# The (minimum) rank of typical fooling-set matrices 

Mozhgan Pourmoradnasseri, Dirk Oliver Theis*<br>Institute of Computer Science of the University of Tartu<br>Ülikooli 17, 51014 Tartu, Estonia<br>\{dotheis, mozhgan\}@ut.ee

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#### Abstract

A fooling-set matrix has nonzero diagonal, but at least one in every pair of diagonally opposite entries is 0 . Dietzfelbinger et al. ' 96 proved that the rank of such a matrix is at least $\sqrt{n}$. It is known that the bound is tight (up to a multiplicative constant).

We ask for the typical minimum rank of a fooling-set matrix: For a fooling-set zerononzero pattern chosen at random, is the minimum rank of a matrix with that zero-nonzero pattern over a field F closer to its lower bound $\sqrt{n}$ or to its upper bound $n$ ? We study random patterns with a given density $p$, and prove an $\Omega(n)$ bound for the cases when (a) $p$ tends to 0 quickly enough; (b) $p$ tends to 0 slowly, and $|\mathrm{F}|=O(1)$; (c) $p \in] 0,1]$ is a constant.

We have to leave open the case when $p \rightarrow 0$ slowly and F is a large or infinite field (e.g., $\left.\mathrm{F}=\mathrm{GF}\left(2^{n}\right), \mathrm{F}=\mathrm{R}\right)$.


## 1 Introduction

Let $f: X \times Y \rightarrow\{0,1\}$ be a function. A fooling set of size $n$ is a family $\left(x_{1}, y_{1}\right), \ldots,\left(x_{n}, y_{n}\right) \in$ $X \times Y$ such that $f\left(x_{i}, y_{i}\right)=1$ for all $i$, and for $i \neq j$, at least one of $f\left(x_{i}, y_{j}\right)$ of $f\left(y_{i}, y_{j}\right)$ is 0 . Sizes of fooling sets are important lower bounds in Communication Complexity (see, e.g., [13, 12]) and the study of extended formulations (e.g., [4, 1]).

There is an a priori upper bound on the size of fooling sets due to Dietzfelbinger et al. [3], based on the rank of a matrix associated with $f$. Let $\mathbb{F}$ be an arbitrary field. The following is a slight generalization of the result in [3] (see the appendix for a proof).

Lemma 1. No fooling set in $f$ is larger than the square of $\min _{A} \operatorname{rk}_{\mathbb{F}}(A)$, where the minimum ranges ${ }^{1}$ over all $X \times Y$-matrices $A$ over $\mathbb{F}$ with $A_{x, y}=0$ iff $f(x, y)=0$.

It is known that, for fields $\mathbb{F}$ with nonzero characteristic, this upper bound is asymptotically attained [6], and for all fields, it is attained up to a multiplicative constant [5]. These results, however, require sophisticated constructions. In this paper, we ask how useful that upper bound is for typical functions $f$.

Put differently, a fooling-set pattern of size $n$ is a matrix $R$ with entries in $\{0,1\} \subseteq \mathbb{F}$ with $R_{k, k}=1$ for all $k$ and $R_{k, \ell} R_{\ell, k}=0$ whenever $k \neq \ell$. We say that a fooling-set pattern of size $n$

[^0]has density $p \in] 0,1]$, if it has exactly $\left\lceil p\binom{n}{2}\right\rceil$ off-diagonal 1-entries. So, the density is roughly the quotient $(|R|-n) /\binom{n}{2}$, where $|\cdot|$ denotes the Hamming weight, i.e., the number of nonzero entries. The densest possible fooling-set pattern has $\binom{n}{2}$ off-diagonal ones (density $p=1$ ).

For any field $\mathbb{F}$ and $y \in \mathbb{F}$, let $\sigma(y):=0$, if $y=0$, and $\sigma(y):=1$, otherwise. For a matrix (or vector, in case $n=1$ ) $M \in \mathbb{F}^{m \times n}$, define the zero-nonzero pattern of $M, \sigma(M)$, as the matrix in $\{0,1\}^{m \times n}$ which results from applying $\sigma$ to every entry of $M$.

This paper deals with the following question: For a fooling-set pattern chosen at random, is the minimum rank of closer to its lower bound $\sqrt{n}$ or to its trivial upper bound $n$ ? The question turns out to be surprisingly difficult. We give partial results, but we must leave some cases open. The distributions we study are the following:
$Q(n)$ denotes a fooling-set pattern drawn uniformly at random from all fooling-set patterns of size $n$;
$R(n, p)$ denotes a fooling-set patterns drawn uniformly at random from all fooling-set patterns of size $n$ with density $p$.

