

Kernelization and Parameterized Algorithms for 3-Path Vertex Cover ^{*}

Mingyu Xiao and Shaowei Kou

School of Computer Science and Engineering,
University of Electronic Science and Technology of China, Chengdu 611731, China
myxiao@gmail.com, kou_sw@163.com

Abstract. A 3-path vertex cover in a graph is a vertex subset C such that every path of three vertices contains at least one vertex from C . The parameterized 3-path vertex cover problem asks whether a graph has a 3-path vertex cover of size at most k . In this paper, we give a kernel of $5k$ vertices and an $O^*(1.7485^k)$ -time polynomial-space algorithm for this problem, both new results improve previous known bounds.

1 Introduction

A vertex subset C in a graph is called an ℓ -path vertex cover if every path of ℓ vertices in the graph contains at least one vertex from C . The ℓ -path vertex cover problem, to find an ℓ -path vertex cover of minimum size, has been studied in the literature [5,6]. When $\ell = 2$, this problem becomes the famous vertex cover problem and it has been well studied. In this paper we study the 3-path vertex cover problem. A 3-path vertex cover is also known as a 1-degree-bounded deletion set. The d -degree-bounded deletion problem [11,25,26] is to delete a minimum number of vertices from a graph such that the remaining graph has degree at most d . The 3-path vertex cover problem is exactly the 1-degree-bounded deletion problem. Several applications of 3-path vertex covers have been proposed in [6,16,27].

It is not hard to establish the NP-hardness of the 3-path vertex cover problem by reduction from the vertex cover problem. In fact, it remains NP-hard even in planar graphs [28] and in C_4 -free bipartite graphs with vertex degree at most 3 [4]. There are several graph classes, in which the problem can be solved in polynomial time [2,3,4,6,7,14,15,18,19,20].

The 3-path vertex cover problem has been studied from approximation algorithms, exact algorithms and parameterized algorithms. There is a randomized approximation algorithm with an expected approximation ratio of $\frac{23}{11}$ [16]. In terms of exact algorithms, Kardoš et al. [16] gave an $O^*(1.5171^n)$ -time algorithm to compute a maximum dissociation set in an n -vertex graph. Chang et al. [8] gave an $O^*(1.4658^n)$ -time algorithm and the result was further improved to $O^*(1.3659^n)$ later [27].

^{*} This is the version accepted by TAMC 2016. To appear in: TAMC 2016, LNCS 9796, pp. 1–15, 2016.

In parameterized complexity, this problem is fixed-parameter tractable by taking the size k of the 3-path vertex cover as the parameter. The running time bound of parameterized algorithm for this problem has been improved at least three times during the last one year. Tu [22] showed that the problem can be solved in $O^*(2^k)$ time. Wu [24] improved the result to $O^*(1.882^k)$ by using the measure-and-conquer method. The current best result is $O^*(1.8172^k)$ by Katrenič [17]. In this paper we will further improve the bound to $O^*(1.7485^k)$.

Another important issue in parameterized complexity is kernelization. A kernelization algorithm is a polynomial-time algorithm which, for an input graph with a parameter (G, k) either concludes that G has no 3-path vertex cover of size k or returns an equivalent instance (G', k') , called a *kernel*, such that $k' \leq k$ and the size of G' is bounded by a function of k . Kernelization for the d -degree-bounded deletion problem has been studied in the literature [11,25]. For $d = 1$, Fellows et al.'s algorithm [11] implies a kernel of $15k$ vertices for the 3-path vertex cover problem, and Xiao's algorithm [25] implies a kernel of $13k$ vertices. There is another closed related problem, called the *3-path packing* problem. In this problem, we are going to check if a graph has a set of at least k vertex-disjoint 3-paths. When we discuss kernelization algorithms, most structural properties of the 3-path vertex cover problem and the 3-path packing problem are similar. Several previous kernelization algorithms for the 3-path packing problem are possible to be modified for the 3-path vertex cover problem. The bound of the kernel size of the 3-path packing problem has been improved for several times from the first bound of $15k$ [21] to $7k$ [23] and then to $6k$ [10]. Recently, there is a paper claiming a bound of $5k$ vertices for the 3-path packing problem in net-free graphs [9]. Although the paper [9] provides some useful ideas, the proof in it is incomplete and the algorithm may not stop. Several techniques for the 3-path packing problem in [23] and [9] will be used in our kernelization algorithm. We will give a kernel of $5k$ vertices for the 3-path vertex cover problem.

Omitted proofs in this extended abstract can be found in the full version of this paper.

