## A Productivity Checker for Logic Programming

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**Abstract.** Automated analysis of recursive derivations in logic programming is known to be a hard problem. Both termination and non-termination are undecidable problems in Turing-complete languages. However, some declarative languages offer a practical work-around for this problem, by making a clear distinction between whether a program is meant to be understood inductively or coinductively. For programs meant to be understood inductively, termination must be guaranteed, whereas for programs meant to be understood coinductively, productive nontermination (or "productivity") must be ensured. In practice, such classification helps to better understand and implement some non-terminating computations. Logic programming was one of the first declarative languages to make this distinction: in the 1980's, Lloyd and van Emden's "computations at infinity" captured the big-step operational semantics of derivations that produce infinite terms as answers. In modern terms, computations at infinity describe "global productivity" of computations in logic programming. Most programming languages featuring coinduction also provide an observational, or small-step, notion of productivity as a computational counterpart to global productivity. This kind of productivity is ensured by checking that finite initial fragments of infinite computations can always be observed to produce finite portions of their infinite answer terms. In this paper we introduce a notion of *observational productivity* for logic programming as an algorithmic approximation of global productivity, give an effective procedure for semi-deciding observational productivity, and offer an implemented automated observational productivity checker for logic programs.

Keywords: Logic programming, corecursion, coinduction, termination, productivity.

### 1 Introduction

Induction is pervasive in programming and program verification. It arises in definitions of finite data (e.g., lists, trees, and other algebraic data types), in program semantics (e.g., of finite iteration and recursion), and proofs (e.g., of properties of finite data and processes). Coinduction, too, is important in these arenas, arising in definitions of infinite data (e.g., lazily defined infinite streams), in program semantics (e.g., of concurrency), and in proofs (e.g., of observational equivalence, or bisimulation, of potentially infinite processes). It is thus desirable to have good support for both induction and coinduction in systems for reasoning about programs.

Given a logic program P and a term A, SLD-resolution provides a mechanism for automatically (and inductively) inferring that  $P \vdash A$  holds, i.e., that P logically entails A. The "answer" for a program P and a query ?  $\leftarrow A$  is a substitution  $\sigma$  computed from *P* and *A* by SLD-resolution. Soundness of SLD-resolution ensures that  $P \vdash \sigma(A)$  holds, so we also say that *P* computes  $\sigma(A)$ .

Example 1 (Inductive logic program). The program  $P_1$  codes the Peano numbers:

 $\begin{array}{l} 0. \ \mathtt{nat}(\mathtt{0}) \leftarrow \\ 1. \ \mathtt{nat}(\mathtt{s}(\mathtt{X})) \leftarrow \ \mathtt{nat}(\mathtt{X}) \end{array}$ 

To answer the question "Does  $P_1 \vdash \operatorname{nat}(\mathfrak{s}(X))$  hold?", we represent it as the logic programming (LP) query ?  $\leftarrow \operatorname{nat}(\mathfrak{s}(X))$  and resolve it with  $P_1$ . It is standard in implementations of traditional LP to use a topmost clause selection strategy, which resolves goals against clauses in the order in which they appear in the program. Topmost clause selection gives the derivation  $\operatorname{nat}(\mathfrak{s}(X)) \to \operatorname{nat}(X) \to \operatorname{true}$  for  $P_1$  and ?  $\leftarrow \operatorname{nat}(\mathfrak{s}(X))$ , which computes the answer  $\{X \mapsto 0\}$  in its last step. Since  $P_1$  computes  $\operatorname{nat}(\mathfrak{s}(0))$ , one answer to our question is "Yes, provided X is 0."

While inductive properties of terminating computations are quite well understood [14], non-terminating LP computations are notoriously difficult to reason about, and can arise even for programs that are intended to be inductive:

Example 2 (Coinductive meaning of inductive logic program). If  $P'_1$  is obtained by reversing the order of the clauses in the program  $P_1$  from Example 1, then the SLD-derivation for program  $P'_1$  and query ?  $\leftarrow \operatorname{nat}(\mathbf{s}(X))$  does not terminate under standard topmost clause selection. Instead, it results in an attempt to compute the "answer"  $\{X \mapsto \mathbf{s}(\mathbf{s}(...))\}$  by repeatedly resolving with Clause 1. Nevertheless,  $P'_1$  is still computationally meaningful, since it computes the first limit ordinal at infinity [14].