We allow that the density depends on the size of the matrix: $p=p(n)$. From now on, $Q=Q(n)$ and $R=R(n, p)$ will denote these random fooling-set patterns.

Our first result is the following. As customary, we use the terminology "asymptotically almost surely, a.a.s.," to stand for "with probability tending to 1 as $n$ tends to infinity".

Theorem 2. (a) For every field $\mathfrak{F}$, if $p=O(1 / n)$, then, a.a.s., the minimum rank of a matrix with zero-nonzero pattern $R(n, p)$ is $\Omega(1)$.
(b) Let $\mathbb{F}$ be a finite field and $F:=|\mathbb{F}|$. (We allow $F$ to grow with n.) If $100 \max (1, \ln \ln F) / n \leq$ $p \leq 1$, then the minimum rank of a matrix over $\mathbb{F}$ with zero-nonzero pattern $R(n, p)$ is

$$
\Omega\left(\frac{\log (1 / p)}{\log (1 / p)+\log (F)} n\right)=\Omega(n / \log (F))
$$

(c) For every field $\mathbb{F}$, if $p \in] 0,1]$ is a constant, then the minimum rank of a matrix with zero-nonzero pattern $R(n, p)$ is $\Omega(1)$. (The same is true for zero-nonzero pattern $Q(n)$.)

Since the constant in the big- $\Omega$ in Thereom 2(c) tends to 0 with $p \rightarrow 0$, the proof technique used for constant $p$ does not work for $p=o(1)$; moreover, the bound in (b) does not give an $\Omega(n)$ lower bound for infinite fields, or for large finite fields, e.g., $\mathrm{GF}\left(2^{n}\right)$. We conjecture that the bound is still true (see Lemma 4 for a lower bound):

Conjecture 3. For every field $\mathbb{F}$ and for all $p=p(n)$, the minimum rank of a fooling-set matrix with random zero-nonzero pattern $R(n, p)$ is $\Omega(n)$.

The bound in Thereom 2(b) is similar to that in [8], but it is better by roughly a factor of $\log n$ if $p$ is (constant or) slowly decreasing, e.g., $p=1 / \log n$. (Their minrank definition gives a lower bound to fooling-set pattern minimum rank.)

The next three sections hold the proofs for Theorem 2.

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## 2 Proof of Theorem 2(a)

It is quite easy to see (using, e.g., Turán's theorem) that in the region $p=O(1 / n), R(n, p)$ contains a triangular submatrix with nonzero diagonal entries of order $\Omega(n)$, thus lower bounding the rank over any field. Here, we prove the following stronger result, which also gives a lower bound (for arbitrary fields) for more slowly decreasing $p$.

Lemma 4. For $p(n)=d(n) / n=o(1)$, if $d(n)>C$ for some constant $C$, then zero-nonzero pattern $R(n, p)$ contains a triangular submatrix with nonzero diagonal entries of size

$$
\Omega\left(\frac{\ln d}{d} \cdot n\right) .
$$

We prove the lemma by using the following theorem about the independence number of random graphs in the Erdös-Rényi model. Let $G_{n, q}$ denote the random graph with vertex set $[n]$ where each edge is chosen independently with probability $q$.

Theorem 5 (Theorem 7.4 in [11]). Let $\epsilon>0$ be a constant, $q=q(n)$, and define

$$
k_{ \pm \epsilon}:=\left\lfloor\frac{2}{q}(\ln (n q)-\ln \ln (n q)+1-\ln 2 \pm \epsilon)\right\rfloor .
$$

There exists a constant $C_{\epsilon}$ such that for $C_{\epsilon} / n \leq q=q(n) \leq \ln ^{-2} n$, a.a.s., the largest independent set in $G_{n, q}$ has size between $k_{-\epsilon}$ and $k_{+\epsilon}$.

Proof of Lemma 4. Construct a graph $G$ with vertex set $[n]$ from the fooling-set pattern matrix $R(n, p)$ in the following way: There is an edge between vertices $k$ and $\ell$ with $k>\ell$, if and only if $M_{k, \ell} \neq 0$. This gives a random graph $G=G_{n, m, 1 / 2}$ which is constructed by first drawing uniformly at random a graph from all graphs with vertex set $[n]$ and exactly $m$ edges, and then deleting each edge, independently, with probability $1 / 2$. Using standard results in random graph theory (e.g., Lemma 1.3 and Theorem 1.4 in [7]), this random graph behaves similarly to the Erdős-Rényi graph with $q:=p / 2$. In particular, since $G_{n, p / 2}$ has an independent set of size $\Omega(n)$, so does $G_{n, m, 1 / 2}$.

