

On the computational complexity of read once resolution decidability in 2CNF formulas

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Abstract

In this paper, we analyze 2CNF formulas from the perspectives of Read-Once resolution (ROR) refutation schemes. We focus on two types of ROR refutations, viz., variable-once refutation and clause-once refutation. In the former, each variable may be used at most once in the derivation of a refutation, while in the latter, each clause may be used at most once. We show that the problem of checking whether a given 2CNF formula has an ROR refutation under both schemes is **NP-complete**. This is surprising in light of the fact that there exist polynomial refutation schemes (tree-resolution and DAG-resolution) for 2CNF formulas. On the positive side, we show that 2CNF formulas have copy-complexity 2, which means that any unsatisfiable 2CNF formula has a refutation in which any clause needs to be used at most twice.

1 Introduction

Resolution is a refutation procedure that was introduced in [15] to establish the unsatisfiability of clausal boolean formulas. Resolution is a sound and complete procedure, although it is not efficient in general [14]. Resolution is one among many proof systems (refutation systems) that have been discussed in the literature [18]; indeed it is among the weaker proof systems [11] in that there exist propositional formulas for which short proofs exist (in powerful proof systems) but resolution proofs of unsatisfiability are exponentially long. Resolution remains an attractive option for studying the complexity of constraint classes on account of its simplicity and wide applicability; it is important to note that resolution is the backbone of a range of automated theorem provers [10].

Resolution refutation techniques often arise in proof complexity. Research in proof complexity is primarily concerned with the establishment of non-trivial lower bounds on the proof lengths of propositional tautologies (alternatively refutation lengths of propositional contradictions). An essential aspect of establishing a lower bound is the proof system used to establish the bound. For instance, super-polynomial bounds for tautologies have been established for weak proof systems such as resolution [14]. Establishing that there exist short refutations for all contradictions in a given proof system causes the classes **NP** and **coNP** to coincide [13].

There are a number of different types of resolution refutation that have been discussed in the literature [16]. The most important types of resolution refutation are *tree-like*, *dag-like* and *read-once*. Each type of resolution is characterized by a restriction on input clause combination. One of the simplest types of resolution is Read-once Resolution (ROR). In an ROR refutation, each input clause and each derived clause may be used at most once. There are several reasons to prefer a ROR proof over a generalized resolution proof, not the least of which is that ROR proofs must *necessarily* be of length polynomial (actually, linear) in the size of the input. It follows that ROR cannot be a complete proof system unless **NP** = **coNP**. That does not preclude the possibility that we could check in polynomial time whether or not a given CNF formula

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has a ROR refutation. Iwama [5] showed that even in case of 3CNF formulas, the problem of checking ROR existence (henceforth, ROR decidability) is **NP-complete**.

It is well-known that 2CNF satisfiability is decidable in polynomial time. There are several algorithms for 2CNF satisfiability, most of which convert the clausal formula into a directed graph and then exploit the connection between the existence of labeled paths in the digraph and the satisfiability of the input formula. A natural progression of this research is to establish the ROR complexity of 2CNF formulas. We show that the problem of deciding whether an arbitrary 2CNF formula has a read-once refutation is **NP-complete**. Although ROR is an **incomplete** refutation technique, we may be able to find a refutation if clauses can be copied. We show that every 2CNF formula has an ROR refutation, if every clause can be copied once.

The principal contributions of this paper are as follows:

1. Establishing the computational complexity of ROR decidability for 2CNF formulas.
2. Establishing the clause copy complexity for ROR existence in 2CNF formulas.

The rest of this paper is organized as follows: In Section 2, we discuss problem preliminaries and formally define the various types of refutations discussed in this paper. The minimal unsatisfiable subset problems is detailed in Section 3. In Section 4, the Variable Read Once Resolution (VAR-ROR) refutation problem is detailed. We also establish the computational complexity of this problem in 2CNF. We show that ROR decidability for 2CNF formulas is **NP-complete** in Section 5. The copy complexity of 2CNF formulas is established in Section 6. Finally, we conclude in Section 7, by summarizing our contributions and outlining avenues for future research.

2 Preliminaries

In this section, we briefly discuss the terms used in this paper. We assume that the reader is familiar with elementary propositional logic. A literal is a variable x or its complement $\neg x$. x is termed a positive and $\neg x$ is termed a negative literal. A clause is a disjunction of literals. The empty clause, which is always false, is denoted as \sqcup .

A Boolean formula Φ is in CNF, if the formula is a conjunction of clauses. Please note, that a CNF is a set of clauses and written as $\{\alpha_1, \dots, \alpha_n\}$, $\alpha_1 \wedge \dots \wedge \alpha_n$, or simply as $\Phi = \alpha_1, \dots, \alpha_n$ for clauses α_i . A formula in CNF is in k -CNF, if it is of the form $\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_m$, where each α_i is a clause of at most k literals.

For a single resolution step with parent clauses $(\alpha \vee x)$ and $(\neg x \vee \beta)$ with resolvent $(\alpha \vee \beta)$, we write

$$(\alpha \vee x), (\neg x \vee \beta) \xrightarrow[\text{RES}]{1} (\alpha \vee \beta).$$

The variable x is termed matching or resolution variable. If for initial clauses $\alpha_1, \dots, \alpha_n$, a clause π can be generated by a sequence of resolution steps we write

$$\alpha_1, \dots, \alpha_n \xrightarrow[\text{RES}]{} \pi.$$

We now formally define the types of resolution refutation discussed in this paper.

Definition 2.1 A formula is in **Var-ROR** (variable-read once resolution), if and only if there is a resolution refutation for which every variable is used at most once as a matching variable.

A resolution derivation $\Phi \xrightarrow[\text{RES}]{} \pi$ is a Var-ROR derivation, if the matching variables are used at most once. We denote this as $\Phi \xrightarrow[\text{Var-RO-Res}]{} \pi$.

Definition 2.2 A formula Φ is said to be **minimally Var-ROR**, if and only if $\Phi \in \text{Var-ROR}$ and every proper sub-formula is not in Var-ROR.

Definition 2.3 A **Read-Once** resolution refutation is a refutation in which each clause, π , can be used in only one resolution step. This applies to clauses present in the original formula and those derived as a result of previous resolution steps.

