$$
 (2)

If $v_{G'} \leq k$, then $e_{G'}/(v_{G'}-1) \leq v_{G'}/2 \leq k/2$. Now let $v_{G'} > k$. We know that $\chi_{(m,n)}(G') \leq \chi_{(m,n)}(G) = k$. So

$$\begin{split} \log_{p} k &\geq \frac{e_{G'}}{v_{G'}} - \frac{k(k-1)}{2v_{G'}} \\ &\geq \frac{e_{G'}}{(v_{G'}-1)} - \frac{e_{G'}}{v_{G'}(v_{G'}-1)} - \frac{k-1}{2} \\ &\geq \frac{e_{G'}}{(v_{G'}-1)} - 1/2 - k/2 + 1/2 \\ &\geq \frac{e_{G'}}{(v_{G'}-1)} - k/2. \end{split}$$

Therefore, $\frac{e_{G'}}{(v_{G'}-1)} \leq \log_p k + k/2.$

We have seen that the (m, n)-colored mixed chromatic number of a graph G is bounded by a function of the acyclic chromatic number of G. Here we show that it is possible to bound the acyclic chromatic number of a graph in terms of its (m, n)-colored mixed chromatic number and arboricity. Our result is a generalization of a similar result proved for (m, n) = (1, 0) by Kostochka, Sopena and Zhu [3].

Theorem 3.3. Let G be an (m,n)-colored mixed graph with arb(G) = r and $\chi_{(m,n)}(G) = k$ where $p = 2m + n \ge 2$. Then $\chi_a(G) \le k^{\lceil \log_p r \rceil + 1}$.

Proof. First we rename the following symbols: $\alpha_1 = a_0, -\alpha_1 = a_1, \alpha_2 = a_2, -\alpha_2 = a_3, ..., \alpha_m = a_{2m-2}, -\alpha_m = a_{2m-1}, \beta_1 = a_{2m}, \beta_2 = a_{2m+1}, ..., \beta_n = a_{2m+n-1}.$

Let G be a graph with $\chi_{(m,n)}(G) = k$ where 2m + n = p. Let $v_1, v_2, ..., v_t$ be some ordering of the vertices of G. Now consider the (m, n)-colored mixed graph G_0 with underlying graph G such that for any i < j we have $v_j \in N^{a_0}(v_i)$ whenever $v_i v_j$ is an edge of G.

Note that the edges of G can be covered by r edge disjoint forests $F_1, F_2, ..., F_r$ as arb(G) = r. Let s_i be the number *i* expressed with base *p* for all $i \in \{1, 2, ..., r\}$. Note that s_i can have at most $s = \lceil log_p r \rceil$ digits.

Now we will construct a sequence of (m, n)-colored mixed graphs $G_1, G_2, ..., G_s$ each having underlying graph G. For a fixed $l \in \{1, 2, ..., s\}$ we will describe the construction of G_l . Let i < jand $v_i v_j$ is an edge of G. Suppose $v_i v_j$ is an edge of the forest $F_{l'}$ for some $l' \in \{1, 2, ..., r\}$. Let the l^{th} digit of $s_{l'}$ be $s_{l'}(l)$. Then G_l is constructed in a way such that we have $v_j \in N^{a_{s_{l'}(l)}}(v_i)$ in G_l .

Note that there is a homomorphism $f_l : G_l \to H_l$ for each $l \in \{1, 2, ..., s\}$ such that H_l is an (m, n)-colored mixed graph on k vertices. Now we claim that $f(v) = (f_0(v), f_1(v), ..., f_s(v))$ for each $v \in V(G)$ is an acyclic coloring of G.

For adjacent vertices u, v in G clearly we have $f(v) \neq f(u)$ as $f_0(v) \neq f_0(u)$. Let C be a cycle in G. We have to show that at least 3 colors have been used to color this cycle with respect to the coloring given by f. Note that in C there must be two incident edges uv and vw such that they belong to different forests, say, F_i and $F_{i'}$, respectively. Now suppose that Creceived two colors with respect to f. Then we must have $f(u) = f(w) \neq f(v)$. In particular we must have $f_0(u) = f_0(w) \neq f_0(v)$. To have that we must also have $u, w \in N^{a_i}(v)$ for some $i \in \{0, 1, ..., p-1\}$ in G_0 . Let s_i and $s_{i'}$ differ in their j^{th} digit. Then in G_j we have $u \in N^{a'_i}(v)$ and $w \in N^{a''_i}(v)$ for some $i' \neq i''$. Then we must have $f_j(u) \neq f_j(w)$. Therefore, we also have $f(u) \neq f(w)$. Thus, the cycle C cannot be colored with two colors under the coloring f. So fis indeed an acyclic coloring of G.