It is easy to see that the independent sets in $G$ are just the lower-triangular principal submatrices of $R_{n, p}$.

As already mentioned, Theorem 2(a) is completed by noting that for $p<C / n$, an easy application of Turán's theorem (or ad-hoc methods) gives us an independent set of size $\Omega(n)$.

## 3 Proof of Theorem 2(b)

Let $\mathbb{F}$ be a finite field with $F:=|\mathbb{F}|$. As mentioned in Theorem 2, we allow $F=F(n)$ to depend on $n$. In this section, we need to bound some quantities away from others, and we do that generously.

Let us say that a tee shape is a set $T=I \times[n] \cup[n] \times I$, for some $I \subset[n]$. A tee matrix is a tee shape $T$ together with a mapping $N: T \rightarrow \mathbb{F}$ which satisfies

$$
\begin{equation*}
N_{k, k}=1 \text { for all } k \in I \text {, and } \quad N_{k, \ell} N_{\ell, k}=0 \text { for all }(k, \ell) \in I \times[n], k \neq \ell . \tag{1}
\end{equation*}
$$

The order of the tee shape/matrix is $|I|$, and the rank of the tee matrix is the rank of the matrix $N_{I \times I}$.

For a matrix $M$ and a tee matrix $N$ with tee shape $T$, we say that $M$ contains the tee matrix $N$, if $M_{T}=N$.

Lemma 6. Let $M$ be a matrix with rank $s:=\operatorname{rk} M$, which contains a tee matrix $N$ of rank $s$. Then $M$ is the only matrix of rank $s$ which contains $N$.

In other words, the entries outside of the tee shape are uniquely determined by the entries inside the tee shape.

Proof. Let $T=I \times[n] \cup[n] \times I$ be the tee shape of a tee matrix $N$ contained in $M$.
Since $N_{I \times I}=M_{I \times I}$ and $\operatorname{rk} N_{I \times I}=s=\operatorname{rk} M$, there is a row set $I_{1} \subseteq I$ of size $s=\operatorname{rk} M$ and a column set $I_{2} \subseteq I$ of size $s$ such that $\mathrm{rk} M_{I_{1} \times I_{2}}=s$. This implies that $M$ is uniquely determined, among the matrices of rank $s$, by $M_{T^{\prime}}$ with $T^{\prime}:=I_{1} \times[n] \cup[n] \times I_{2} \subseteq T$. (Indeed,
since the rows of $M_{I_{1} \times[n]}$ are linearly independent and span the row space of $M$, every row in $M$ is a unique linear combination of the rows in $M_{I_{1} \times[n]}$; since the rows in $M_{I_{1} \times I_{2}}$ are linearly independent, this linear combination is uniquely determined by the rows of $M_{[n] \times I_{2}}$.)

Hence, $M$ is the only matrix $M^{\prime}$ with rk $M^{\prime}=s$ and $M_{T^{\prime}}^{\prime}=M_{T^{\prime}}$. Trivially, then, $M$ is the only matrix $M^{\prime}$ with $\operatorname{rk} M^{\prime}=s$ and $M_{T}^{\prime}=M_{T}=N$.

Lemma 7. For $r \leq n / 5$ and $m \leq 2 r(n-r) / 3$, there are at most

$$
O(1) \cdot\binom{n}{2 r} \cdot\binom{2 r(n-r)}{m} \cdot(2 F)^{m}
$$

matrices of rank at most $r$ over $\mathbb{F}$ which contain a tee matrix of order $2 r$ with at most $m$ nonzeros.

Proof. By the Lemma 6, the number of these matrices is upper bounded by the number of tee matrices (of all ranks) of order $2 r$ with at most $k$ nonzeros.

The tee shape is uniquely determined by the set $I \subseteq[n]$. Hence, the number of tee shapes of order $2 r$ is

$$
\begin{equation*}
\binom{n}{2 r} \tag{*}
\end{equation*}
$$

The number of ways to choose the support a tee matrix. Suppose that the tee matrix has $h$ nonzeros. Due to (1), $h$ nonzeros must be chosen from $\binom{2 r}{2}+2 r(n-2 r) \leq 2 r(n-r)$ opposite pairs. Since $h<2 r(n-r) / 2$, we upper bound this by

$$
\binom{2 r(n-r)}{h}
$$

For each of the $h$ opposite pairs, we have to pick one side, which gives a factor of $2^{h}$. Finally, picking, a number in $\mathbb{F}$ for each of the entries designated as nonzero gives a factor of $(F-1)^{h}$.