2 Preliminaries

We let $G = (V, E)$ denote a simple and undirected graph with $n = |V|$ vertices and $m = |E|$ edges. A singleton $\{v\}$ may be simply denoted by v . The vertex set and edge set of a graph G' are denoted by $V(G')$ and $E(G')$, respectively. For a subgraph (resp., a vertex subset) X , the subgraph induced by $V(X)$ (resp., X) is simply denoted by $G[X]$, and $G[V \setminus V(X)]$ (resp., $G[V \setminus X]$) is also written as $G \setminus X$. A vertex in a subgraph or a vertex subset X is also called a *X-vertex*. For a vertex subset X , let $N(X)$ denote the set of *open neighbors* of X , i.e., the vertices in $V \setminus X$ adjacent to some vertex in X , and $N[X]$ denote the set of *closed neighbors* of X , i.e., $N(X) \cup X$. The *degree* of a vertex v in a graph G , denoted by $d(v)$, is defined to be the number of vertices adjacent to v in G . Two vertex-disjoint subgraphs X_1 and X_2 are *adjacent* if there is an edge with one endpoint in X_1 and the other in X_2 . The number of connected components in

a graph G is denoted by $Comp(G)$ and the number of connected components of size i in a graph G is denoted by $Comp_i(G)$. thus, $Comp(G) = \sum_i Comp_i(G)$.

A 3-path, denoted by P_3 , is a simple path with three vertices and two edges. A vertex subset C is called a 3-path vertex cover or a P_3VC -set if there is no 3-path in $G \setminus C$. Given a graph $G = (V, E)$, a P_3 -packing $\mathcal{P} = \{L_1, L_2, \dots, L_t\}$ of size t is a collection of vertex-disjoint P_3 in G , i.e., each element $L_i \in \mathcal{P}$ is a 3-path in G and $V(L_{i_1}) \cap V(L_{i_2}) = \emptyset$ for any two different 3-paths $L_{i_1}, L_{i_2} \in \mathcal{P}$. A P_3 -packing is *maximal* if it is not properly contained in any strictly larger P_3 -packing in G . The set of vertices in 3-paths in \mathcal{P} is denoted by $V(\mathcal{P})$.

Let \mathcal{P} be a P_3 -packing and A be a vertex set such that $A \cap V(\mathcal{P}) = \emptyset$ and A induces a graph of maximum degree 1. We use A_i to denote the set of degree- i vertices in the induced graph $G[A]$ for $i = 0, 1$. A component of two vertices in $G[A]$ is called an A_1 -edge. For each $L_i \in \mathcal{P}$, we use $A(L_i)$ to denote the set of A -vertices that are in the components of $G[A]$ adjacent to L_i . For a 3-path $L_i \in \mathcal{P}$, the degree-2 vertex in it is called the *middle vertex* of it and the two degree-1 vertices in it are called the *ending vertices* of it.

3 A Parameterized Algorithm

In this section we will design a parameterized algorithm for the 3-path vertex cover problem. Our algorithm is a branch-and-reduce algorithm that runs in $O^*(1.7485^k)$ time and polynomial space, improving all previous results. In branch-and-reduce algorithms, the exponential part of the running time is determined by the branching operations in the algorithm. In a branching operation, the algorithm solves the current instance I by solving several smaller instances. We will use the parameter k as the measure of the instance and use $T(k)$ to denote the maximum size of the search tree generated by the algorithm running on any instance with parameter at most k . A branching operation, which generates l small branches with measure decrease in the i -th branch being at least c_i , creates a recurrence relation $T(k) \leq T(k - c_1) + T(k - c_2) + \dots + T(k - c_l) + 1$. The largest root of the function $f(x) = 1 - \sum_{i=1}^l x^{-c_i}$ is called the *branching factor* of the recurrence. Let γ be the maximum branching factor among all branching factors in the algorithm. The running time of the algorithm is bounded by $O^*(\gamma^k)$. More details about the analysis and how to solve recurrences can be found in the monograph [13]. Next, we first introduce our branching rules and then present our algorithm.

3.1 Branching Rules

We have four branching rules. The first branching rule is simple and easy to observe.

Branching rule (B1): Branch on a vertex v to generate $|N[v]| + 1$ branches by either

(i) deleting v from the graph, including it to the solution set, and decreasing k by 1, or

- (ii) deleting $N[v]$ from the graph, including $N(v)$ to the solution set, and decreasing k by $|N(v)|$, or
- (iii) for each neighbor u of v , deleting $N[\{u, v\}]$ from the graph, including $N(\{u, v\})$ to the solution set, and decreasing k by $|N(\{u, v\})|$.

A vertex v is *dominated* by a neighbor u of it if v is adjacent to all neighbors of u . The following property of dominated vertices has been proved and used in [27].

Lemma 1. *Let v be a vertex dominated by u . If there is a minimum 3-path vertex cover C not containing v , then there is a minimum 3-path vertex cover C' of G such that $v, u \notin C'$ and $N(\{u, v\}) \subseteq C'$.*

Based on this lemma, we design the following branching rule.

Branching rule (B2): *Branch on a vertex v dominated by another vertex u to generate two instances by either*

- (i) deleting v from the graph, including it to the solution set, and decreasing k by 1, or
- (ii) deleting $N[\{u, v\}]$ from the graph, including $N(\{u, v\})$ to the solution set, and decreasing k by $|N(\{u, v\})| = |N(v)| - 1$.

