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Chapter 13
Rule Base Reduction Using Conflicting and Reinforcement Measures

Luca Anzilli and Silvio Giove

Abstract In this paper we present an innovative procedure to reduce the number of rules in a Mamdani rule-based fuzzy systems. First of all, we extend the similarity measure or degree between antecedent and consequent of two rules. Subsequently, we use the similarity degree to compute two new measures of conflicting and reinforcement between fuzzy rules. We apply these conflicting and reinforcement measures to suitably reduce the number of rules. Namely, we merge two rules together if they are redundant, i.e. if both antecedent and consequence are similar together, repeating this operation until no similar rules exist, obtaining a reduced set of rules. Again, we remove from the reduced set the rule with conflict with other, i.e. if antecedent are similar and consequence not; among the two, we remove the one characterized by higher average conflict with all the rules in the reduced set.

Keywords Fuzzy systems · Rule base reduction · Rule base simplification · Conflicting and reinforcement measures

13.1 Introduction

The number of rules in a fuzzy system (FIS, Fuzzy Inference System) exponentially increases with the number of the input variables and the number of the linguistic values that these inputs can take (antecedent fuzzy terms) [1, 2]. Several approaches for reducing fuzzy rule base have been proposed using different techniques such as interpolation methods, orthogonal transformation methods, clustering techniques [3–8]. A typical tool to perform model simplification is merging similar fuzzy sets and rules using similarity measures [9–14].

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In this paper we propose a new procedure for simplifying rule-based fuzzy systems. Starting from similarity measures we introduce two new measures of conflicting and reinforcement between fuzzy sets. Then we develop a simplification methodology using the introduced conflicting and reinforcement measures to merge similar rules and to remove redundant rules from the rule set.

The paper is organized as follows. In Sect. 13.2 we briefly review the basic notions of fuzzy systems. In Sect. 13.3 we define conflicting and reinforcement measures. In Sect. 13.4 we present the merging methodology and, finally, in Sect. 13.5 we illustrate our rule-base reduction method.

13.2 Fuzzy Systems

The *knowledge* of a FIS can be obtained from available data using some optimization tool as a neural approach, or by direct elicitation from one or a group of Experts. In the latter case, the Experts represent their knowledge by defining a set of inferential rules. The input variables are processed by these rules to generate an appropriate output.

In the case of a FIS with n input variables, $x_1, \ldots, x_n$ and a single output y (miso fuzzy system, [2]) every rule has the form

$$R_i: \quad \text{IF } x_1 \text{ is } A_{i,1} \text{ and}, \ldots, \text{ and } x_n \text{ is } A_{i,n} \text{ THEN } y \text{ is } B_i \qquad i = 1, \ldots, N$$

where $A_{i,j}$ is a fuzzy sets of universe space X_j and B_i is a fuzzy set of universe space Y, and N is the number of rules. The fuzzy set $A_{i,j}$ is the *linguistic label* associated with j-th antecedent in the i-th rule and B_i is the linguistic label associated with the consequent in the i-th rule. We recall that a linguistic label can be easily represented by a fuzzy set [15]. The rule i, R_i, can be represented by the ordered couple $R_i = \left(\bigcap_{j=1}^{n} A_{i,j}(x_j), B_i\right)$, being $A_{i,j}(x_j)$ the j-th component of the antecedent and B_i the consequent, $i = 1, \ldots, N$, and $\bigcap$ is the conjunction operator. Every Rule R_i in the data base is characterized by a confidence degree e_i (or rule weight), with $e_i > 0$ (see [8, 16])). Rule weights can be applied to complete rules or only to the consequent part of the rules [16]. In the first case, the weight is used to modulate the activation degree of a rule, and in second to modulate the rule conclusion.

13.3 Conflicting and Supporting Rules

13.3.1 Similarity Measures Between Fuzzy Sets

We denote by $Sim(A, B)$ the similarity between fuzzy sets A and B with respect to a similarity measure Sim. Different similarity measures for fuzzy sets have been proposed in literature [17–23]. They can be classified into two main groups:

geometric and *set-theoretic* similarity measures. Axiomatic definitions of similarity can be found in [18, 23].

An example of a geometric similarity measure based on distance between two fuzzy sets is

$$Sim_1(A, B) = \frac{1}{1 + D(A, B)}$$

being $D(A, B)$ a suitable distance among the two fuzzy sets A and B.

An example of a similarity measure between two fuzzy sets, based on the set-theoretic operations of intersection and union, is (see [15])

$$Sim_2(A, B) = \frac{M(A \cap B)}{M(A \cup B)} \tag{13.1}$$

where

$$M(A) = \int_{-\infty}^{+\infty} A(x)\, dx\,. \tag{13.2}$$

is the size of fuzzy set A. Details for computing (13.1) are given in [4, 14].