Some programs do not admit terminating computations under *any* selection strategy:

*Example 3 (Coinductive logic program).* No derivation for the query  $? \leftarrow \texttt{stream}(X)$  and the program  $P_2$  comprising the clause

0.  $stream(scons(0, Y)) \leftarrow stream(Y)$ 

terminates with an answer, be it success or otherwise. Nevertheless,  $P_2$  has computational meaning: it computes the infinite stream of 0s at infinity.

The importance of developing sufficient infrastructure to support coinduction in automated proving has been argued across several communities; see, e.g., [13,17,21]. In LP, the ability to work with non-terminating and coinductive programs depends crucially on understanding the structural properties of non-terminating SLD-derivations. To illustrate, consider the non-terminating programs  $P_3$ ,  $P_4$ , and  $P_5$ :

Program	Program definition	For query $? \leftarrow p(X)$ , computes the answer:
$P_3$	$\mathtt{p}(\mathtt{X}) \gets \mathtt{p}(\mathtt{X})$	id
$P_4$	$\mathtt{p}(\mathtt{X}) \leftarrow \mathtt{p}(\mathtt{f}(\mathtt{X}))$	id
$P_5$	$\mathtt{p}(\mathtt{f}(\mathtt{X})) \gets \mathtt{p}(\mathtt{X})$	$\{\mathtt{X} \mapsto \mathtt{f}(\mathtt{f})\}$

Programs  $P_3$  and  $P_4$  each loop without producing any substitutions at all; only  $P_5$  computes an infinite term at infinity. It is of course not a coincidence that only  $P_5$  resembles a (co)inductive data definition by pattern matching on a constructor, as is commonly used in functional programming.

When an infinite SLD-derivation computes an infinite object, and this object can be successively approximated by applying to the initial query the substitutions computed at each step of the derivation, the derivation is said to be globally productive. The only derivation for program  $P_5$  and the query ?  $\leftarrow p(X)$  is globally productive since it approximates, in the sense just described, the infinite term p(f(f...)). In terminology of [14], it computes p(f(f...)) at infinity. Programs  $P_2$  and  $P'_1$  similarly give rise to globally productive derivations. But no derivations for  $P_3$  or  $P_4$  are globally productive.

Since global productivity determines which non-terminating logic programs can be seen as defining coinductive data structures, we would like to identify exactly when a program is globally productive. But porting functional programming methods of ensuring productivity by static syntactic checks is hardly possible. Unlike pattern matching in functional programming, SLD-resolution is based on *unification*, which has very different operational properties — including termination and productivity properties — from pattern matching. For example, programs  $P_1$ ,  $P'_1$ ,  $P_2$ , and  $P_5$  are all terminating by termmatching SLD-resolution, i.e., resolution in which unifiers are restricted to matchers, as in term rewriting. We thus call this kind of derivations *rewriting derivations*.

Example 4 (Coinductive program defining an irrational infinite term). The program  $P_6$  comprises the single clause

 $0.\;\texttt{from}(\texttt{X},\texttt{scons}(\texttt{X},\texttt{Y})) \gets \texttt{from}(\texttt{s}(\texttt{X}),\texttt{Y})$ 

For  $P_6$  and the query ?  $\leftarrow from(0, Y)$ , SLD-resolution computes at infinity the answer substitution  $\{Y \mapsto [0, \mathbf{s}(0), \mathbf{s}(\mathbf{s}(0)), \ldots]\}$ . Here  $[t_1, t_2, \ldots]$  abbreviates  $\mathbf{scons}(t_1, \mathbf{scons}(t_2, \ldots))$ , and similarly in the remainder of this paper. This derivation depends crucially on unification since variables occurring in the two arguments to **from** in the clause head overlap. If we restrict to rewriting, then there are no successful derivations (terminating or nonterminating) for this choice of program and query.