For a vertex v , a vertex $s \in N_2(v)$ is called a *satellite* of v if there is a neighbor p of v such that $N[p] - N[v] = \{s\}$. The vertex p is also called the *parent* of the satellite s at v .

Lemma 2. *Let v be a vertex that is not dominated by any other vertex. If v has a satellite, then there is a minimum 3-path vertex cover C such that either $v \in C$ or $v, u \notin C$ for a neighbor u of v .*

Branching rule (B3): *Let v be a vertex that has a satellite but is not dominated by any other vertex. Branch on v to generate $|N[v]|$ instances by either*

- (i) deleting v from the graph, including it to the solution set, and decreasing k by 1, or
- (ii) for each neighbor u of v , deleting $N[\{u, v\}]$ from the graph, including $N(\{u, v\})$ to the solution set, and decreasing k by $|N(\{u, v\})|$.

Lemma 3. *Let v be a degree-3 vertex with a degree-1 neighbor u_1 and two adjacent neighbors u_2 and u_3 . There is a minimum 3-path vertex cover C such that either $C \cup \{u_1, v\} = \emptyset$ or $C \cup \{u_1, u_2, u_3\} = \emptyset$.*

Branching rule (B4): *Let v be a degree-3 vertex with a degree-1 neighbor u_1 and two adjacent neighbors u_2 and u_3 . Branch on v to generate two instances by either*

- (i) deleting $N[\{u_1, v\}]$ from the graph, including $\{u_2, u_3\}$ to the solution set, and decreasing k by 2, or
- (ii) deleting $N[\{u_2, u_3\}] \cup \{u_1\}$ from the graph, including $N(\{u_2, u_3\})$ to the solution set, and decreasing k by $|N(\{u_2, u_3\})|$.

3.2 The Algorithm

We will use $\text{P3VC}(G, k)$ to denote our parameterized algorithm. The algorithm contains 7 steps. When we execute one step, we assume that all previous steps are not applicable anymore on the current graph. We will analyze each step after describing it.

Step 1 (Trivial cases) *If $k \leq 0$ or the graph is an empty graph, then return the result directly. If the graph has a component of maximum degree 2, find a minimum 3-path vertex cover S of it directly, delete this component from the graph, and decrease k by the size of S .*

After Step 1, each component of the graph contains at least four vertices. A degree-1 vertex v is called a *tail* if its neighbor u is a degree-2 vertex. Let v be a tail, u be the degree-2 neighbor of v , and w be the other neighbor of u . We show that there is a minimum 3-path vertex cover containing w but not containing any of u and v . At most one of u and v is contained in any minimum 3-path vertex cover C , otherwise $C \cup \{w\} \setminus \{u, v\}$ would be a smaller 3-path vertex cover. If none of u and v is in a minimum 3-path vertex cover C , then w must be in C to cover the 3-path uvw and then C is a claimed minimum 3-path vertex cover. If exactly one of u and v is contained in a minimum 3-path vertex cover C , then $C' = C \cup \{w\} \setminus \{u, v\}$ is a claimed minimum 3-path vertex cover.

Step 2 (Tails) *If there is a degree-1 vertex v with a degree-2 neighbor u , then return $\text{p3vc}(G \setminus N[\{v, u\}], k - 1)$.*

Step 3 (Dominated vertices of degree ≥ 3) *If there is a vertex v of degree ≥ 3 dominated by u , then branch on v with Rule (B2) to generate two branches*

$$\text{p3vc}(G \setminus \{v\}, k - 1) \quad \text{and} \quad \text{p3vc}(G \setminus N[\{v, u\}], k - |N(\{v, u\})|).$$

Lemma 1 guarantees the correctness of this step. Note that $|N(\{v, u\})| = d(v) - 1$. This step gives a recurrence

$$T(k) \leq T(k - 1) + T(k - (d(v) - 1)) + 1, \tag{1}$$

where $d(v) \geq 3$. For the worst case that $d(v) = 3$, the branching factor of it is 1.6181.

A degree-1 vertex with a degree-1 neighbor will be handled in Step 1, a degree-1 vertex with a degree-2 neighbor will be handled in Step 2, and a degree-1 vertex with a neighbor of degree ≥ 3 will be handled in Step 3. So after Step 3, the graph has no vertex of degree ≤ 1 . Next we consider degree ≥ 4 vertices.

Step 4 (Vertices of degree ≥ 4 with satellites) *If there is a vertex v of $d(v) \geq 4$ having a satellite, then branch on v with Rule (B3) to generate $d(v) + 1$ branches*

$$\text{p3vc}(G \setminus \{v\}, k - 1) \quad \text{and} \quad \text{p3vc}(G \setminus N[\{v, u\}], k - |N(\{v, u\})|) \quad \text{for each } u \in N(v).$$

The correctness of this step is guaranteed by lemma 2. Note that there is no dominated vertex after Step 3. Each neighbor u of v is adjacent to at least one vertex in $N_2(v)$ and then $|N(\{v, u\})| \geq d(v)$.