13.3.2 Similarity Measures Between Rules

Let us consider a fuzzy system with n input variables (= number of antecedents of each rule) and N rules

$$R_i = \left(\bigcap_{j=1}^{n} A_{i,j}(x_j), B_i \right), \qquad i = 1, \ldots, N$$

being $A_{i,j}(x_j)$ the j-th component of the antecedent and B_i the consequent. Each Rule R_i in the data base will be characterized also by a confidence degree e_i.

Definition 13.1 Following measures will be considered (see [11, 14]):

(1) Similarity between the antecedent of two rules, R_k, R_ℓ $(k, \ell = 1, 2 \ldots, N)$

$$\mu_{k,l} = Sim(Ant_k, Ant_\ell) = \mathrm{T}_{j=1}^{n}\, Sim(A_{k,j}, A_{\ell,j})$$

where T is a t-norm (in [11, 14] $T = \min$);

(2) Similarity between the consequent of two rules, R_k, R_ℓ

$$\nu_{k,l} = Sim(Cons_k, Cons_\ell) = Sim(B_k, B_\ell)\,.$$

13.3.3 Conflicting and Reinforcement Degrees

Definition 13.2 Based on the two above measures, $\mu_{k,\ell}$ and $\nu_{k,\ell}$, we can propose the following conflicting and reinforcing degrees:

(i) *Conflicting degree*, a measure of the conflict among a couple of rules:

$$c(k,\ell) = \mu_{k,\ell}\,(1-\nu_{k,\ell})f(e_k, e_\ell)\,; \tag{13.3}$$

(ii) *Reinforcement degree*, a measure of the agreement among a couple of rules:

$$r(k,\ell) = \mu_{k,\ell}\,\nu_{k,\ell}\,f(e_k, e_\ell) \tag{13.4}$$

being $f(e_k, e_\ell)$ a suitable aggregation function, symmetric and idempotent, not decreasing in both its two arguments.

Proposition 13.1 *Conflicting and reinforcement degrees satisfy the following properties: for any $k, \ell = 1, 2 \ldots, N$*

(i) $0 \le c(k,\ell) \le 1$, $0 \le r(k,\ell) \le 1$
(ii) $c(k,\ell) = c(\ell,k)$, $r(k,\ell) = r(\ell,k)$
(iii) $c(k,k) = 0$, $r(k,k) = f(e_k, e_k) = e_k$
(iv) $c(k,\ell) = 1 \implies r(k,\ell) = 0$
(v) $r(k,\ell) = 1 \implies c(k,\ell) = 0$
(vi) $0 \le c(k,\ell) + r(k,\ell) \le \min\{\mu_{k,\ell}, f(e_k, e_\ell)\} \le 1$
(vii) *if the aggregation function f is such that $f(e_k, e_\ell) = 0$ only if $c_k = 0$ or $c_\ell = 0$, that is the only annihilator element (see [24]) of f is* 0*, then:*

$$\mu_{k,\ell} > 0 \implies c(k,\ell) + r(k,\ell) > 0\,.$$

Proof Property (ii) holds since similarity measure is symmetric. Property (iii) follows taking into account that $\mu_{k,k} = \nu_{k,k} = 1$ since $Sim(A, A) = 1$ (assuming that *Sim* is a normal similarity measure). Properties (vi) and (vii) follow from the relation

$$c(k,\ell) + r(k,\ell) = \mu_{k,\ell} \cdot f(e_k, e_\ell)$$

and observing that, since $e_\ell > 0$ for any ℓ, we have $f(e_k, e_\ell) > 0$. □

We observe that both $c(k,\ell)$ and $r(k,\ell)$ can be equal to zero, but if one is close to one the other is close to zero.

13.4 Merging Methodology

13.4.1 Merging Fuzzy Sets

Different shape of membership functions exist in the specialized literature. Among them we recall trapezoidal, triangular, bell-shaped fuzzy number. A trapezoidal fuzzy number is defined[1] by the 4-ple $A = (a_1, a_2, a_3, a_4)$, with $a_1 < a_2 \leq a_3 < a_4$, and has membership function

$$A(x) = \begin{cases} 0 & x \leq a_1 \text{ or } x \geq a_4 \\ \dfrac{x - a_1}{a_2 - a_1} & a_1 \leq x \leq a_2 \\ 1 & a_2 \leq x \leq a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & a_3 \leq x \leq a_4 \end{cases}$$