Thus, combining Theorem 3.2 and 3.3 we have $\chi_a(G) \leq k^{\lceil \log_p \lceil \log_p k + k/2 \rceil \rceil + 1}$ for $\chi_{(m,n)}(G) = k$ where $p = 2m + n \geq 2$. However, we managed to obtain the following better bound.

Theorem 3.4. Let G be an (m, n)-colored mixed graph with $\chi_{(m,n)}(G) = k \ge 4$ where $p = 2m + n \ge 2$. Then $\chi_a(G) \le k^2 + k^{2+\lceil \log_2 \log_p k \rceil}$.

Proof. Let t be the maximum real number such that there exists a subgraph G' of G with $v_{G'} \ge k^2$ and $e_{G'} \ge t.v_{G'}$. Let G'' be the biggest subgraph of G with $e_{G''} > t.v_{G''}$. Thus, by maximality of t, $v_{G''} < k^2$.

Let $G_0 = G - G''$. Hence $\chi_a(G) \le \chi_a(G_0) + k^2$. By maximality of G'', for each subgraph H of G_0 , we have $e_H \le t.v_H$.

If $t \leq \frac{v_H - 1}{2}$, then $e_H \leq (t + 1/2)(v_H - 1)$. If $t > \frac{v_H - 1}{2}$, then $\frac{v_H}{2} < t + 1/2$. So $e_H \leq \frac{(v_H - 1).v_H}{2} \leq (t + 1/2)(v_H - 1)$. Therefore, $e_H \leq (t + 1/2)(v_H - 1)$ for each subgraph H of G_0 .

By Nash-Williams' Theorem [4], there exists $r = \lfloor t+1/2 \rfloor$ forests F_1, F_2, \cdots, F_r which covers all the edges of G_0 . We know from Theorem 3.3 $\chi_a(G_0) \leq k^{s+1}$ where $s = \lfloor log_pr \rfloor$.

Using inequality (2) we get $log_p k \ge t - 1/2$. Therfore

$$s = \lceil \log_p(\lceil t + 1/2 \rceil) \rceil \leq \lceil \log_p(1 + \lceil \log_p k \rceil) \rceil \leq 1 + \lceil \log_p \log_p k \rceil.$$

Hence $\chi_a(G) \leq k^2 + k^{2 + \lceil \log_p \log_p k \rceil}.$

Our bound, when restricted to the case of (m, n) = (1, 0), slightly improves the existing bound [3].

4 On graphs with bounded maximum degree

Recall that \mathcal{G}_{Δ} is the family of graphs with maximum degree Δ . It is known that $\chi_{(1,0)}(\mathcal{G}_{\Delta}) \leq 2\Delta^2 2^{\Delta}$ [3]. Here we prove that $\chi_{(m,n)}(\mathcal{G}_{\Delta}) \leq 2(\Delta - 1)^p p (\Delta^{-1}) + 2$ for all $p = 2m + n \geq 2$ and $\Delta \geq 5$. Our result, restricted to the case (m, n) = (1, 0), slightly improves the upper bound of Kostochka, Sopena and Zhu [3].

Theorem 4.1. For the family \mathcal{G}_{Δ} of graphs with maximum degree Δ we have $p^{\Delta/2} \leq \chi_{(m,n)}(\mathcal{G}_{\Delta}) \leq 2(\Delta-1)^p \cdot p^{(\Delta-1)} + 2$ for all $p = 2m + n \geq 2$ and for all $\Delta \geq 5$.

If every subgraph of a graph G have at least one vertex with degree at most d, then G is d-degenerated. Minimum such d is the degeneracy of G. To prove the above theorem we need the following result.