A resolution derivation $\Phi \vdash_{RES} \pi$ is a read-once resolution derivation, if for all resolution steps $\pi_1 \wedge \pi_2 \vdash_{RES}^1 \pi$, we delete the clauses π_1 and π_2 from, and add the resolvent π to, the current set of clauses. In other words, if U is the current set of clauses, we obtain $U = (U \setminus \{\pi_1, \pi_2\}) \cup \{\pi\}$. An example of this refutation method can be found in Appendix A.

ROR is the set of formulas in CNF, for which a read-once resolution refutation exists ($\Phi \in \text{ROR}$ if and only if $\Phi \vdash_{RO-Res} \sqcup$).

Example (1): Consider the following 2CNF formula:

$$\begin{array}{ccc} (x_1, x_2) & (x_3, x_4) & (\neg x_1, \neg x_3) \\ (\neg x_1, \neg x_4) & (\neg x_2, \neg x_3) & (\neg x_2, \neg x_4) \end{array}$$

We now show that this formula does not have a read-once refutation.

To derive (x_1) we need to derive $(\neg x_2)$. Similarly, to derive (x_2) we need to derive (x_1) . However, the derivations of both $(\neg x_1)$ and $(\neg x_2)$ require the use of the clause (x_3, x_4) .

To derive (x_3) we need to derive $(\neg x_4)$. Similarly, to derive (x_4) we need to derive $(\neg x_3)$. However, the derivations of both $(\neg x_3)$ and $(\neg x_4)$ require the use of the clause (x_1, x_2) .

Definition 2.4 A formula, Φ , is **minimally ROR** if and only if the formula is in ROR and every proper sub-formula is not in ROR.

It is important to note that both types of Read-Once resolution (Var-ROR and ROR) are **incomplete** refutation procedures. Furthermore, ROR is a strictly more powerful refutation procedure than VAR-ROR. For instance, consider the formula:

$$\alpha = \{(a \vee b), (a \vee \neg b), (\neg a \vee b), (\neg a \vee \neg b)\}.$$

It is not hard to see that α has a ROR refutation but not Var-ROR refutation.

After a resolution step $(\pi_1 \vee x) \wedge (\neg x \vee \pi_2) \vdash_{RES}^1 (\pi_1 \vee \pi_2)$ in a Var-ROR refutation no clause with a literal over x can be used. Thus, we can delete all clauses which include the variable x . We see that $\text{Var-ROR} \subseteq \text{ROR}$. For 2-CNF formulas, Var-ROR is a proper subset of ROR.

Example (2): Let $\alpha = \{(a \vee b), (a \vee \neg b), (\neg a \vee b), (\neg a \vee \neg b)\}$. We have that $\alpha \in \text{ROR}$, as shown by the following series of refutation steps:

1. $(a \vee b) \wedge (\neg a \vee b) \vdash_{RES}^1 (b)$.
2. $(a \vee \neg b) \wedge (\neg a \vee \neg b) \vdash_{RES}^1 \neg(b)$.
3. $(b) \wedge (\neg b) \vdash_{RES}^1 \sqcup$.

However, α is not in Var-ROR. After the first resolution step, $(a \vee b) \wedge (\neg a \vee b) \vdash_{RES}^1 (b)$, the variable a can not be used as the matching variable in any other resolution step. Thus, we cannot derive $(\neg b)$. Hence there is no Var-ROR refutation.

3 Minimal Unsatisfiability

In the characterization of various read-once classes and in the proofs, we make use of minimal unsatisfiable formulas and splitting of these formulas.

First, we recall some notions and results.

Definition 3.1 A formula in CNF is **minimal unsatisfiable**, if and only if the formula is unsatisfiable and every proper sub-formula is satisfiable. The set of minimal unsatisfiable formulas is denoted as MU .

Definition 3.2 The **deficiency** of a formula Φ , written as $d(\Phi)$, is the number of clauses minus the number of variables. For fixed k , $MU(k)$ is the set of MU -formulas with deficiency k .

The problem of deciding whether a formula is minimal unsatisfiable is **D^P-complete** [8]. **D^P** is the class of problems which can be represented as the difference of two **NP**-problems. Every minimal unsatisfiable formula has a deficiency greater or equal than 1 [1].

For fixed k , deciding if $\Phi \in \text{MU}(k)$ can be solved in polynomial time [3]. For formulas in 2CNF, there is no constant upper bound for the deficiency of minimal unsatisfiable formulas.

The proofs in this paper make use of so-called splitting formulas for MU-formulas.

Definition 3.3 *Let*

$$\begin{aligned}\Phi = & (x \vee \pi_1), \dots, (x \vee \pi_r), \sigma_1, \dots, \sigma_t, \\ & (\neg x \vee \phi_1), \dots, (\neg x \vee \phi_q)\end{aligned}$$

*be a minimal unsatisfiable formula, where neither the literal x nor the literal $\neg x$ occur in the clauses σ_i . A pair of formulas $(F_x, F_{\neg x})$ with $F_x = \pi_1, \dots, \pi_r, \sigma_{i_1}, \dots, \sigma_{i_s}$ and $F_{\neg x} = \sigma_{j_1}, \dots, \sigma_{j_k}, \phi_1, \dots, \phi_q$ is called a **splitting** of Φ over x , if F_x and $F_{\neg x}$ are minimal unsatisfiable.*

Definition 3.4 *A splitting $(F_x, F_{\neg x})$ is **disjunctive**, if $\{\sigma_{i_1}, \dots, \sigma_{i_s}\} \cap \{\sigma_{j_1}, \dots, \sigma_{j_k}\}$ is empty. That is, F_x and $F_{\neg x}$ have no clause σ_i in common. If additionally F_x and $F_{\neg x}$ do not share any variables, then we say that the splitting is **variable-disjunctive**.*

We can continue to split both F_x and $F_{\neg x}$ to obtain a splitting tree. Splitting stops when the formula contains only one variable.

Definition 3.5 *A splitting tree is **complete** if every leaf of the tree is a formula that contains at most one variable.*

Note that, after splitting on the variable x , neither F_x nor $F_{\neg x}$ contains the variable x . Thus, the splitting tree can have depth at most $(n - 1)$.