For summing over $h=0, \ldots, m$, first of all, remember that $\sum_{i=0}^{(1-\varepsilon) j / 2}\binom{j}{i}=O_{\varepsilon}(1) \cdot\binom{j}{(1-\varepsilon) j / 2}$ (e.g., Theorem 1.1 in [2], with $p=1 / 2, u:=1+\varepsilon$ ). Since $m \leq 2 r(n-r) / 3$, we conclude

$$
\sum_{h=0}^{m}\binom{2 r(n-r)}{h}=O(1) \cdot\binom{2 r(n-r)}{m}
$$

(with an absolute constant in the big-Oh). Hence, we find that the number of tee matrices (with fixed tee shape) is at most

$$
\sum_{h=0}^{m}\binom{2 r(n-r)}{h} 2^{h}(F-1)^{h} \leq(2 F)^{m} \sum_{h=0}^{m}\binom{2 r(n-r)}{h}=O(1) \cdot(2 F)^{m} \cdot\binom{2 r(n-r)}{m}
$$

Multiplying by ( $*$ ), the statement of the lemma follows.
Lemma 8. Let $r \leq n / 5$. Every matrix $M$ of rank at most $r$ contains a tee matrix of order $2 r$ and rank rk $M$.

Proof. There is a row set $I_{1}$ of size $s:=\operatorname{rk} M$ and a column set $I_{2}$ of size $s$ such that rk $M_{I_{1} \times I_{2}}=$ $s$. Take $I$ be an arbitrary set of size $2 r$ containing $I_{1} \cup I_{2}$, and $T:=I \times[n] \cup[n] \times I$. Clearly, $M$ contains the tee matrix $N:=M_{T}$, which is of order $2 r$ and rank $s=\operatorname{rk} M$.

Lemma 9. Let $100 \max (1, \ln \ln F) / n \leq p \leq 1$, and $n /(1000(\max (1, \ln F)) \leq r \leq n / 100$. A.a.s., every tee shape of order $2 r$ contained in the random matrix $R(n, p)$ has fewer than $15 p r(n-r)$ nonzeros.

Proof. We take the standard Chernoff-like bound for the hypergeometric distribution of the intersection of uniformly random $p\binom{n}{2}$-element subset (the diagonally opposite pairs of $R(n, p)$ which contain a 1-entry) of a $\binom{n}{2}$-element ground set (the total number of diagonally opposite pairs) with a fixed $2 r(n-r)$-element subset (the opposite pairs in $T$ ) of the ground set: ${ }^{2}$ With $\lambda:=p 2 r(n-r)$ (the expected size of the intersection), if $x \geq 7 \lambda$, the probability that the intersection has at least $x$ elements is at most $e^{-x}$.

Hence, the probability that the support of a fixed tee shape of order $2 r$ is greater than than $15 p r(n-r) \geq 14 p r(n-r)+r$ is at most

$$
e^{-14 p r(n-r)} \leq e^{-r \cdot 14 \cdot 99 \cdot \max (1, \ln \ln F)} \leq e^{-r \cdot 1000 \cdot \max (1, \ln \ln F))}
$$

Since the number of tee shapes is

$$
\binom{n}{r} \leq e^{r(1+\ln (n / r))} \leq e^{r(11+\ln \max (1, \ln F))}, \leq e^{r(11+\max (1, \ln \ln F))}
$$

we conclude that the probability that a dense tee shape exists in $R(n, p)$ is at most $e^{-\Omega(r)}$.
We are now ready for the main proof.
Proof of Theorem 2(b). Call a fooling-set matrix $M$ regular, if $M_{k, k}=1$ for all $k$. The minimum rank over a fooling-set pattern is always attained by a regular matrix (divide every row by the corresponding diagonal element).

Consider the event that there is a regular matrix $M$ over $\mathbb{F}$ with $\sigma(M)=R(n, p)$, and rk $M \leq r:=n /(2000 \ln F)$. By Lemma $8, M$ contains a tee matrix $N$ of order $2 r$ and rank rk $M$. If the size of the support of $N$ is larger than $15 \operatorname{pr}(n-r)$, then we are in the situation of Lemma 9.