More in general, if the (continuous) fuzzy number A is characterized by the membership $A(x)$, its α-cuts are by the intervals $A(\alpha) = \{x|A(x) \geq \alpha\} = [a_1(\alpha), a_2(\alpha)]$, with $\alpha \in [0, 1]$. The size $M(A)$ of fuzzy set A, as defined in (13.2), can be computed using α-cuts by

$$M(A) = \int_0^1 M(A(\alpha))\, d\alpha$$

where $M(A(\alpha))$ is the size of α-cut $A(\alpha)$. We extend this concept by defining

$$M_l(A) = \int_0^1 M(A(\alpha))\, l(\alpha)\, d\alpha$$

being $l(\alpha)$ a suitable *weighting* function, $l(\alpha) : [0, 1] \to [0, 1]$ (see [25, 26]). Then we introduce the following extended version of the similarity (13.1)

$$Sim_2^l(A, B) = \frac{M_l(A \cap B)}{M_l(A \cup B)}\,.$$

In order to compute the similarity, we observe that a discrete representation of A can be done through a finite subset of its α-cuts, see [15]. Particularly useful is an equally spaced grid for α, as $\alpha_i = \frac{i}{T}$, $i = 0, 1, .., T$, with step size $\frac{1}{T}$. For a discretized fuzzy number A we have

$$M_l(A) = \sum_{i=1}^{T} \int_{\alpha_{i-1}}^{\alpha_i} M(A(\alpha))\, l(\alpha)\, d\alpha \approx \frac{1}{T} \sum_{i=1}^{T} M(A(\alpha_i))\, l(\alpha_i)\,.$$

[1]A triangular fuzzy number is a sub-case of a trapezoidal one, with $a_2 = a_3$, while a *bell-shape* recalls a gaussian distribution.

Then, taking into account that $(A \cap B)(\alpha) = A(\alpha) \cap B(\alpha)$ and $(A \cup B)(\alpha) = A(\alpha) \cup B(\alpha)$, we get the following formulas

$$M_l(A \cap B) = \frac{1}{T}\sum_{i=1}^{T} M((A(\alpha_i) \cap B(\alpha_i))\, l(\alpha_i)$$
$$M_l(A \cup B) = \frac{1}{T}\sum_{i=1}^{T} M((A(\alpha_i) \cup B(\alpha_i))\, l(\alpha_i)$$

where $M((A(\alpha_i) \cap B(\alpha_i)) = \max\{min(a_2(\alpha_i), b_2(\alpha_i)) - max(a_1(\alpha_i), b_1(\alpha_i)), 0\}$ and $M((A(\alpha_i) \cup B(\alpha_i)) = M(A(\alpha_i)) + M(B(\alpha_i)) - M(A(\alpha_i) \cap B(\alpha_i))$. As a consequence, the similarity degree among two discretized fuzzy numbers A and B is given by

$$Sim_2^l(A, B) = \frac{\sum_{i=1}^{T} M((A(\alpha_i) \cap B(\alpha_i))\, l(\alpha_i)}{\sum_{i=1}^{T} M((A(\alpha_i) \cup B(\alpha_i))\, l(\alpha_i)}\,.$$

Two fuzzy sets A and B can be merged into a fuzzy set $C = \lambda A + (1 - \lambda) B$, where $0 < \lambda < 1$ is a suitably selected parameter. If $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ are trapezoidal fuzzy numbers, the merged (trapezoidal) fuzzy number $C = (c_1, c_2, c_3, c_4)$ is given by[2]

$$\begin{aligned} c_1 &= \lambda\, a_1 + (1 - \lambda)\, b_1 \\ c_2 &= \lambda\, a_2 + (1 - \lambda) b_2 \\ c_3 &= \lambda\, a_3 + (1 - \lambda) b_3 \\ c_4 &= \lambda\, a_4 + (1 - \lambda)\, b_4\,. \end{aligned}$$

13.4.2 Merging Rules

Let us fix a pre-specified antecedent-similarity threshold $\bar{\mu} > 0$. If $\mu_{k,\ell} > \bar{\mu}$ then we can merge Rules R_k, R_ℓ into a single rule $R_{(k,\ell)}$ with confidence degree $e_{k,l}$ given by

$$e_{k,l} = h(c(k,\ell), r(k,\ell)) \cdot f(e_k, e_\ell)\,. \tag{13.5}$$

We require that function $h : [0, 1]^2 \to [0, 1]$ satisfy the following properties:

(i) $h(c, r)$ is decreasing with respect to c
(ii) $h(c, r)$ is increasing with respect to r

[2] Alternatively, in [9] the following merging procedure is proposed: if $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ are trapezoidal fuzzy numbers, the merged (trapezoidal) fuzzy number $C = (c_1, c_2, c_3, c_4)$ is obtained by $c_1 = \min(a_1, b_1)$, $c_2 = \lambda_2 a_2 + (1 - \lambda_2) b_2$, $c_3 = \lambda_3 a_3 + (1 - \lambda_3) b_3$, $c_4 = \max(a_4, b_4)$, where $\lambda_2, \lambda_3 \in [0, 1]$.