In case of disjunctive splittings, we speak about disjunctive splitting trees. It is known that formulas in $\text{MU}(1)$ have variable-disjunctive splitting trees [2]. Moreover, every minimal unsatisfiable formula with a read-once resolution refutation has a disjunctive splitting tree and vice versa. [6].

Let Φ be a minimal unsatisfiable 2CNF formula. We will now prove several properties of Φ .

Lemma 3.1 *If Φ contains a unit clause, then $\Phi \in \text{MU}(1)$.*

Proof: (By induction on the number n of variables) If Φ has only one variable, then $\Phi = (x) \wedge (\neg x)$. Clearly, $\Phi \in \text{MU}(1)$.

Now assume that Φ has $(n + 1)$ variables. Let $(x) \in \Phi$ be a unit clause. Thus, Φ has the form

$$\Phi = (x) \wedge (\neg x \vee L_1) \wedge \dots \wedge (\neg x \vee L_k) \wedge \sigma,$$

where x and $\neg x$ do not occur in any clause in σ . Thus, the formula

$$\Phi_x = (L_1) \wedge \dots \wedge (L_k) \wedge \sigma$$

is minimal unsatisfiable. We have that $(L_1), \dots, (L_k)$ are unit clauses. Thus, by the induction hypothesis $\Phi_x \in \text{MU}(1)$. This means that $\Phi \in \text{MU}(1)$, since Φ contains one more clause and one more variable than Φ_x . \square

Lemma 3.2 *For every variable x , the splitting formulas $F_x, F_{\neg x}$ are in $\text{MU}(1)$.*

Proof: We have that the splitting formulas F_x and $F_{\neg x}$ contain a unit clause and are minimal unsatisfiable. Thus, from lemma 3.1, $F_x, F_{\neg x}$ are in $\text{MU}(1)$. \square

Lemma 3.3 [7] *If for a variable x , there is a disjunctive splitting, then the splitting is unique.*

Lemma 3.4 *The problem of determining if Φ has a complete disjunctive splitting is in **P**. Furthermore, if Φ has a disjunctive splitting for some variable x , then there exists a complete splitting tree.*

Proof: Let $\Phi = (x \vee L_1) \wedge (x \vee L_k) \wedge \sigma \wedge (\neg x \vee K_1) \wedge (\neg x \vee K_r)$. For x , we take the formula $(L_1) \wedge (L_k) \wedge \sigma$ and reduce the formula to a minimal unsatisfiable formula $F_x = (L_1) \wedge (L_k) \wedge \sigma_1$ where $\sigma_1 \subseteq \sigma$. This can be done in polynomial time, because the satisfiability problem for 2-CNF is solvable in linear time. Similarly, for $\neg x$, we can compute $F_{\neg x}$.

To decide whether the formula has a disjunctive splitting tree, it suffices to look for a variable x for which $(F_x, F_{\neg x})$ is a disjunctive splitting. From lemma 3.2, F_x and $F_{\neg x}$ are in MU(1) and therefore have disjunctive splitting trees [2].

For each x , we compute a splitting $(F_x, F_{\neg x})$. If this splitting is disjunctive, we have found the desired splitting. If it is not disjunctive, then for x there is no disjunctive splitting. If x did have a disjunctive splitting, then, by lemma 3.3, that splitting would be unique meaning that x would have no non-disjunctive splittings.

Hence, the problem is solvable in polynomial time. \square

4 The Complexity of Var-ROR for 2CNF

In this section, we show that determining if a formula in 2CNF has a variable read-once refutation is **NP-complete**. The Var-ROR problem is **NP-complete** for CNF formulas in general [17]. We show that this result holds even when restricted to 2CNF formulas.

Let Φ be a formula in 2CNF.

Theorem 4.1 $\Phi \in \text{Var-ROR}$, if and only if there exists a sub-formula $\Phi' \subseteq \Phi$ such that $\Phi' \in \text{MU}(1)$.

Theorem 4.1 follows immediately from lemma 4.1.

Lemma 4.1 Φ is minimally Var-ROR, if and only if $\Phi \in \text{MU}(1)$.

Proof: Let Φ be minimally Var-ROR. We will show that $\Phi \in \text{MU}(1)$ by induction on the number of variables.

First, assume that Φ is a CNF formula over 1 variable. Thus, Φ has the form $(x) \wedge (\neg x)$. Obviously, $\Phi \in \text{MU}(1)$.

Now assume that Φ is a CNF formula over $(n + 1)$ variables. Let

$$(\alpha \vee x), (\neg x \vee \beta) \xrightarrow[\text{RES}]{1} (\alpha \vee \beta)$$

be the first resolution step in a Var-ROR refutation of Φ . Thus, the formula

$$\Phi' = (\Phi \setminus \{(\alpha \vee x), (\neg x \vee \beta)\} \cup \{(\alpha \vee \beta)\})$$

is minimally Var-ROR and contains no clause with x or $\neg x$. x has already been used as a matching variable. Thus, if any other clauses of Φ , or Φ' , used x then Φ would not be minimally Var-ROR.

Φ' has n variables. Thus, by the induction hypothesis $\Phi' \in \text{MU}(1)$. Since Φ' is minimal unsatisfiable and consists of $(n + 1)$ clauses, Φ is minimal unsatisfiable and consists of $(n + 2)$ clauses. This means that $\Phi \in \text{MU}(1)$.

Now let Φ be a formula in MU(1). We will show that Φ is minimally Var-ROR by induction on the number of variables.

For every formula in MU(1), there exists a variable-disjunctive splitting tree [2]. Thus, we can easily construct a Var-ROR refutation for Φ .

First, assume that Φ is a CNF formula over 1 variable. Thus, $\Phi = (x) \wedge (\neg x)$ and is minimally Var-ROR.

Now assume that Φ is a CNF formula over $(n + 1)$ variables. Let $(F_x, F_{\neg x})$ be the first variable-disjunctive splitting in a variable-disjunctive splitting tree. Without loss of generality we assume that neither F_x nor $F_{\neg x}$ is the empty clause. By the induction hypothesis, both formulas are minimally Var-ROR, because, by lemma 3.2, $F_x, F_{\neg x} \in \text{MU}(1)$.