Theorem 4.2. Let \mathcal{G}'_{Δ} be the family of graphs with maximum degree Δ and degeneracy $(\Delta - 1)$. Then $\chi_{(m,n)}(\mathcal{G}'_{\Delta}) \leq 2(\Delta - 1)^p p^{(\Delta - 1)}$ for all $p = 2m + n \geq 2$ and for all $\Delta \geq 5$.

To prove the above theorem we need the following lemma.

Lemma 4.3. There exists an (m, n)-colored complete mixed graph with property $Q_{1+(t-j)(t-2)}^{t-1,j}$ on $c = 2(t-1)^p p^{(t-1)}$ vertices where $p = 2m + n \ge 2$ and $t \ge 5$.

Proof. Let C be a random (m, n)-colored mixed graph with underlying complete graph. Let u, v be two vertices of C and the events $u \in N^a(v)$ for $a \in \{\pm \alpha_1, \pm \alpha_2, ..., \pm \alpha_m, \beta_1, \beta_2, ..., \beta_n\}$ are equiprobable and independent with probability $\frac{1}{2m+n} = \frac{1}{p}$. We will show that the probability of C not having property $Q_{1+(t-j)(t-2)}^{t-1,j}$ is strictly less than 1 when $|C| = c = 2(t-1)^p p^{(t-1)}$. Let $P(J, \vec{a})$ denote the probability of the event $|N^{\vec{a}}(J)| < 1 + (t-j)(t-2)$ where J is a j-tuple of C and \vec{a} is a j-vector for some $j \in \{0, 1, ..., t-1\}$. Call such an event a bad event. Thus,

$$P(J,\vec{a}) = \sum_{i=0}^{(t-j)(t-2)} {\binom{c-j}{i}} p^{-ij} (1-p^{-j})^{c-i-j}$$

$$< (1-p^{-j})^c \sum_{i=0}^{(t-j)(t-2)} \frac{c^i}{i!} (1-p^{-j})^{-i-j} p^{-ij}$$

$$< 2e^{-cp^{-j}} \sum_{i=0}^{(t-j)(t-2)} c^i$$

$$< e^{-cp^{-j}} c^{(t-j)(t-2)+1}.$$
(3)

Let P(B) denote the probability of the occurrence of at least one bad event. To prove this lemma it is enough to show that P(B) < 1. Let T^j denote the set of all *j*-tuples and W^j denote the set of all *j*-vectors. Then

$$P(B) = \sum_{j=0}^{t-1} \sum_{J \in T^{j}} \sum_{\vec{a} \in W^{j}} P(J, \vec{a}) < \sum_{j=0}^{t-1} {\binom{c}{j}} p^{j} e^{-cp^{-j}} c^{(t-j)(t-2)+1} < \sum_{j=0}^{t-1} \frac{c^{j}}{j!} p^{j} e^{-cp^{-j}} c^{(t-j)(t-2)+1} = 2 \sum_{j=0}^{t-1} \frac{p^{j}}{2^{j}} \frac{2^{j-1}}{j!} c^{j} e^{-cp^{-j}} c^{(t-j)(t-2)+1} < 2 \sum_{j=0}^{t-1} \frac{p^{j}}{2^{j}} e^{-cp^{-j}} c^{(t-j)(t-2)+1+j}.$$

$$(4)$$

Consider the function $f(j) = 2(p/2)^j e^{-cp^{-j}} c^{(t-j)(t-2)+1+j}$. Observe that f(j) is the j^{th} summand of the last sum from equation (4). Now

$$\frac{f(j+1)}{f(j)} = \frac{p}{2} \frac{e^{(p-1)cp^{-j-1}}}{c^{t-3}} > \frac{p}{2} \frac{e^{(p-1)cp^{-(t-1)}}}{c^{t-3}} > \frac{p}{2} \left(\frac{e^{2(p-1)(t-1)^{p-1}}}{c}\right)^{t-3}$$
(5)

As $\frac{p-1}{p} > \frac{1}{2}$, $\frac{(k-1)^{p-1}}{2} > \ln(k-1) \implies (p-1)(k-1)^{p-1} > \ln(k-1)^p$.