(iii) $h(1, r) = 0$
(iv) $h(0, r) = r$.

Examples of functions $h(c, r)$ are:

(a) $h_1(c, r) = r(1 - c)$
(b) $h_2(c, r) = T(n(c), r)$, where T is a t-norm and n is a fuzzy complement (that is a decreasing function $n : [0, 1] \to [0, 1]$ such that $n(0) = 1$ and $n(1) = 0$).
We observe that h_1 is a special case of h_2 with the product t-norm $T = T_P$ and the standard fuzzy complement $n(c) = 1 - c$.
(c) $h_3(c, r) = \frac{r}{c+r}$ (we may set $h_2(c, r) = 1$ if $c + r = 0$; note that for $\mu_{k,\ell} > \bar{\mu} > 0$ we have $c(k, \ell) + r(k, \ell) > 0$). We observe that $h_3(c, r) = \nu_{k,\ell}$.

13.5 Rule Base Reduction Method

We now present a reduction algorithm to perform a rule base simplification of a Mamdani fuzzy system [15]. The main idea is the following: if two rules have similar antecedents and similar consequents we merge them into a single rule; if two rules have similar antecedents but dissimilar consequents we remove the rule which have the greater conflict. Our method consists two steps.

First step. We merge two Rules characterized by a value of similarity $\mu_{k,\ell}$ greater than a pre-specified threshold $\bar{\mu}$ and a value $\nu_{k,\ell}$ greater than a pre-specified threshold $\bar{\nu}$ in a single rule. The antecedent (consequent) of the merged Rule will be obtained merging together the antecedents (consequents) of the two Rules using a suitable averaging operator. The confidence degree of the merged Rule will be increased in the case of reinforcement, but decreased in the case of conflicting. The merging algorithm thus will proceed considering every couple of Rules, selecting the most antecedent-similar couple and merge the two Rules into a single one, consequently modifying the aggregated confidence. The merging procedure will continue until two Rules with antecedent-similarity and consequent-similarity greater that a specified threshold exist in the data base.

Second step. We consider the reduced rule set $\mathscr{R}$. If two Rules R_k, R_ℓ have a value of similarity $\mu_{k,\ell}$ greater than a pre-specified threshold $\bar{\mu}$ then we compute the total conflicting degrees

$$c_k = \sum_{R_m \in \mathscr{R}} c(k, m), \qquad c_\ell = \sum_{R_m \in \mathscr{R}} c(k, m)$$

and remove the Rule having the greater conflict degree. The removing procedure will continue until two Rules with antecedent-similarity greater than $\bar{\mu}$ and different total conflicting degree exist in the data base.

The previous methodology can be formalized in the following algorithm:

1. calculate $\mu_{k,\ell}$ and $\nu_{k,\ell}$ for $k, \ell = 1, \dots, N$;
2. calculate the values of $c(k, \ell)$ and $r(k, \ell)$ for $k, \ell = 1, \dots, N$;
3. for each $k, \ell = 1, \dots, N, k \neq \ell$: if $\mu_{k,\ell} > \bar{\mu}$ and $\nu_{k,\ell} > \bar{\nu}$ then we merge R_k and R_ℓ and assign to merged Rule $R_{(k,\ell)}$ a confidence degree $e_{(k,\ell)}$ computed according to (13.5);
4. let $\mathscr{R} = \{R_1, \dots, R_p\}$, $p \leq N$, be the new (reduced) rule set;
5. for each $k, \ell = 1, \dots, p, k \neq \ell$: if $\mu_{k,\ell} > \bar{\mu}$ (and thus $\nu_{k,\ell} \leq \bar{\nu}$) then
    - if $c_k > c_\ell$ we remove R_k,
    - if $c_k < c_\ell$ we remove R_ℓ.

13.6 Conclusion

In this paper we proposed a novel methodology for Rule base reduction of Mamdani fuzzy systems based on conflicting and reinforcement measures. The reduction is achieved by merging antecedents and consequents of two Rules and assigning to the merged Rule an increased (decreased) confidence degree in the case of reinforcement (conflicting).

As a future development, we intend to investigate the properties of conflicting and reinforcement measures and, moreover, to apply the proposed simplification procedure to Takagi-Sugeno fuzzy systems.

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