Example 4 shows that any analysis of global productivity must necessarily rely on specific properties of the operational semantics of LP, rather than on program syntax alone. It has been observed in [11,9] that one way to distinguish globally productive programs operationally is to identify those that admit infinite SLD-derivations, but for which rewriting derivations always terminate. We call this program property observational productivity. The programs  $P_1$ ,  $P'_1$ ,  $P_2$ ,  $P_5$ ,  $P_6$  are all observationally productive.

The key observation underlying observational productivity is that terminating rewriting derivations can be viewed as points of finite observation in infinite derivations. Consider again program  $P_6$  and query ?  $\leftarrow \texttt{from}(0, \texttt{Y})$  from Example 4. Drawing rewriting derivations vertically and unification-based resolution steps horizontally, we see that each unification substitution applied to the original query effectively observes a further fragment of the stream computed at infinity:

$$\begin{array}{ccc} {}^{\{X\mapsto [0,X']\}} & {}^{\{X'\mapsto [s(0),X'']\}} & \rightarrow \\ \texttt{from}(0,X) & \texttt{from}(0,[0,X']) & \texttt{from}(0,[0,s(0),X'']) & \\ & \texttt{from}(s(0),X') & \texttt{from}(s(0),[s(0),X'']) & \\ & \texttt{from}(s(s(0)),X'') & \\ \end{array}$$

If we compute unifiers only when rewriting derivations terminate, then the resulting derivations exhibit consumer-producer behaviour: rewriting steps consume structure (here, the constructor scons), and unification steps produce more structure (here, new sconses)

for subsequent rewriting steps to consume. This style of interleaving matching and unification steps was called *structural resolution* (or S-resolution) in [9,12].

Model-theoretic properties of S-resolution relative to least and greatest Herbrand models of programs were studied in [12]. In this paper, we provide a suitable algorithm for semi-deciding observational productivity of logic programs, and present its implementation [19], see also Appendix B online. As exemplified above, observational productivity of a program P is in fact a conjunction of two properties of P:

- 1. universal observability: termination of all rewriting derivations, and
- 2. *existential liveness*: existence of *at least one* non-terminating S-resolution or SLD-resolution derivation.

While the former property is universal, the latter must be existential. For example, the program  $P_1$  defining the Peano numbers can have both inductive and coinductive meaning. When determining that a program is observationally productive, we must certify that the program actually *does* admit derivations that produce infinite data, i.e., that it actually *can* be seen as a coinductive definition. Our algorithm for semi-deciding observational productivity therefore combines two checks:

- 1. guardedness checks that semi-decide universal observability: if a program is guarded, then it is universally observable. (The converse is not true in general.)
- 2. *liveness invariant checks* ensuring that, if a program is guarded and exhibits an invariant in its consumption-production of constructors, then it is existentially live.

This is the first work to develop productivity checks for LP. An alternative approach to coinduction in LP, known as CoLP [7,21], detects loops in derivations and closes them coinductively. However, loop detection was not intended as a tool for the study of productivity and, indeed, is insufficient for that purpose: programs  $P_3$ ,  $P_4$  and  $P_5$ , of which only the latter is productive, are all treated similarly by CoLP, and all give coinductive proofs via its loop detection mechanism.

Our approach also differs from the usual termination checking algorithms in termrewriting systems (TRS) [22,1,8] and LP [3,16,18,20,15]. Indeed, these algorithms focus on guaranteeing termination, rather than productivity, see Section 5. And although the notion of productivity has been studied in TRS [4,5], the actual technical analysis of productivity is rather different there because it considers infinitary properties of rewriting, whereas observational productivity relies on termination of rewriting.

The rest of this paper is organised as follows. In Section 2 we introduce a *contrac*tion ordering on terms that extends the more common lexicographic ordering, and argue that this extension is needed for our productivity analysis. We also recall that static guardedness checks do not work for LP. In Section 3 we employ contraction orderings in dynamic guardedness checks and present a decidable property, called GC2, that characterises guardedness of a single rewriting derivation, and thus certifies existential observability. In Section 4 we employ GC2 to develop an algorithm, called GC3, that analyses *consumer-producer* invariants of S-resolution derivations to certify universal observability. For universally observable programs, these invariants also serve as liveness invariant checks. We also prove that GC3 indeed semi-decides observational productivity. In Section 5 we discuss related work and in Section 6 – implementation and applications of the productivity checker. In Section 7 we conclude the paper.