This step gives a recurrence

$$T(k) \leq T(k-1) + d(v) \cdot T(k-d(v)) + 1, \quad (2)$$

where $d(v) \geq 4$. For the worst case that $d(v) = 4$, the branching factor of it is 1.7485.

After Step 4, if there is still a vertex of degree ≥ 4 , we use the following branching rule. Note that now each neighbor u of v is adjacent to at least two vertices in $N_2(v)$ and then $|N(\{v, u\})| \geq d(v) + 1$.

Step 5 (Normal vertices of degree ≥ 4) *If there is a vertex v of $d(v) \geq 4$, then branch on v with Rule (B1) to generate $d(v) + 2$ branches*

$$\begin{aligned} & \text{p3vc}(G \setminus \{v\}, k-1), \quad \text{p3vc}(G \setminus N[v], k - |N(v)|) \\ & \text{and } \text{p3vc}(G \setminus N[\{v, u\}], k - |N(\{v, u\})|) \text{ for each } u \in N(v). \end{aligned}$$

Since $|N(\{v, u\})| \geq d(v) + 1$, this step gives a recurrence

$$T(k) \leq T(k-1) + T(k-d(v)) + d(v) \cdot T(k-(d(v)+1)) + 1, \quad (3)$$

which $d(v) \geq 4$. For the worst case that $d(v) = 4$, the branching factor of it is 1.6930.

After Step 5, the graph has only degree-2 and degree-3 vertices. We first consider degree-2 vertices.

A path $u_0u_1u_2u_3$ of four vertices is called a *chain* if the first vertex u_0 is of degree ≥ 3 and the two middle vertices are of degree 2. Note that there is no chain with $u_0 = u_3$ after Step 3. So when we discuss a chain we always assume that $u_0 \neq u_3$. A chain can be found in linear time if it exists. In a chain $u_0u_1u_2u_3$, u_2 is a satellite of u_0 with a parent u_1 .

Step 6 (Chains) *If there is a chain $u_0u_1u_2u_3$, then branch on u_0 with Rule (B3). In the branch where u_0 is deleted and included to the solution set, u_1 becomes a tail and we further handle the tail as we do in Step 2.*

We get the following branches

$$\begin{aligned} & \text{p3vc}(G \setminus N[\{u_1, u_2\}], k-2) \\ & \text{and } \text{p3vc}(G \setminus N[\{u_0, u\}], k - |N(\{u_0, u\})|) \text{ for each } u \in N(u_0). \end{aligned}$$

Note that $|N(\{u_0, u\})| \geq d(u_0)$ since there is no dominated vertex. We get a recurrence

$$T(k) \leq T(k-2) + d(u_0) \cdot T(k-d(u_0)) + 1,$$

where $d(u_0) \geq 3$. For the worst case that $d(u_0) = 3$, the branching factor of it is 1.6717.

After Step 6, each degree-2 vertex must have two nonadjacent degree-3 vertices. Note that no degree-2 is in a triangle if there is no dominated vertex.

Step 7 (Degree-2 vertices with a neighbor in a triangle) *If there is a degree-2 vertex v with $N(v) = \{u, w\}$ such that a neighbor u of it is in a triangle uu_1u_2 , then branch on w with Rule (B1) and then in the branch w is deleted and included in the solution set further branch on u with Rule (B4). We get the following branches*

$$\begin{aligned} & \text{p3vc}(G \setminus N[\{u, v\}], k - |N(\{u, v\})|), \\ & \text{p3vc}(G \setminus N[\{u_1, u_2\}] \cup \{u, w\}, k - |N(\{u_1, u_2\}) \cup \{w\}|), \text{ and} \\ & \text{p3vc}(G \setminus N[\{w, u'\}], k - |N(\{w, u'\})|) \text{ for each } u' \in N(w). \end{aligned}$$

There two neighbors u and w of v are degree-3 vertices. Since there is no dominated vertex, for any edge v_1v_2 it holds $|N(\{v_1, v_2\})| \geq \min\{d(v_1), d(v_2)\}$. We know that $|N(\{u, v\})| \geq d(u) = 3$, $|N(\{u_1, u_2\}) \cup \{w\}| \geq |N(\{u_1, u_2\})| \geq 3$ (since no degree-2 vertex is in a triangle) and $|N(\{w, u'\})| \geq d(w)$ for each $u' \in N(w)$. We get the following recurrence

$$T(k) \leq T(k-3) + T(k-3) + 3 \cdot T(k-3) + 1.$$

The branching factor of it is 1.7100.

After Step 7, no degree-3 vertex in a triangle is adjacent to a degree-2 vertex.