Otherwise, $M$ is one of the

$$
O(1) \cdot\binom{n}{2 r} \cdot\binom{2 r(n-r)}{15 p r(n-r)} \cdot(2 F)^{15 p r(n-r)}
$$

matrices of Lemma 7.
Hence, the probability of said event is $o(1)$ (from Lemma 9) plus at most an $O(1)$ factor of the following (with $m:=p n^{2} / 2$ and $\varrho:=r / n$ ) a constant

$$
\left.\begin{array}{rl}
\frac{\binom{n}{2 r} \cdot\binom{2 r(n-r)}{15 p r(n-r)} \cdot(2 F)^{15 p r(n-r)}}{\binom{\binom{n}{2}}{p\binom{n}{2}} 2^{p\binom{n}{2}} 2^{-O(p n)}} & =\frac{\binom{n}{2 r} \cdot\binom{2 r(n-r)}{15 p r(n-r)} \cdot(2 F)^{15 p r(n-r)}}{\binom{n^{2} / 2}{p n^{2} / 2} 2^{p n^{2} / 2-O(p n)}} \\
=\frac{\binom{n}{2 \varrho n} \cdot\binom{4 \varrho(1-\varrho) n^{2} / 2}{30 p \varrho(1-\varrho) n^{2} / 2} \cdot(2 F)^{30 p \varrho(1-\varrho) n^{2} / 2}}{\binom{n^{2} / 2}{p n^{2} / 2} 2^{p n^{2} / 2-O(p n)}} \\
& =\frac{\binom{n}{2 \varrho n} \cdot\binom{4 \varrho(1-\varrho) n^{2} / 2}{30 \varrho(1-\varrho) p n^{2} / 2} \cdot(2 F)^{30 \varrho(1-\varrho) p n^{2} / 2}}{\binom{n^{2} / 2}{p n^{2} / 2} 2^{p n^{2} / 2-O(p n)}}
\end{array}=: Q\right)
$$

Abbreviating $\alpha:=30 \varrho(1-\varrho)<30 \varrho$, denoting $H(t):=-t \ln t-(1-t) \ln (1-t)$, and using

$$
\begin{equation*}
\binom{a}{t a}=\Theta\left((t a)^{-1 / 2}\right) e^{H(t) a}, \text { for } t \leq 1 / 2 \tag{2}
\end{equation*}
$$

[^1](for a large, " $\leq$ " holds instead of " $=\Theta$ "), we find (the $O(p n)$ exponent comes from replacing $\binom{n}{2}$ by $n^{2} / 2$ in the denominator)
\[

$$
\begin{aligned}
\frac{\binom{n}{2 \varrho n} 2^{30 \varrho(1-\varrho) p n^{2} / 2}}{2^{p n^{2} / 2-O(p n)}} & \leq e^{H(1 / 2 \varrho) n-(\ln 2)(1-\alpha) p n^{2} / 3} \\
& \leq e^{H(1 / 2 \varrho) n-(\ln 2)(1-\alpha) p n^{2} / 3} \\
& =e^{n(H(1 / 2 \varrho)-(\ln 2)(1-\alpha) p n / 3)} \\
& \leq e^{n(H(1 / 2 \varrho)-(\ln 2) 33(1-30 \varrho))} \\
& =o(1)
\end{aligned}
$$
\]

as $p n / 2 \geq 30$ and $1-\alpha>1-30 \varrho$, and the expression in the parentheses is negative for all $\varrho \in[0,3 / 100]$.

For the rest of the fraction $Q$ above, using (2) again, we simplify

$$
\frac{\binom{4 \varrho(1-\varrho) n^{2} / 2}{30 \varrho(1-\varrho) p n^{2} / 2} F^{30 \varrho(1-\varrho) p n^{2} / 2}}{\binom{n^{2} / 2}{p n^{2} / 2}} \leq \frac{\binom{\alpha n^{2} / 2}{\alpha p n^{2} / 2} F^{30 \varrho(1-\varrho) p n^{2} / 2}}{\binom{n^{2} / 2}{p n^{2} / 2}}=O(1) \cdot e^{n^{2} / 2 \cdot((\alpha-1) H(p)+p \alpha \ln F)}
$$

Setting the expression in the parentheses to 0 and solving for $\varrho$, we find

$$
\alpha \geq \frac{\ln (1 / p)}{\ln (1 / p)+\ln F}
$$

suffices for $Q=o(1)$; as $\alpha \leq \varrho$, the same inequality with $\alpha$ replaced by $\varrho$ is sufficient. This completes the proof of the theorem.