Thus, there is a Var-ROR derivation $F_x \xrightarrow[\text{Var-RO-Res}]{x} (x)$ and $F_{\neg x} \xrightarrow[\text{Var-RO-Res}]{\neg x} (\neg x)$, where $F_x (F_{\neg x})$ is the formula we obtain by adding the removed literal x ($\neg x$) to the clauses in F_x ($F_{\neg x}$). The final step is to resolve (x) and $(\neg x)$. Note that the variable x has not been used as a matching variable in the derivations $F_x \xrightarrow[\text{Var-RO-Res}]{x} (x)$ and $F_{\neg x} \xrightarrow[\text{Var-RO-Res}]{\neg x} (\neg x)$. Hence, Φ is in Var-ROR and is minimally Var-ROR. \square

For a formula in MU(1), there always exists a complete variable-disjunctive splitting. Furthermore, a complete variable-disjunctive splitting tree can be computed in polynomial time. Thus, every formula in MU(1) has a Var-ROR refutation that can be computed in polynomial time.

Corollary 4.1 Every formula in MU(1) has a Var-ROR refutation that can be computed in polynomial time.

Next, we will show that determining if a 2CNF formula has a Var-ROR refutation is **NP-complete**. It can easily be seen that this problem is in **NP**. If the formula has n variables, then at most n resolution steps can be performed. The **NP-hardness** will be shown by a reduction from the vertex-disjoint path problem for directed graphs.

Definition 4.1 Given a directed graph G and pairwise distinct vertexes s_1, t_1, s_2, t_2 , the **vertex-disjoint path problem (2-DPP)** consists of finding a pair of vertex-disjoint paths in G , one from s_1 to t_1 and the other from s_2 to t_2 .

The problem is known to be **NP-complete** [4]. Now we modify the problem as follows.

Definition 4.2 Given a directed graph G and two distinct vertexes s and t , the **vertex-disjoint cycle problem (C-DPP)** consists of finding a pair of vertex-disjoint paths in G , one from s to t and the other from t to s .

Note that the paths are vertex-disjoint, if the inner vertexes of the path from s to t are disjoint from the inner vertexes of the path from t to s .

Lemma 4.2 C-DPP is **NP-complete**.

Proof: Obviously, the problem is in **NP**. We will show **NP-hardness** by a reduction from 2-DPP.

From $G = (V, E)$, s_1, t_1, s_2 , and t_2 we construct the new graph

$$G' = (V \cup \{s, t\}, E \cup \{(s, s_1), (t_2, s), (t_1, t), (t, s_2)\}).$$

Assume that G has two vertex-disjoint paths, w_1 from s_1 to t_1 , and w_2 from s_2 to t_2 . Thus, the paths $(s, s_1), w_1, (t_1, t)$ and $(t, s_2), w_2, (t_2, s)$ in G' are vertex-disjoint. Note that s_1, s_2, t_1, t_2 are pairwise distinct. Thus, G' has the desired vertex-disjoint cycle.

Now assume that G' has two vertex-disjoint paths, w_1 from s to t , and w_2 from t to s . By construction, w_1 must contain a path from s_1 to t_1 . Similarly, w_2 must contain a path from s_2 to t_2 . Since w_1 and w_2 are vertex-disjoint these new paths must also be vertex-disjoint. Thus, G has the desired vertex-disjoint paths. \square

Theorem 4.2 Determining if a 2CNF formula has a Var-ROR refutation is **NP-complete**.

Proof: As previously stated, we only need to show **NP-hardness**. That will be done by a reduction from C-DPP.

From $G = (V, E)$, s , and t we construct a formula Φ in 2CNF as follows:

1. For each vertex $v_i \in V - \{s, t\}$, create the variable x_i .
2. Create the variable x_0 .
3. Let $v_i, v_j \in V - \{s, t\}$.
 - (a) If $(s, v_i) \in E$ add the clause $(x_0 \rightarrow x_i)$ to Φ .
 - (b) If $(t, v_i) \in E$ add the clause $(\neg x_0 \rightarrow x_i)$ to Φ .
 - (c) If $(v_i, s) \in E$ add the clause $(x_i \rightarrow x_0)$ to Φ .
 - (d) If $(v_i, t) \in E$ add the clause $(x_i \rightarrow \neg x_0)$ to Φ .
 - (e) If $(v_i, v_j) \in E$ add the clause $(x_i \rightarrow x_j)$ to Φ .

Assume that G has two vertex-disjoint paths,

$$w_1 = s, v_{i_1}, \dots, v_{i_j}, t \text{ and } w_2 = t, v_{i_{j+1}}, \dots, v_{i_k}, s.$$

Thus, there exist 2CNF formulas Φ_1 and Φ_2 such that:

$$\begin{aligned} \Phi_1 &= \{(x_0 \rightarrow x_{i_1}), (x_{i_1} \rightarrow x_{i_2}), \dots, (x_{i_j} \rightarrow \neg x_0)\} \\ \Phi_2 &= \{(\neg x_0 \rightarrow x_{i_{j+1}}), (x_{i_{j+1}} \rightarrow x_{i_{j+2}}), \dots, (x_{i_k} \rightarrow x_0)\}. \end{aligned}$$

Clearly, $\Phi_1 \vdash_{\text{Var-ROR-Res}} (\neg x_0)$ and $\Phi_2 \vdash_{\text{Var-ROR-Res}} (x_0)$. Note that x_0 has not been used as a matching variable. Since w_1 and w_2 are vertex-disjoint, we have that

$$\{x_{i_1}, \dots, x_{i_j}\} \cap \{x_{i_{j+1}}, \dots, x_{i_k}\} = \emptyset.$$

Thus, $\Phi_1 \cup \Phi_2 \vdash_{\text{Var-ROR-Res}} \perp$. This means that $\Phi \supseteq \Phi_1 \cup \Phi_2$ is in Var-ROR.

Now assume that Φ is in Var-ROR. Let $\Phi' \subseteq \Phi$ be minimally Var-ROR. We have that Φ' contains clauses with x_0 and $\neg x_0$. Otherwise, the formula would be satisfiable by setting each x_i to **true**.

We proceed by an induction on the number of clauses in Φ' .

The shortest formula is $\Phi' = (x_0 \rightarrow \neg x_0) \wedge (\neg x_0 \rightarrow x_0)$. This Φ' is generated when $(s, t), (t, s) \in E$. These edges form the desired vertex-disjoint paths.