Step 8 (Degree-2 vertices v with a degree-3 vertex in $N_2(v)$) *If there is a degree-2 vertex v such that at least one of its neighbors u and w , say u , has a degree-3 neighbor u_1 , then branch on u with Rule (B1) and in the branch where u is deleted and included to the solution set, branch on w with Rule (B2). We get the branches*

$$\begin{aligned} & \text{p3vc}(G \setminus \{u, v, w\}, k - 2), \text{ p3vc}(G \setminus N[\{w, v\}], k - |N(\{w, v\})|), \\ & \text{and } \text{p3vc}(G \setminus N[\{u, u'\}], k - |N(\{u, u'\})|) \text{ for each } u' \in N(u). \end{aligned}$$

Note that $d(u) = d(w) = 3$. It holds $|N(\{w, v\})| \geq d(w) = 3$ and $|N(\{u, u'\})| \geq d(u) = 3$ for $u' \in N(u)$. Furthermore, we have that $|N(\{u, u_1\})| \geq 4$ because u and u_1 are degree-3 vertices not in any triangle. We get the following recurrence

$$T(k) \leq T(k-2) + T(k-3) + 2 \cdot T(k-3) + T(k-4).$$

The branching factor of it is 1.7456.

Lemma 4. *After Step 8, if the graph is not an empty graph, then each component of the graph is either a 3-regular graph or a bipartite graph with one side of degree-2 vertices and one side of degree-3 vertices.*

Lemma 5. *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph such that all vertices in V_1 are of degree 2 and all vertices in V_2 are of degree 3. The set V_1 is a minimum 3-path vertex cover of G .*

Step 9 (Bipartite graphs) *If the graph has a component H being a bipartite graph with one side V_1 of degree-2 vertices and one side V_2 of degree-3 vertices, then return $\text{p3vc}(G \setminus H, k - |V_1|)$.*

Step 10 (3-regular graphs) *If the graph is a 3-regular graph, pick up an arbitrary vertex v and branch on it with Rule (B1).*

Lemma 4 shows that the above steps cover all the cases, which implies the correctness of the algorithm. Note that all the branching operations except Step 10 in the algorithm have a branching factor at most 1.7485. We do not analyze the branching factor for Step 10, because this step will not exponentially increase the running time bound of our algorithm. Any proper subgraph of a connected 3-regular graph is not a 3-regular graph. For each connected component of a 3-regular graph, Step 10 can be applied for at most one time and all other branching operations have a branching factor at most 1.7485. Thus each connected component of a 3-regular graph can be solved in $O^*(1.7485^k)$ time. Before getting a connected component of a 3-regular graph, the algorithm always branches with branching factors of at most 1.7485. Therefore,

Theorem 1. *The 3-path vertex cover problem can be solved in $O^*(1.7485^k)$ time and polynomial space.*

4 Kernelization

In this section, we show that the parameterized 3-path vertex cover problem allows a kernel with at most $5k$ vertices.

4.1 Graph decompositions

The kernelization algorithm is based on a vertex decomposition of the graph, called *good decomposition*, which can be regarded as an extension of the crown decomposition [1]. Based on a good decomposition we show that an optimal solution to a special local part of the graph is contained in an optimal solution to the whole graph. Thus, once we find a good decomposition, we may be able to reduce the graph by adding some vertices to the solution set directly. We only need to find good decompositions in polynomial time in graphs with a large size to get problem kernels. Some previous rules to kernels for the parameterized 3-path packing problem [10,12,23] are adopted here to find good decompositions in an effective way.

Definition 1. *A good decomposition of a graph $G = (V, E)$ is a decomposition (I, C, R) of the vertex set V such that*

1. *the induced subgraph $G[I]$ has maximum degree at most 1;*
2. *the induced subgraph $G[I \cup C]$ has a P_3 -packing of size $|C|$;*
3. *no vertex in I is adjacent to a vertex in R .*

Lemma 6. *A graph G that admits a good decomposition (I, C, R) has a P_3 -vertex cover (resp., P_3 -packing) of size k if and only if $G[R]$ has a P_3 -vertex cover (resp., P_3 -packing) of size $k - |C|$.*

Lemma 6 provides a way to reduce instances of the parameterized 3-path vertex cover problem based on a good decomposition (I, C, R) of the graph: deleting $I \cup C$ from the graph and adding C to the solution set. Here arise a question: how to effectively find good decompositions? It is strongly related to the quality of our kernelization algorithm. The kernel size will be smaller if we can polynomially compute a good decomposition in a smaller graph. Recall that we use $Comp(G')$ and $Comp_i(G')$ to denote the number of components and number of components with i vertices in a graph G' , respectively. For a vertex subset A that induces a graph of maximum degree at most 1 and $j = \{1, 2\}$, we use $N_j(A) \subseteq N(A)$ to denote the set of vertices in $N(A)$ adjacent to at least one component of size j in $G[A]$, and $N'_2(A) \subseteq N_2(A)$ be the set of vertices in $N(A)$ adjacent to at least one component of size 2 but no component of size 1 in $G[A]$. We will use the following lemma to find good decompositions, which was also used in [9] to design kernel algorithms for the 3-path packing problem.