## 4 Proof of Theorem 2(c)

In this section, following the idea of [9], we apply a theorem of Ronyai, Babai, and Ganapathy [15] on the maximum number of zero-patterns of polynomials, which we now describe.

Let $f=\left(f_{j}\right)_{j=1, \ldots, h}$ be an $h$-tuple of polynomials in $n$ variables $x=\left(x_{1}, x_{2}, \cdots, x_{n}\right)$ over an arbitrary field $\mathbb{F}$. In line with the definitions above, for $u \in \mathbb{F}^{n}$, the zero-nonzero pattern of $f$ at $u$ is the vector $\sigma(f(u)) \in\{0,1\}^{h}$.

Theorem 10 ([15]). If $h \geq n$ and each $f_{j}$ has degree at most $d$ then, for all $m$, the set

$$
\mid\left\{y \in\{0,1\}^{h}| | y \mid \leq m \text { and } y=\sigma(f(u)) \text { for some } u \in \mathbb{F}^{n}\right\} \left\lvert\, \leq\binom{ n+m d}{n}\right.
$$

In other words, the number of zero-nonzero patterns with Hamming weight at most $m$ is at $\operatorname{most}\binom{n+m d}{n}$.

As has been observed in [9], this theorem is implicit in the proof of Theorem 1.1 of [15] (for the sake of completeness, the proof is repeated in the appendix). It has been used in the context of minimum rank problems before (e.g., [14, 9]), but our use requires slightly more work.

Given positive integers $r<n$, let us say that a G-pattern is an $r \times n$ matrix whose entries are the symbels 0,1 , and $*$, with the following properties.
(1) Every column contains at most one 1 , and every column containing a 1 contains no $* s$.
(2) In every row, the leftmost entry different from 0 is a 1 , and every row contains at most one 1.
(3) Rows containing a 1 (i.e., not all-zero rows) have smaller row indices than rows containing no 1 (i.e., all-zero rows). In other words, the all-zero rows are at the bottom of $P$.
We say that an $r \times n$ matrix $Y$ has $G$-pattern $P$, if $Y_{j, \ell}=0$ if $P_{j, \ell}=0$, and $Y_{j, \ell}=1$ if $P_{j, \ell}=1$. There is no restriction on the $Y_{j, \ell}$ for which $P_{j, \ell}=*$.
"G" stands for "Gaussian elimination using row operations". We will need the following tree easy lemmas.

Lemma 11. Any $r \times n$ matrix $Y^{\prime}$ can be transformed, by Gaussian elimination using only row operations, into a matrix $Y$ which has some G-pattern.

Proof (sketch). If $Y^{\prime}$ has no nonzero entries, we are done. Otherwise start with the left-most column containing a nonzero entry, say $(j, \ell)$. Scale row $j$ that entry a 1 , permute the row to the top, and add suitable multiples of it to the other rows to make every entry below the 1 vanish.

If all columns $1, \ldots, \ell$ have been treated such that column $\ell$ has a unique 1 in row, say $j(\ell)$, consider the remaining matrix $\{j(\ell)+1, \ldots r\} \times\{\ell+1, \ldots, n\}$. If every entry is a 0 , we are down. Otherwise, find the leftmost nonzero entry in the block; suppose it is in column $\ell^{\prime}$ and row $j^{\prime}$. Scale row $j^{\prime}$ to make that entry a 1 , permute row $j^{\prime}$ to $j(\ell)+1$, and add suitable multiples of it to all other rows $\{1, \ldots, r\} \backslash\{j(\ell)+1\}$ to make every entry below the 1 vanish.

Lemma 12. For every $r \times n$-pattern matrix $P$, the number of $*$-entries in $P$ is at most $r(n-r / 2)$.

Proof (sketch). The G-pattern matrix $P$ is uniquely determined by $c_{1}<\cdots<c_{s}$, the (sorted) list of columns of $P$ which contain a 1 . With $c_{0}:=0$, for $i=1, \ldots, s$, if $c_{i-1}<c_{i}-1$, then replacing $c_{i}$ by $c_{i}-1$ gives us a G-pattern matrix with one more $*$ entry. Hence, we may assume that $c_{i}=i$ for $i=1, \ldots, s$. If $s<r$, then adding $s+1$ to the set of 1 -columns cannot decrease the number of $*$-entries (in fact, it increases the number, unless $s+1=n$ ). Hence, we may assume that $s=r$. The number of $*$-entries in the resulting (unique) G-pattern matrix is