Let y be the variable for which $(y) \wedge (\neg y) \vdash_{\text{RES}} \perp$ is the last resolution step in $\Phi' \vdash_{\text{Var-ROR-Res}} \perp$. Thus, Φ' can be divided into two variable-disjoint sets of clauses, Φ'_1 and Φ'_2 , such that $\Phi'_1 \vdash_{\text{Var-ROR-Res}} (y)$ and $\Phi'_2 \vdash_{\text{Var-ROR-Res}} (\neg y)$. Otherwise, a variable would be used twice in $\Phi' \vdash_{\text{Var-ROR-Res}} \perp$.

Let $(L \vee x_i) \wedge (\neg x_i \vee K) \vdash_{\text{RES}} (L \vee K)$ a resolution step in $\Phi'_1 \vdash_{\text{Var-ROR-Res}} (y)$ such that $(L \vee x_i) \in \Phi'_1$ and $(\neg x_i \vee K) \in \Phi'_1$. Thus, no clause with x_i occurs in Φ'_2 or Φ_1 (except $(L \vee x_i)$ and $(\neg x_i \vee K)$). Moreover, we see that the formula

$$(\Phi' \setminus \{(L \vee x_i), (\neg x_i \vee K)\}) \cup \{(L \vee K)\}$$

is in Var-ROR. This formula represents the reduced graph where the edges $L \rightarrow a_i$ and $a_i \rightarrow K$ are replaced with the edge $L \rightarrow K$. By the induction hypothesis, there exists a vertex-disjoint cycle in this reduced graph. Thus, a vertex-disjoint cycle exists in G . \square

For arbitrary formulas in CNF, the problem of deciding whether a formula Φ has a sub-formula Φ' such that $\Phi' \in \text{MU}(1)$ is known to be **NP-complete**. But it was only known for arbitrary CNF. Based on the Theorems above, we obtain as a corollary that the MU(1) sub-formula problem is **NP-complete** for 2CNF, too.

Corollary 4.2 *The problem of deciding whether a formula in 2CNF contains a minimal unsatisfiable formula with deficiency 1 is NP-complete.*

5 The Complexity of ROR for 2CNF

In this section, we show that determining if a formula in 2CNF has a read-once refutation is **NP-complete**. It was established in [17] that the Var-ROR problem for 2CNF can be reduced to the ROR problem for 2CNF. Together with Theorem 4.2, this establishes that the ROR problem for 2CNF is **NP-complete**. We now present an alternate way of obtaining this result.

Unlike minimally Var-ROR formulas, minimally ROR formulas are not necessarily minimal unsatisfiable. They also can have deficiencies other than 1. An example of such a formula can be seen in Appendix B.

We now prove some properties of minimal unsatisfiable formulas in 2-CNF with one or two unit clauses. It can easily be seen that such formulas contain at most two unit clauses.

Lemma 5.1 *Let Φ be a minimal unsatisfiable 2CNF formula.*

1. *If Φ contains two unit clauses, then Φ has the form*

$$(L), (\neg L \vee L_1), \dots, (\neg L_{t-1} \vee L_t), (\neg L_t \vee \neg K), (K)$$

where L, L_1, \dots, L_t, K are pairwise distinct.

2. *If Φ contains exactly one unit clause, then Φ has the form*

$$\begin{aligned} (L), (\neg L \vee L_1), (\neg L_1 \vee L_2), \dots, (\neg L_t \vee K), \\ (\neg K \vee S_1), (\neg S_1 \vee S_2), \dots, (\neg S_q \vee R), \\ (\neg K \vee P_1), (\neg P_1 \vee P_2), \dots, (\neg P_m \vee \neg R) \end{aligned}$$

where the literals are all pairwise distinct.

3. If Φ contains at least one unit clause, then Φ has a read-once resolution refutation.
4. If Φ is in $MU(1)$, then Φ has a ROR refutation.

Proof: We prove each part of the lemma separately.

The proofs of part 1 and 2 are straightforward because no minimal unsatisfiable 2CNF formula contains more than two unit clauses.

3. If the formula has two unit clauses, then structure of the formula leads immediately to a ROR refutation.

If the formula has one unit clause, then we can perform the desired resolution refutation as follows:

- (a) First, we resolve $(\neg K \vee S_1), (\neg S_1 \vee S_2), \dots, (\neg S_q \vee R)$ to obtain $(\neg K \vee R)$.
- (b) Then, we resolve $(\neg K \vee P_1), (\neg P_1 \vee P_2), \dots, (\neg P_m \vee \neg R)$ to obtain $(\neg K \vee \neg R)$.
- (c) Next, we perform the resolution step

$$(\neg K \vee R) \wedge (\neg K \vee \neg R) \xrightarrow{1}_{RES} (\neg K).$$

- (d) Finally, the unit clause $(\neg K)$ together with the chain

$$(L), (\neg L \vee L_1), (\neg L_2 \vee L_3), \dots, (\neg L_t \vee K)$$

resolve to finish the ROR refutation.

4. Every 2CNF formula in $MU(1)$ has a complete disjunctive splitting tree. This guarantees the existence of a ROR refutation [6].

□

Theorem 5.1 Let Φ be in 2CNF. Φ is in ROR, if and only if there exists a sub-formula $\Phi' \subseteq \Phi$ for which there exists a variable x and a disjunctive splitting $(F_x, F_{\neg x})$ over x , such that $F_x, F_{\neg x}$ are in $MU(1)$.

Proof: Suppose, there exists a sub-formula $\Phi' \subseteq \Phi$ with disjunctive splitting $(F_x, F_{\neg x})$, where $F_x, F_{\neg x} \in MU(1)$. We have that F_x and $F_{\neg x}$ each contain at least one unit clause. Now we reconstruct the clauses of F_x and $F_{\neg x}$ by adding the removed literal x (resp. $\neg x$) to the clauses in F_x (resp. $F_{\neg x}$). These new formulas are denoted as F_x^x and $F_{\neg x}^{\neg x}$.