Furthermore,

$$\frac{(p-1)}{\ln p}(k-1)^{p-1} > \frac{\ln 2}{\ln p} + (k-1) \implies (p-1)(k-1)^{p-1} > \ln(2p^{k-1}).$$

Adding the above two inequalities we get

$$e^{2(p-1)(t-1)^{p-1}} > 2(t-1)^p p^{t-1} = c.$$

Hence $\frac{f(j+1)}{f(j)} > \frac{p}{2}$. Thus, using inequality (4) we get $P(B) < \sum_{j=0}^{t-1} f(j)$. This implies

$$P(B) < \begin{cases} \frac{(p/2)^t - 1}{(p/2) - 1} f(0), & \text{if } p > 2\\ tf(0), & \text{if } p = 2 \end{cases}$$

Case.1: p > 2.

$$P(B) < 2 \cdot \frac{(p/2)^{t} - 1}{(p/2) - 1} \cdot \frac{c^{(t-1)^{2}}}{e^{2(t-1)^{p}p^{t-1}}} < 4 \cdot \frac{(p/2)^{t} - 1}{p - 2} \cdot \left(\frac{c}{e^{2p^{t-1}}}\right)^{(t-1)^{p}} < 4 \cdot (p/2)^{t} \cdot \left(\frac{c}{e^{2p^{t-1}}}\right)^{(t-1)^{p}} < \left(\frac{pc}{e^{2p^{t-1}}}\right)^{(t-1)^{p}}$$

$$(6)$$

Now, we observe that

$$\begin{aligned} \ln(pc) &< \ln p + \ln 2 + p \ln(t-1) + (t-1) \ln p \\ &= t \ln p + p \ln(t-1) + \ln 2 \\ &< tp + p(t-1) + 2 \\ &< 2tp < 2p^{t-1} \end{aligned}$$

So from the inequality (6), we can say that P(B) < 1 for p > 2. Case.2: p = 2.

$$P(B) < 2t. \frac{c^{(t-1)^2}}{e^{(t-1)^2 2^t}} = 2t. \left(\frac{c}{e^{2^t}}\right)^{(t-1)^2}$$

$$< \left(\frac{2tc}{e^{2^t}}\right)^{(t-1)^2}$$
(7)

Observe that, $\ln c = 2\ln(t-1) + t\ln 2 < 2(t-1) + 2t = 4t - 2$. Now, we see that

$$\ln(2tc) < 4t-2+2t < 6t < 2^t \implies 2tc < e^{2^t} \implies \frac{2tc}{e^{2^t}} < 1$$

So from the inequality (7), we can say that P(B) < 1 for p = 2.

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. Suppose that G is an (m, n)-colored mixed graph with maximum degree Δ and degeneracy $(\Delta - 1)$. By Lemma 4.3 we know that there exists an (m, n)-colored mixed graph C with property $Q_{1+(\Delta-j)(\Delta-2)}^{\Delta-1,j}$ on $2(\Delta-1)^p p^{(\Delta-1)}$ vertices where $p = 2m + n \ge 2$ and $\Delta \ge 5$. We will show that G admits a homomorphism to C.

As G has degeneracy $(\Delta - 1)$, we can provide an ordering $v_1, v_2, ..., v_k$ of the vertices of G in such a way that each vertex v_j has at most $(\Delta - 1)$ neighbors with lower indices. Let G_l be the (m, n)-colored mixed graph induced by the vertices $v_1, v_2, ..., v_l$ from G for $l \in \{1, 2, ..., k\}$. Now we will recursively construct a homomorphism $f : G \to C$ with the following properties:

- (i) The partial mapping $f(v_1), f(v_2), ..., f(v_l)$ is a homomorphism of G_l to C for all $l \in \{1, 2, ..., k\}$.
- (*ii*) For each i > l, all the neighbors of v_i with indices less than or equal to l has different images with respect to the mapping f.