$$
n-1+\cdots+n-r=r n-r(r+1) / 2 \leq r(n-r / 2),
$$

as promised.
Lemma 13. Let $\varrho \in] 0, .49]$. The number of $n \times$ @n G-pattern matrices is at most

$$
O(1) \cdot\binom{n}{\varrho n}
$$

(with an absolute constant in the big-O).
Proof (sketch). A G-pattern matrix is uniquely determined by the set of columns containing a 1 , which can be between 0 and $\varrho n$. Hence, the number of $n \times \varrho n$ G-pattern matrices is

$$
\begin{equation*}
\sum_{j=0}^{\varrho n}\binom{n}{j} \tag{*}
\end{equation*}
$$

From here on, we do the usual tricks. As in the previous section, we use the helpful fact (Theorem 1.1 in [2]) that

$$
(*) \leq \frac{1}{1-\frac{\varrho}{1-\varrho}}\binom{n}{\varrho n} .
$$

A swift calculation shows that $1 /(1-\varrho /(1-\varrho)) \leq 30$, which completes the proof.
We are now ready to complete the Proof of Theorem 2(c).

Proof of Theorem 2(c). Let $M$ be a fooling-set matrix of size $n$ and rank at most $r$. It can be factored as $M=X Y$, for an $n \times r$ matrix $X$ and an $r \times n$ matrix $Y$. By Lemma 11, through applying row operations to $Y$ and corresponding column operations to $X$, we can assume that $Y$ has a G-pattern.

Now we use Theorem 10, for every G-pattern matrix separately. For a fixed G-pattern matrix $P$, the variables of the polynomials are

- $X_{k, j}$, where $(k, j)$ ranges over all pairs $\{1, \ldots, n\} \times\{1, \ldots, r\}$; and
- $Y_{j, \ell}$, where $(j, \ell)$ ranges over all pairs $\{1, \ldots, r\} \times\{1, \ldots, n\}$ with $P_{j, \ell}=*$.

The polynomials are: for every $(k, \ell) \in\{1, \ldots, n\}^{2}$, with $k \neq \ell$,

$$
f_{k, \ell}=\sum_{\substack{j \\ P_{j, \ell}=1}} X_{k, j}+\sum_{\substack{j \\ P_{j, \ell}=*}} X_{k, j} Y_{j, \ell} .
$$

Clearly, there are $n(n-1)$ polynomials; the number of variables is $2 r n-r^{2} / 2$, by Lemma 12 (and, if necessary, using "dummy" variables which have coefficient 0 always). The polynomials have degree at most 2.

By Theorem 10, we find that the number of zero-nonzero patterns with Hamming weight at most $m$ of fooling-set matrices with rank at most $r$ which result from this particular Gpattern matrix $P$ is at most

$$
\binom{2 r n-r^{2} / 2+2 m}{2 r n-r^{2} / 2}
$$

Now, take a $\varrho<1 / 2$, and let $r:=\varrho n$. Summing over all G-pattern matrices $P$, and using Lemma 13, we find that the number of zero-nonzero patterns with Hamming weight at most $m$ of fooling-set matrices with rank at most $\varrho n$ is at most an absolute constant times

$$
\binom{n}{\varrho n}\binom{\left(2 \varrho-\varrho^{2} / 2\right) n^{2}+2 m}{\left(2 \varrho-\varrho^{2} / 2\right) n^{2}} .
$$

Now, take a constant $p \in] 0,1]$, and let $m:=\left\lceil p\binom{n}{2}\right\rceil$. The number of fooling-set patterns of size $n$ with density $p$ is

$$
\left(\begin{array}{c}
n \\
2 \\
m
\end{array}\right) 2^{m},
$$

and hence, the probability that the minimum rank of a fooling-set matrix with zero-nonzero pattern $R(n, p)$ has rank at most $r$ is at most

$$
\frac{\binom{n}{\varrho n}\binom{\left(2 \varrho-\varrho^{2} / 2\right) n^{2}+2 m}{\left(2 \varrho-\varrho^{2} / 2\right) n^{2}}}{\left(\begin{array}{c}
\left(\begin{array}{c}
n \\
2 \\
m
\end{array}\right)
\end{array} 2^{m}\right.} \leq \frac{\binom{n}{\varrho n}\binom{\left(2 \varrho-\varrho^{2} / 2\right) n^{2}+2 p n^{2} / 2}{\left(2 \varrho-\varrho^{2} / 2\right) n^{2}}}{\binom{n^{2} / 2}{p n^{2} / 2} 2^{p n^{2} / 2+O(p n)}}=\frac{\binom{n}{\varrho n}\binom{\alpha n^{2}+p n^{2}}{\alpha n^{2}}}{\binom{n^{2} / 2}{p n^{2} / 2} 2^{p n^{2} / 2+O(p n)}}
$$