From lemma 5.1, every formula in $MU(1)$ with a unit-clause has a read-once resolution refutation. We also have that (x) and $(\neg x)$ do not occur in the splitting formulas. Thus, we get $F_x \xrightarrow{RO-Res} \perp$, $F_{\neg x} \xrightarrow{RO-Res} \perp$, $F_x^x \xrightarrow{RO-Res} (x)$, and $F_{\neg x}^{\neg x} \xrightarrow{RO-Res} (\neg x)$. Now we have to guarantee that there is a read-once resolution for Φ . $(F_x, F_{\neg x})$ is disjunctive splitting. Thus, no clause of Φ occurs in both F_x^x and in $F_{\neg x}^{\neg x}$. We can combine the resolutions $F_x^x \xrightarrow{RO-Res} (x)$ and $F_{\neg x}^{\neg x} \xrightarrow{RO-Res} (\neg x)$, with the resolution step $(x) \wedge (\neg x) \xrightarrow{1}_{RES} \perp$ to yield $\Phi \xrightarrow{RO-Res} \perp$, since $F_x^x, F_{\neg x}^{\neg x} \subseteq \Phi$.

Now suppose, $\Phi \in ROR$ and without loss of generality is minimally ROR. We will show that Φ contains the desired splitting. Let $x \wedge \neg x \xrightarrow{RES} \perp$ the last resolution step in the read-once resolution refutation. Furthermore, let F_x^x ($F_{\neg x}^{\neg x}$ respectively) be the set of original clauses from Φ used in the derivation of x ($\neg x$ respectively). These sets have no clause in common because together they form a read-once resolution refutation for Φ .

The formulas have the form

$$F_x^x = (x \vee L_1) \wedge \dots \wedge (x \vee L_t) \wedge \sigma_1 \text{ and } F_{\neg x}^{\neg x} = (\neg x \vee K_1) \wedge \dots \wedge (\neg x \vee K_r) \wedge \sigma_2,$$

where $\sigma_1 \cap \sigma_2 = \emptyset$.

Thus, we can construct the formulas

$$F_x = (L_1) \wedge \dots \wedge (L_t) \wedge \sigma_1 \text{ and } F_{\neg x} = (K_1) \wedge \dots \wedge (K_r) \wedge \sigma_2,$$

where $\sigma_1 \cap \sigma_2 = \emptyset$.

By construction, $F_x \xrightarrow{RO-Res} \perp$ and $F_{\neg x} \xrightarrow{RO-Res} \perp$. Both F_x and $F_{\neg x}$ are minimal unsatisfiable. Otherwise, Φ would not be minimally ROR. Thus, $(F_x, F_{\neg x})$ is a disjunctive splitting. Both F_x and $F_{\neg x}$ contain unit clauses. Thus, by lemma 3.1, F_x and $F_{\neg x}$ are $MU(1)$. □

We will now prove the **NP-completeness** or the ROR problem for 2CNF formulas. Instead of using the vertex-disjoint cycle problem, we will be reducing from the edge-disjoint cycle problem for directed graphs.

Definition 5.1 Given a directed graph G and two distinct vertexes s and t , the **edge-disjoint cycle problem (C-DEP)** consists of finding a pair of edge-disjoint paths in G , one from s to t and the other from t to s .

The problem is **NP-complete**. For two pairs of vertexes, the edge-disjoint path problem is **NP-complete** [9]. We can reduce the edge-disjoint path problem to C-DEP the same way we reduced 2-DPP to C-DPP.

Theorem 5.2 The ROR problem for 2CNF formulas is **NP-complete**.

Proof: ROR is in **NP** for arbitrary formulas in CNF [5]. Thus, we only need to show **NP-hardness**. That will be done by a reduction from C-DEP.

From $G = (V, E)$, s , and t we construct a formula Φ in 2CNF as follows:

1. For each vertex $v_i \in V - \{s, t\}$, create the variable x_i .
2. Create the variable x_0 .
3. Let $v_i, v_j \in V - \{s, t\}$.
 - (a) If $(s, v_i) \in E$ add the clause $(x_0 \rightarrow x_i)$ to Φ .
 - (b) If $(t, v_i) \in E$ add the clause $(\neg x_0 \rightarrow x_i)$ to Φ .
 - (c) If $(v_i, s) \in E$ add the clause $(x_i \rightarrow x_0)$ to Φ .
 - (d) If $(v_i, t) \in E$ add the clause $(x_i \rightarrow \neg x_0)$ to Φ .
 - (e) If $(v_i, v_j) \in E$ add the clause $(x_i \rightarrow x_j)$ to Φ .

Assume that G has two edge-disjoint paths,

$$w_1 = s, v_{i_1}, \dots, v_{i_j}, t \text{ and } w_2 = t, v_{i_{j+1}}, \dots, v_{i_k}, s.$$

Thus, there exist 2CNF formulas Φ_1 and Φ_2 such that:

$$\begin{aligned} \Phi_1 &= \{(x_0 \rightarrow x_{i_1}), (x_{i_1} \rightarrow x_{i_2}), \dots, (x_{i_j} \rightarrow \neg x_0)\} \\ \Phi_2 &= \{(\neg x_0 \rightarrow x_{i_{j+1}}), (x_{i_{j+1}} \rightarrow x_{i_{j+2}}), \dots, (x_{i_k} \rightarrow x_0)\}. \end{aligned}$$

Obviously, $\Phi_1 \vdash_{RO-Res} (\neg x_0)$ and $\Phi_2 \vdash_{RO-Res} (x_0)$. Note that x_0 has not been used as a matching variable. Since w_1 and w_2 are edge-disjoint, we have that $\Phi_1 \cap \Phi_2 = \emptyset$. Thus, $\Phi_1 \cup \Phi_2 \vdash_{RO-Res} \perp$. This means that $\Phi \supseteq \Phi_1 \cup \Phi_2$ is in ROR.

Now assume that Φ is in ROR.

Let $\Phi' \subseteq \Phi$ be minimally ROR. We have that Φ' contains clauses with x_0 and $\neg x_0$. Otherwise, the formula would be satisfiable by setting each x_i to **true**.

We proceed by an induction on the number of clauses in Φ' .

The shortest formula is $\Phi' = (x_0 \rightarrow \neg x_0) \wedge (\neg x_0 \rightarrow x_0)$. This Φ' is generated when $(s, t), (t, s) \in E$. These edges form the desired edge-disjoint paths.

Let $(L \rightarrow K) \wedge (K \rightarrow R) \vdash_{RES}^1 (L \vee R)$ be a resolution step where $(L \rightarrow K) \in \Phi'$ and $(K \rightarrow R) \in \Phi'$. Note that $(L \rightarrow R) \notin \Phi'$. Otherwise, Φ' would not be minimally ROR.