## 2 Contraction Orderings on Terms

In this section, we will introduce the contraction ordering on first-order terms, on which our productivity checks will rely. We work with the standard definition of first-order logic programs. A signature  $\Sigma$  consists of a set  $\mathcal{F}$  of function symbols  $f, g, \ldots$  each equipped with an arity. Nullary (0-ary) function symbols are constants. We also assume a countable set Var of variables, and a set  $\mathcal{P}$  of predicate symbols each equipped with an arity. We have the following standard definition for terms, formulae and Horn clauses:

#### Definition 1 (Syntax of Horn clauses and programs).

Terms Term ::=  $Var \mid \mathcal{F}(Term, ..., Term)$ Atomic formulae (or atoms) At ::=  $\mathcal{P}(Term, ..., Term)$ (Horn) clauses CH ::=  $At \leftarrow At, ..., At$ Logic programs Prog ::= CH, ..., CH

In what follows, we will use letters A, B with subscripts to refer to elements of At. Given a program P, we assume all clauses are indexed by natural numbers starting from 0. When we need to refer to *i*th clause of program P, we will use notation P(i). To refer to the head of clause P(i), we will use notation head(P(i)).

A substitution is a total function  $\sigma : Var \to Term$ . Substitutions are extended from variables to terms as usual: if  $t \in Term$  and  $\sigma$  is a substitution, then the *application*  $\sigma(t)$  is a result of applying  $\sigma$  to all variables in t. A substitution  $\sigma$  is a *unifier* for t, u if  $\sigma(t) = \sigma(u)$ , and is a *matcher* for t against u if  $\sigma(t) = u$ . A substitution  $\sigma$  is a *most* general unifier (mgu) for t and u if it is a unifier for t and u and is more general than any other such unifier. A most general matcher (mgm)  $\sigma$  for t against u is defined analogously.

We can view every term and atom as a tree. Following standard definitions [2,14], such trees can be indexed by elements of a suitably defined tree language. Let  $\mathbb{N}^*$  be the set of all finite words (i.e., sequences) over the set  $\mathbb{N}$  of natural numbers. A set  $L \subseteq \mathbb{N}^*$  is a *(finitely branching) tree language* if the following two conditions hold: (i) for all  $w \in \mathbb{N}^*$ and all  $i, j \in \mathbb{N}$ , if  $wj \in L$  then  $w \in L$  and, for all  $i < j, wi \in L$ , and (ii) for all  $w \in L$ , the set of all  $i \in \mathbb{N}$  such that  $wi \in L$  is finite. A tree language L is *finite* if it is a finite subset of  $\mathbb{N}^*$ , and *infinite* otherwise. Term trees (for terms and atoms) are defined as mappings from a tree language L to the given signature, see [2,14,9]. Informally speaking, every symbol occurring in a term or an atom receives an index from L.

In what follows, we will work with term tree representation of all terms and atoms, and for brevity we will refer to all term trees simply as *terms*. We will use notation t(w) when we need to talk about the element of the term tree t indexed by a word  $w \in L$ . Note that leaf nodes are always given by variables or constants.

*Example 5.* Given  $L = \{\epsilon, 0, 00, 01\}$ , the atom stream(scons(0, Y)) can be seen as a term tree t given by the map  $t(\epsilon) =$ stream, t(0) =scons, t(00) =0, t(01) =Y.

We can use such indexing to refer to subterms, and notation subterm(t, w) will refer to a subterm of term t starting at node w. In the above example, where t = stream(scons(0, Y)), subterm(t, 0) is scons(0, Y).

Two most popular tools for termination analysis of declarative programs are lexicographic ordering and (recursive) path ordering of terms. Informally, the idea can be adopted to LP setting as follows. Suppose we have a clause  $A \leftarrow B_1, \ldots, B_i, \ldots, B_n$ . We may want to check whether each  $B_i$  sharing the predicate with A is "smaller"' than A, since this guarantees that no infinite rewriting derivation is triggered by this clause. For lexicographic ordering we will write  $B_i <_l A$  and for path ordering we will write  $B_i <_p A$ .