where we have set $\alpha:=2 \varrho-\varrho^{2} / 2$. As in the previous section, we use (2) to estimate this expression, and we obtain

$$
\ln \left(\frac{\binom{n}{\varrho n}\binom{\alpha n^{2}+p n^{2}}{\alpha n^{2}}}{\binom{n^{2} / 2}{p n^{2} / 2} 2^{p n^{2} / 2+O(p n)}}\right)=n H(\varrho)+n^{2}\left(\alpha H(\alpha /(\alpha+p))-\frac{1}{2} H(p)-(\ln 2) p / 2\right)+O(p n) .
$$

The dominant term is the one where $n$ appears quadratic. The expression $\frac{1}{2} H(p)+(\ln 2) p / 2$ takes values in $] 0,1[$. For every fixed $p$, the function $g: \alpha \mapsto \alpha H(\alpha /(\alpha+p))$ is strictly increasing on $[0,1 / 2]$ and satisfies $g(0)=0$. Hence, for every given constant $p$, there exists an $\alpha$ for which the coefficient after the $n^{2}$ is negative.
(As indicated in the introduction, such an $\alpha$ must tend to 0 with $p \rightarrow 0$.)

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## A Proof of Lemma 1

Let $\left(x_{1}, y_{1}\right), \ldots,\left(x_{n}, y_{n}\right) \in X \times Y$ be a fooling set in $f$, and let $A$ be a matrix over $\mathbb{F}$ with $A_{x, y}=0$ iff $f(x, y)=0$. Consider the matrix $B:=A \otimes A^{\top}$. This matrix $B$ contains a permutation matrix of size $n$ as a submatrix: for $i=1, \ldots, n, B_{\left(x_{i}, x_{i}\right),\left(y_{i}, y_{i}\right)}=A_{x_{i}, y_{i}} A_{y_{i}, x_{i}}=1$ but for $i \neq j$, $B_{\left(x_{i}, x_{i}\right),\left(y_{j}, y_{j}\right)}=A_{x_{i}, y_{j}} A_{y_{i}, x_{j}}=0$. Hence,

$$
n \leq \operatorname{rk}(B)=\operatorname{rk}(A)^{2}
$$

## B Proof of Theorem 10

Since Theorem 10 is not explicitly proven in [15], we give here the slight modification of the proof of Theorem 1.1 from Theorem 10 which proves Theorem 10. The only difference between the following proof and that in [15] is where the proof below upper-bounds the degrees of the polynomials $g_{y}$.

Proof of Theorem 10. Consider the set

$$
S:=\left\{y \in\{0,1\}^{h}| | y \mid \leq m \text { and } y=\sigma(f(u)) \text { for some } u \in \mathbb{F}^{n}\right\} .
$$

For each such $y$, let $u_{y} \in \mathbb{F}^{n}$ be such that $\sigma\left(f\left(u_{y}\right)\right)=y$, and let

$$
g_{y}:=\prod_{j, y_{j}=1} f_{j}
$$

Now define a square matrix $A$ whose row- and column set is $S$, and whose $(y, z)$ entry is $g_{y}\left(u_{z}\right)$. We have

$$
g_{y}\left(u_{z}\right) \neq 0 \Longleftrightarrow z \geq y
$$

with entry-wise comparison, and " $1>0$ ". Hence, if the rows and columns are arranged according to this partial ordering of $S$, the matrix is upper triangular, with nonzero diagonal, so it has full rank, $|S|$. This implies that the $g_{y}, y \in S$, are linearly independent.

Since each $g_{y}$ has degree at most $|y| \cdot d \leq m d$, and the space of polynomials in $n$ variables with degree at most $m d$ has dimension $\binom{n+m d}{m d}$, it follows that $S$ has at most that many elements.


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    ${ }^{1}$ This concept of minimum rank differs from the definition used in the context of index coding [10, 8]. It is closer to the minimum rank of a graph, but there the matrix $A$ has to be symmetric while the diagonal entries are unconstrained.

[^1]:    ${ }^{2}$ Specifically, we use Theorem 2.10 applied to (2.11) in [11]