Lemma 7. *Let A be a vertex subset of a graph G such that each connected component of the induced graph $G[A]$ has at most 2 vertices. If*

$$Comp(G[A]) > 2|N(A)| - |N'_2(A)|, \quad (4)$$

then there is a good decomposition (I, C, R) of G such that $\emptyset \neq I \subseteq A$ and $C \subseteq N(A)$. Furthermore, the good decomposition (I, C, R) together with a P_3 -packing of size $|C|$ in $G[I \cup C]$ can be computed in $O(\sqrt{nm})$ time.

By using Lemma 7, we can get a linear kernel for the parameterized 3-path vertex cover problem quickly. We find an arbitrary maximal P_3 -packing S and let $A = V \setminus V(S)$. We assume that S contains less than k 3-paths and then $|V(S)| < 3k$, otherwise the problem is solved directly. Note that $|N(A)| \subseteq |V(S)|$. If $|A| > 12k$, then $Comp(G[A]) \geq \frac{|A|}{2} > 6k > 2|V(S)| \geq 2|N(A)|$ and we reduce the instance by Lemma 7. So we can get a kernel of $15k$ vertices. This bound can be improved by using a special case of Lemma 7.

For a vertex subset A such that $G[A]$ has maximum degree at most 1. Let A_0 be the set of degree-1 vertices in $G[A]$. Note that $Comp(G[A_0]) = Comp_2(G[A])$ and $|N(A_0)| = |N_2(A_0)| = |N_2(A)|$. By applying Lemma 7 on A_0 , we can get

Corollary 1 *Let A be a vertex subset of a graph G such that each connected component of the induced graph $G[A]$ has at most 2 vertices. Let $N_2(A) \subseteq N(A)$ be the set of vertices in $N(A)$ adjacent to at least one vertex in a component of size 2 in $G[A]$. If*

$$Comp_2(G[A]) > |N_2(A)|, \quad (5)$$

then there is a good decomposition (I, C, R) of G such that $\emptyset \neq I \subseteq A$ and $C \subseteq N(A)$. Furthermore, the good decomposition (I, C, R) together with a P_3 -packing of size $|C|$ in $G[I \cup C]$ can be computed in $O(\sqrt{nm})$ time.

Note that $|A| = Comp_1(G[A]) + 2 \cdot Comp_2(G[A])$. If $|A| > 9k$, then $Comp_1(G[A]) + 2 \cdot Comp_2(G[A]) = |A| > 9k > 3|V(S)| \geq 3|N(A)| \geq (2|N(A)| - |N'_2(A)|) +$

$|N_2(A)|$ and at least one of (4) and (5) holds. Then by using Lemma 7 and Corollary 1, we can get a kernel of size $9k + 3k = 12k$. It is possible to bound $|N(A)|$ by k and then to get a kernel of size $3k + 3k = 6k$. To further improve the kernel size to $5k$, we need some sophisticated techniques and deep analyses on the graph structure.

4.2 A $5k$ kernel

In this section, we use “crucial partitions” to find good partitions. A vertex partition (A, B, Z) of a graph is called a *crucial partition* if it satisfies *Basic Conditions* and *Extended Conditions*. Basic Conditions include the following four items:

- (B1) A induces a graph of degree at most 1;
- (B2) B is the vertex set of a P_3 -packing \mathcal{P} ;
- (B3) No vertex in A is adjacent to a vertex in Z ;
- (B4) $|Z| \leq 5 \cdot \gamma(G[Z])$, where $\gamma(G[Z])$ is the size of a minimum P_3VC -set in the induced subgraph $G[Z]$.

Before presenting the definition of Extended Conditions, we give some used definitions. We use \mathcal{P}_j to denote the collection of 3-paths in \mathcal{P} having j vertices adjacent to A -vertices ($j = 0, 1, 2, 3$). Then $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$. We use \mathcal{P}^1 to denote the collection of 3-paths $L \in \mathcal{P}$ such that $|A(L)| = 1$. We also partition $\mathcal{P}_1 \setminus \mathcal{P}^1$ into two parts:

let $\mathcal{P}_M \subseteq \mathcal{P}_1 \setminus \mathcal{P}^1$ be the collection of 3-paths with the middle vertex adjacent to some A -vertices;

let $\mathcal{P}_L \subseteq \mathcal{P}_1 \setminus \mathcal{P}^1$ be the collection of 3-paths L_i such that $|A(L_i)| \geq 2$ and one ending vertex of L_i is adjacent to some A -vertices.

A vertex in a 3-path in \mathcal{P} is *free* if it is not adjacent to any A -vertex. A 3-path in \mathcal{P}_0 is *bad* if it has at least two vertices adjacent to some free-vertex in a 3-path in \mathcal{P}_L and *good* otherwise. A 3-path in \mathcal{P}_L is *bad* if it is adjacent to a bad 3-path in \mathcal{P}_0 and *good* otherwise.