In a read-once refutation, we remove the parent clauses from Φ and add the resolvent $(L \rightarrow R)$. This new formula has a read-once resolution refutation and can be considered as obtained by a reduced graph without the edges $L \rightarrow K, K \rightarrow R$ but with the edge $L \rightarrow R$. By the induction hypothesis, this new graph contains the desired edge-disjoint cycle. If we replace the edge $L \rightarrow R$ in this cycle with $L \rightarrow K$ and $K \rightarrow R$, then we construct the desired edge-disjoint cycle in G . \square

By lemma 3.4, the problem of determining if an MU-formula in 2CNF has a disjunctive splitting whose splitting formulas are in MU(1) can be decided in polynomial time. Hence, for MU-formulas in 2CNF, the ROR problem is solvable in polynomial time.

Corollary 5.1 The ROR problem for minimal unsatisfiable 2CNF formulas is in **P**.

6 Copy Complexity of 2CNF formulas

Let Φ be an unsatisfiable formula in CNF and let Λ be a tree-like resolution refutation of Φ . For each clause $\pi_i \in \Phi$, let Λ_i be the number of times π_i is used in a resolution step of Λ .

Definition 6.1 A CNF formula Φ has a **Copy Complexity** of k , if there exists Λ such that $\Lambda_i \leq k$ for $i = 1, \dots, m$.

Note that in a tree-like refutation reusing a resultant clause requires the reuse of the clauses originally in Φ . Thus, by limiting the number of times Λ can use each clause in Φ we also limit the number of times each resultant clause can be used.

We can equivalently define copy complexity as follows.

Definition 6.2 A CNF formula Φ has a **Copy Complexity** of k , if there exists a multi-set of CNF clauses, Φ' such that:

1. Every clause in Φ appears at most k times in Φ' .
2. Every clause in Φ' appears in Φ .
3. Φ' has a read-once refutation.

Thus, if a formula, Φ , has a copy complexity of 1 then the formula has a read-once resolution refutation.

We can extend the concept of copy complexity to classes of CNF formulas.

Definition 6.3 A class of CNF formulas has a **Copy Complexity** of k , if every formula in that class has a copy complexity of k .

We will now show that 2CNF has a copy complexity of 2. This means that we can always prove unsatisfiability using each clause at most twice.

Lemma 6.1 Let Φ be a formula in 2CNF. If we can prove (x_i) , then we can prove (x_i) using each clause no more than once.

Proof: For a formula Φ , we construct the corresponding implication graph G as follows:

1. For each variable x_i , we create the vertices x_i and \bar{x}_i . These correspond to the literals x_i and $\neg x_i$.
2. For each clause $(L \vee K)$, we create the edges $\bar{L} \rightarrow K$ and $\bar{K} \rightarrow L$.
3. For each unit clause (L) , we create the edge $\bar{L} \rightarrow L$.

We know that we can prove (x_i) in Φ , if and only if there exists a path from \bar{x}_i to x_i in G . Let p denote this path. If no two edges in p correspond to the same clause, then this path already corresponds to a read-once proof of (x_i) .

Let e be the first edge in p such that the other edge corresponding to the same clause as e has already been used in p . We can assume without loss of generality that e is $x_j \rightarrow x_k$. Thus, we can break p up as follows:

1. a path, p_1 , from \bar{x}_i to \bar{x}_k ,
2. the edge $\bar{x}_k \rightarrow \bar{x}_j$ (the edge corresponding to the same clause as e),
3. a path, p_2 , from \bar{x}_j to x_j ,
4. the edge $x_j \rightarrow x_k$ (the edge e),
5. and a path, p_3 , from x_k to x_i .

This can be seen in Figure 1.

By our choice of e , we know no two edges in p_1 or p_2 correspond to the same clause. Thus, p_2 corresponds to a read-once proof of (x_j) . We also have that p_1 combined with the edge $\bar{x}_k \rightarrow \bar{x}_j$ is a read once proof of $(\neg x_i \rightarrow \neg x_j)$. Combining these two yields a read-once proof of $(\neg x_i \rightarrow x_i)$ and therefore for (x_i) . \square

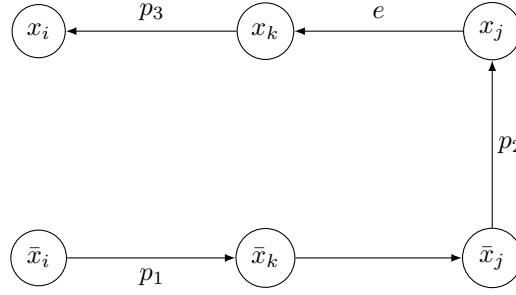


Figure 1: Example of Path p

Theorem 6.1 *If a formula in 2CNF is unsatisfiable, then there exists a resolution refutation in which no clause is used more than twice.*

Proof: Suppose, the formula Φ in 2CNF is unsatisfiable. Then for some x_i , we have that:

1. $\Phi \vdash_{RES} (x_i)$
2. and $\Phi \vdash_{RES} (\neg x_i)$.

From Lemma 6.1, we know that we can derive (x_i) using each clause at most once. Similarly we can derive $(\neg x_i)$ using each clause at most once.

Thus, for Φ , there is a resolution refutation using each clause at most twice. That is once in the refutation derivation of (x_i) and once in the resolution derivation of $(\neg x_i)$. \square

7 Conclusion

In this paper, we discuss the computational complexity of ROR decidability in a well-known classe of CNF formulas, viz., 2CNF. Prior research had established that the ROR decidability problem is **NP-complete** for 3CNF formulas. Likewise, it is known that resolution refutations for 3CNF formulas, even when they exist need not be polynomial sized. For 2CNF formulas, it is well-known that polynomial sized refutations exist in the general case. We showed that ROR decidability in 2CNF formulas is **NP-complete**. Additionally, we showed that the copy complexity of 2CNF formulas is 2. This means that every unsatisfiable 2CNF formula has a refutation in which each input clause is used at most twice. Furthermore, the optimal length resolution when clause copy is permitted can be determined in polynomial time.