Note that the base case is trivial, that is, any partial mapping $f(v_1)$ is enough. Suppose that the function f satisfies the above properties for all $j \leq t$ where $t \in \{1, 2, ..., k-1\}$ is fixed. Now assume that v_{t+1} has s neighbors with indices greater than t+1. Then v_{t+1} has at most $(\Delta - s)$ neighbors with indices less than t+1. Let A be the set of neighbors of v_{t+1} with indices greater than t+1. Let B be the set of vertices with indices at most t and with at least one neighbors in A. Note that as each vertex of A is a neighbor of v_{t+1} and has at most $\Delta - 1$ neighbors with lesser indices, $|B| = (\Delta - 2)|A| = s(\Delta - 2)$. Let D be the set of possible options for $f(v_{t+1})$ such that the partial mapping is a homomorphism of G_{t+1} to C. As C has property $Q_{1+(\Delta - j)(\Delta - 2)}^{\Delta - 1,j}$ we have $|C| \ge 1 + s(\Delta - 1)$. So the set $D \setminus B$ is non-empty. Thus, choose any vertex from $D \setminus B$ as the image $f(v_{t+1})$. Note that this partial mapping satisfies the required conditions.

Finally, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. First we will prove the lower bound. Let G_t be a Δ regular graph on t vertices. Thus, G_t has $\frac{t\Delta}{2}$ edges. Then we have

$$k_t = \chi_{(m,n)}(G_t) \ge \frac{p^{\Delta/2}}{n^{\binom{k_t}{2}/t}}$$

using inequality (1) (see Section 3). If $\chi_{(m,n)}(G_t) \ge p^{\Delta/2}$ for some t, then we are done. Otherwise, $\chi_{(m,n)}(G_t) = k_t$ is bounded. In that case, if t is sufficiently large, then $\chi_{(m,n)}(G_t) \ge p^{\Delta/2}$ as $\chi_{(m,n)}(G_t)$ is a positive integer.

Let $G = (V, A \cup E)$ be a connected (m, n)-colored mixed graph with maximum degree $\Delta \geq 5$ and $p = 2m + n \geq 2$. If G has a vertex of degree at most $(\Delta - 1)$ then it has degeneracy at most $(\Delta - 1)$. In that case by Theorem 4.1 we are done.

Otherwise, G is Δ regular. In that case, remove an edge uv of G to obtain the graph G'. Note that G' has maximum degree at most Δ and has degeneracy at most $(\Delta - 1)$. Therefore, by Theorem 4.1 there exists an (m, n)-colored complete mixed graph C on $2(\Delta - 1)^p \cdot p^{(\Delta - 1)}$ vertices to which G' admits a f homomorphism to. Let G'' be the graph obtained by deleting the vertices u and v of G'. Note that the homomorphism f restricted to G'' is a homomorphism f_{res} of G'' to C. Now include two new vertices u' and v' to C and obtain a new graph C'. Color the edges or arcs between the vertices of C and $\{u', v'\}$ in such a way so that we can extend the homomorphism f_{res} to a homomorphism f_{ext} of G to C' where $f_{ext}(u) = u'$, $f_{ext}(v) = v'$ and $f_{ext}(x) = f_{res}(x)$ for all $x \in V(G) \setminus \{u, v\}$. It is easy to note that the above mentioned process is possible.

Thus, every connected (m, n)-colored mixed graph with maximum degree Δ admits a homomorphism to C'.

References

- N. Alon and T. H. Marshall. Homomorphisms of edge-colored graphs and Coxeter groups. Journal of Algebraic Combinatorics, 8(1):5–13, 1998.
- [2] O. V. Borodin. On acyclic colorings of planar graphs. Discrete Mathematics, 25(3):211–236, 1979.
- [3] A. V. Kostochka, É. Sopena, and X. Zhu. Acyclic and oriented chromatic numbers of graphs. Journal of Graph Theory, 24:331–340, 1997.
- [4] C. S. J. Nash-Williams. Decomposition of finite graphs into forests. Journal of the London Mathematical Society, 1(1):12–12, 1964.
- [5] J. Nešetřil and A. Raspaud. Colored homomorphisms of colored mixed graphs. Journal of Combinatorial Theory, Series B, 80(1):147–155, 2000.
- [6] P. Ochem. Negative results on acyclic improper colorings. In European Conference on Combinatorics (EuroComb 2005), pages 357–362, 2005.
- [7] A. Raspaud and É. Sopena. Good and semi-strong colorings of oriented planar graphs. Information Processing Letters, 51(4):171–174, 1994.
- [8] E. Sopena. Homomorphisms and colourings of oriented graphs: An updated survey. *Discrete Mathematics*, 2015 (in press).