Extended Conditions include the following seven items:

- (E1) For each 3-path $L_i \in \mathcal{P} \setminus \mathcal{P}^1$, at most one vertex in L_i is adjacent to some vertex in A , i.e., $\mathcal{P} \setminus \mathcal{P}^1 = \mathcal{P}_0 \cup \mathcal{P}_1$;
- (E2) No 3-path in \mathcal{P}_M is adjacent to both of A_0 -vertices and A_1 -vertices;
- (E3) No free-vertex in a 3-path in \mathcal{P}_L is adjacent to a free-vertex in another 3-path in \mathcal{P}_L ;
- (E4) No free-vertex in a 3-path in \mathcal{P}_L is adjacent to a free-vertex in a 3-path in \mathcal{P}_M ;
- (E5) Each 3-path in \mathcal{P}^1 has at most one vertex adjacent to a free-vertex in a 3-path in \mathcal{P}_L ;
- (E6) If a 3-path in \mathcal{P}^0 has at least two vertices adjacent to some free-vertex in a 3-path in \mathcal{P}_L , then all those free-vertices are from one 3-path in \mathcal{P}_L , i.e., each bad 3-path in \mathcal{P}^0 is adjacent to free-vertices in only one bad 3-path in \mathcal{P}_L ;

(E7) No free-vertex in a 3-path in \mathcal{P}_L is adjacent to a vertex in Z .

Lemma 8. *A crucial partition of the vertex set of any given graph can be found in polynomial time.*

After obtaining a crucial partition (A, B, Z) , we use the following three reduction rules to reduce the graph. In fact, Extended Conditions are mainly used for the third reduction rule and the analysis of the kernel size.

Reduction Rule 1 *If the number of 3-paths in \mathcal{P} is greater than $k - |Z|/5$, halt and report it as a no-instance.*

Note that each P_3VC -set of the graph G must contain at least $|Z|/5$ vertices in Z by Basic Condition (B4) and each P_3VC -set must contain one vertex from each 3-path in \mathcal{P} . If the number of 3-paths in \mathcal{P} is greater than $k - |Z|/5$, then any P_3VC -set of the graph has a size greater than k .

Reduction Rule 2 *If $Comp_2(G[A]) > |N_2(A)|$ (the condition in Corollary 1) holds, then find a good decomposition by Corollary 1 and reduce the instance based on the good decomposition.*

Reduction Rule 2 is easy to observe. Next, we consider the last reduction rule. Let B^* be the set of free-vertices in good 3-paths in \mathcal{P}_L and let A^* be the set of A_0 -vertices adjacent to 3-paths in \mathcal{P}^1 . Let $A' = A \cup B^* \setminus A^*$. By the definition of crucial decompositions, we can get that

Lemma 9. *The set A' still induces of a graph of maximum degree 1.*

Proof. Vertices in B^* are free-vertices and then any vertex in B^* is not adjacent to a vertex in A . Furthermore, no two free-vertices in B^* from two different 3-paths in \mathcal{P}_L are adjacent by Extended Condition (E3). Since A induces a graph of maximum degree 1, we know that $A \cup B^*$ induces a graph of maximum degree 1. The set $A' = A \cup B^* \setminus A^*$ is a subset of $A \cup B^*$ and then A' induces of a graph of maximum degree 1. \square

Based on Lemma 9, we can apply the following reduction rule.

Reduction Rule 3 *If $Comp(G[A']) > 2|N(A')| - |N'_2(A')|$ (the condition in Lemma 6 on set A') holds, then find a good decomposition by Lemma 6 and reduce the instance based on the good decomposition.*

Next, we assume that none of the three reduction rules can be applied and prove that the graph has at most $5k$ vertices.

We consider a crucial partition (A, B, Z) of the graph. Let k_1 be the number of 3-paths in \mathcal{P} . Since Reduction Rule 1 cannot be applied, we know that

$$k_1 \leq k - |Z|/5. \quad (6)$$

Since Reduction Rule 2 and Reduction Rule 3 cannot be applied, we also have the following two relations

$$\text{Comp}_2(G[A]) \leq |N_2(A)|, \quad (7)$$

and

$$\text{Comp}(G[A']) \leq 2|N(A')| - |N'_2(A')|. \quad (8)$$

By Extended Condition (E1), we know that $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \mathcal{P}^1 = \mathcal{P}_0 \cup \mathcal{P}_L \cup \mathcal{P}_M \cup \mathcal{P}^1$. Let x_1 and x_2 be the numbers of good and bad 3-paths in \mathcal{P}_L , respectively. Let y_i ($i = 0, 1$) be the number of 3-paths in \mathcal{P}_0 with i vertices adjacent to some free-vertex in a 3-path in \mathcal{P}_L , and y_2 be the number of 3-paths in \mathcal{P}_0 with at least two vertices adjacent to some free-vertex in a 3-path in \mathcal{P}_L , i.e., the number of bad 3-paths in \mathcal{P}_0 . Let z_1 and z_2 be the numbers of 3-paths in \mathcal{P}_M adjacent to only A_0 -vertices and only A_1 -vertices, respectively. Let w_1 be the number of 3-paths in \mathcal{P}^1 adjacent to some free-vertex in a 3-path in \mathcal{P}_L and w_2 be the number of 3-paths in \mathcal{P}^1 not adjacent to any free-vertex in a 3-path in \mathcal{P}_L . We get that