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A ROR Refutation Example

We now apply read-once resolution refutation to generate a refutation of the 2SAT instance specified by Formula (1).

$$\begin{array}{lll}
 (x_1, x_2) & (\neg x_1, x_3) & (\neg x_1, x_4) \\
 (\neg x_2, x_3) & (\neg x_2, x_4) & (\neg x_3, x_5) \\
 (\neg x_3, x_6) & (\neg x_4, \neg x_5) & (\neg x_4, \neg x_6)
 \end{array} \tag{1}$$

The application of this read-once resolution refutation to Formula (1) can be seen in Figure 2.

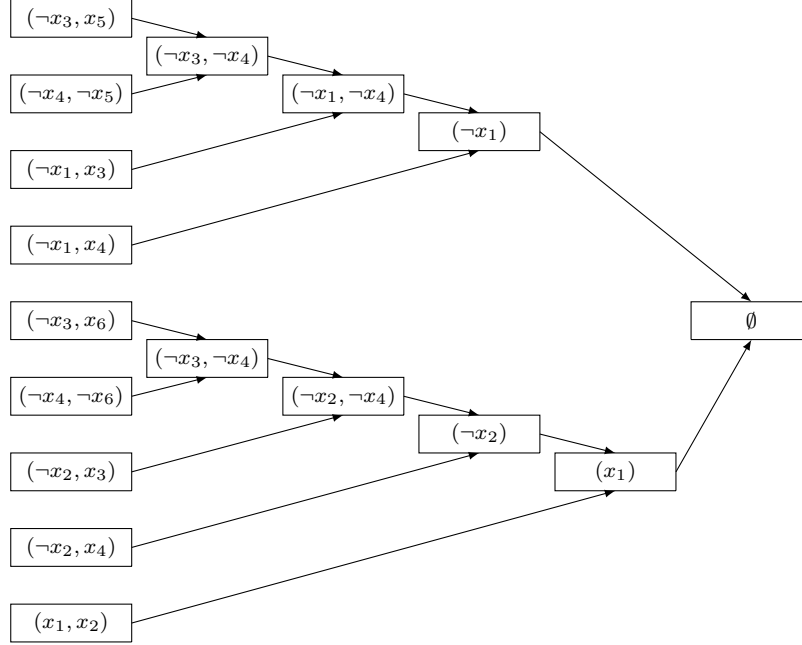


Figure 2: Read-Once Refutation

Note that the clause $(\neg x_3, \neg x_4)$ is used twice. However, this is still a read-once refutation since each time the clause $(\neg x_3, \neg x_4)$ is derived different clauses from the original formula are used.

B Non MU(1) ROR refutation

Example (3): Let

$$\begin{aligned}
 \Phi = \{ & (\neg x \rightarrow a), (a \rightarrow b), (a \rightarrow c), (c \rightarrow b), \\
 & (b \rightarrow x), (b \rightarrow \neg x), (x \rightarrow a) \}.
 \end{aligned}$$

Φ is not minimal unsatisfiable and has deficiency 2. Minimal unsatisfiable sub-formulas of Φ are:

$$\begin{aligned}
 \Phi_1 &= \{(\neg x \rightarrow a), (a \rightarrow b), (b \rightarrow x), (b \rightarrow \neg x), (x \rightarrow a)\} \\
 \Phi_2 &= \{(\neg x \rightarrow a), (a \rightarrow c), (c \rightarrow b), \\
 &\quad (b \rightarrow x), (b \rightarrow \neg x), (x \rightarrow a)\}.
 \end{aligned}$$

Note that Φ_1 and Φ_2 have deficiency 1. There is no read-once resolution refutation for Φ_1 or Φ_2 . However, Φ has the following read-once resolution refutation:

1. $(\neg x \rightarrow a) \wedge (a \rightarrow c) \stackrel{1}{\text{RES}} (\neg x \rightarrow c).$
2. $(\neg x \rightarrow c) \wedge (c \rightarrow b) \stackrel{1}{\text{RES}} (\neg x \rightarrow b).$
3. $(\neg x \rightarrow b) \wedge (b \rightarrow x) \stackrel{1}{\text{RES}} (x).$
4. $(x \rightarrow a) \wedge (a \rightarrow b) \stackrel{1}{\text{RES}} (x \rightarrow b).$
5. $(x \rightarrow b) \wedge (b \rightarrow \neg x) \stackrel{1}{\text{RES}} (\neg x).$
6. $(\neg x) \wedge (x) \stackrel{1}{\text{RES}} \perp.$

From Φ we can construct the graph G as follows:

1. For each literal in Φ , create a vertex in G .
2. For each implication $(x \rightarrow a)$ in Φ , create the edge $(x \rightarrow a)$ in G .

The result of this can be seen in Figure 3.

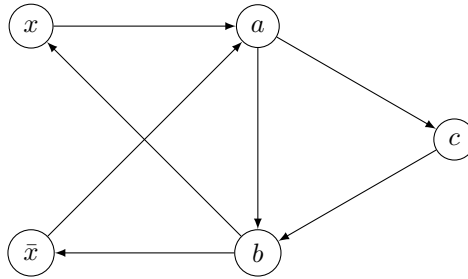


Figure 3: Edge-Disjoint Cycle: x to \bar{x} and \bar{x} to x

The paths $\neg x, a, b, x$ and $x, a, c, b, \neg x$ form an edge-disjoint cycle from $\neg x$ to x and from x to $\neg x$. These paths correspond to the follow sub-formulas of Φ :

$$\begin{aligned}
 F_x^x &= (\neg x \rightarrow a) \wedge (b \rightarrow x) \wedge (a \rightarrow b) \\
 F_{\neg x}^{\neg x} &= (a \rightarrow c) \wedge (c \rightarrow b) \wedge (x \rightarrow a) \wedge (b \rightarrow \neg x).
 \end{aligned}$$

The following summarizes our observations:

1. The formula Φ is in ROR, but not minimal unsatisfiable. Moreover, any minimal unsatisfiable sub-formula of Φ has no read-once resolution refutation.
2. The formula Φ is minimally ROR and Φ has deficiency greater than 1.
3. The formula Φ has two sub-formulas formulas in MU(1).
4. The formula Φ , when considered as directed graph, has an edge-disjoint cycle from $\neg x$ to x and from x to $\neg x$. Each path in this cycle corresponds to a sub-formula of Φ .