$$k_1 = x_1 + x_2 + y_0 + y_1 + y_2 + z_1 + z_2 + w_1 + w_2. \quad (9)$$

By Extended Conditions (E1) and (E2), we know that

$$|N(A)_2| \leq x_1 + x_2 + z_2. \quad (10)$$

Extended Condition (E6) implies the number of bad 3-paths in \mathcal{P}_L is at most the number of bad 3-paths in \mathcal{P}_0 , i.e.,

$$x_2 \leq y_2. \quad (11)$$

Each 3-path in \mathcal{P}^1 is adjacent to only one A_0 -vertex. Since A^* is the set of A_0 -vertices adjacent to 3-paths in \mathcal{P}^1 , we know that $|A^*|$ is not greater than $w_1 + w_2$, i.e., the number of 3-paths in \mathcal{P}^1 . By the definition of A' , we know that

$$\text{Comp}(G[A']) \geq \text{Comp}(G[A]) + x_1 - (w_1 + w_2). \quad (12)$$

Next, we consider $|N(A')|$ and $|N'_2(A')|$. Note that each 3-path has at most one vertex adjacent to vertices in $A \setminus A^*$ by Extended Condition (E1). This property will also hold for the vertex set $A' = (A \setminus A^*) \cup B^*$. We prove the following two relations

$$|N(A')| \leq x_1 + x_2 + y_0 + y_1 + z_1 + z_2 + w_1, \quad (13)$$

and

$$|N'_2(A')| \geq y_1 + z_2 + w_1. \quad (14)$$

By Extended Conditions (E1) and (E3), we know that each 3-path in \mathcal{P}_L has at most one vertex in $N((A \setminus A^*) \cup B^*) = N(A')$. By the definition of good 3-paths

in \mathcal{P}_0 , we know that each good 3-path in \mathcal{P}_0 has no vertex adjacent to vertices in A and has at most one vertex adjacent to vertices in B^* (which will be in a component of size 2 in $G[A']$). There are exactly y_1 vertices in good 3-paths in \mathcal{P}_0 adjacent to vertices in B^* . No vertex in a bad 3-path in \mathcal{P}_0 is adjacent to a vertex in $A \cup B^*$ by the definitions of bad 3-paths and B^* . Each 3-path in \mathcal{P}_M has at most one vertex adjacent to A' by Extended Conditions (E1) and (E4). Only z_2 vertices in 3-paths in \mathcal{P}_M are adjacent to vertices in A' , all of which are vertices of degree-1 in $G[A']$. No vertex in a 3-path in \mathcal{P}^1 is adjacent to a vertex in $A \setminus A^* \supseteq A'$ by the definition of A^* . Furthermore, each 3-path in \mathcal{P}^1 has at most one vertex adjacent to vertices in B^* (which will be in a component of size 2 in $G[A']$) by Extended Condition (E5) and there are exactly w_1 vertices in 3-paths in \mathcal{P}^1 adjacent to vertices in B^* . No vertex in Z is adjacent to a vertex in $A \cup B^*$ by Basic Condition (B3) and Extended Condition (E7). Summing all above up, we can get (13) and (14).

Relations (8), (12), (13) and (14) imply

$$\text{Comp}(G[A]) \leq 2(x_2 + y_0 + z_1 + w_1) + x_1 + y_1 + z_2 + w_2. \quad (15)$$

According to (7) and (10), we know that

$$\text{Comp}_2(G[A]) \leq x_1 + x_2 + z_2. \quad (16)$$

Note that $|A| = \text{Comp}(G[A]) + \text{Comp}_2(G[A])$, we get

$$\begin{aligned} |A| &= \text{Comp}(G[A]) + \text{Comp}_2(G[A]) \\ &\leq 2(x_1 + x_2 + y_0 + z_1 + z_2 + w_1) + x_2 + y_1 + w_2 && \text{by (15) and (16)} \\ &\leq 2(x_1 + x_2 + y_0 + z_1 + z_2 + w_1) + y_2 + y_1 + w_2 && \text{by (11)} \\ &\leq 2k_1 && \text{by (9)}. \end{aligned}$$

Note that $|B| = 3k_1$ and $k_1 \leq k - |Z|/5$ by (6). We get that

$$\begin{aligned} |V| &= |A| + |B| + |Z| \\ &\leq 5k_1 + |Z| \leq 5k. \end{aligned}$$

Theorem 2. *The parameterized 3-path vertex cover problem allows a kernel of at most $5k$ vertices.*

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