Using standard orderings to prove universal observability works well for program  $P_2$ , since stream(Y)  $<_l$  stream(scons(0,Y)) and stream(Y)  $<_p$  stream(scons(0,Y)), and so any rewriting derivation for  $P_2$  terminates. But universal observability of  $P_6$  from Example 4 cannot be shown by this method. Indeed, none of the four orderings from(X, scons(X,Y))  $<_l$  from(s(X),Y), from(s(X),Y)  $<_l$  from(X, scons(X,Y)), from(s(X),Y), and from(s(X),Y)  $<_p$  from(X, scons(X,Y)) holds because the subterms pairwise disagree on the ordering. This situation is common for LP, where some arguments hold input data and some hold output data, so that some decrease while others increase in recursive calls. Nevertheless,  $P_6$  is universally observable, and we want to be able to infer this. Studying the S-resolution derivation for  $P_6$  in Section 1, we note that universal observability of  $P_6$  is guaranteed by contraction of from's second argument. It is therefore sufficient to establish that terms get smaller in only one argument. This inspires our definition of a *contraction ordering*, which takes advantage of the tree representation of terms.

**Definition 2 (Contraction, recursive contraction).** If  $t_1$  and  $t_2$  are terms, then  $t_2$  is a contraction of  $t_1$  (written  $t_1 \triangleright t_2$ ) if there is a leaf node  $t_2(w)$  on a branch B in  $t_2$ , and there exists a branch B' in  $t_1$  that is identical to B up to node w, however,  $t_1(w)$  is not a leaf. If, in addition, subterm $(t_1, w)$  contains the symbol given by  $t_2(w)$ , then  $t_2$  is a recursive contraction of  $t_1$ .

We distinguish variable contractions and constant contractions according as  $t_2(w)$  is a variable or constant, and call subterm $(t_1, w)$  a reducing subterm for  $t_1 \triangleright t_2$  at node w. We call subterm $(t_1, w)$  a recursive, variable or constant reducing subterm if  $t_1 \triangleright t_2$  is a recursive, variable or constant contraction, respectively.

Example 6 (Contraction orderings). We have  $from(X, scons(X, Y)) \triangleright from(s(X), Y)$ , as the leaf Y in the latter is "replaced" by the term scons(X, Y) in the former. Formally, scons(X, Y) is a recursive and variable reducing subterm. It can be used to certify termination of all rewriting derivations for  $P_6$ . Note that  $from(s(X), Y) \triangleright from(X, scons(X, Y))$  also holds, with (recursive and variable) reducing subterm s(X).

The fact that  $\triangleright$  is not well-founded makes reasoning about termination delicate. Nevertheless, contractions emerge as precisely the additional ingredient needed to formulate our productivity check for a sufficiently general and interesting class of logic programs.

In general, static termination checking for LP suffers serious limitations; see, e.g., [3]. The following example illustrates this phenomenon.

Example 7 (Contraction ordering on clause terms is insufficient for termination checks). The program  $P_7$ , that is not universally observable, is given by mutual recursion: 0.  $p(s(X1), X2, Y1, Y2) \leftarrow q(X2, X2, Y1, Y2)$ 

1.  $q(X1, X2, s(Y1), Y2) \leftarrow p(X1, X2, Y2, Y2)$ 

No two terms from the same clause of  $P_7$  can be related by any contraction ordering because their head symbols differ. But recursion arises for  $P_7$  when a derivation calls its two clauses alternately, so we would like to examine rewriting derivations for queries, such as  $? \leftarrow p(s(X1), X2, s(Y1), Y2)$  and  $? \leftarrow p(s(X1), s(X2), s(Y1), s(Y2))$ , that exhibit its recursive nature. Unfortunately, such queries are not given directly by  $P_7$ 's syntax, and so are not available for static program analysis.

As static checking for contraction ordering in clauses is not sufficient, we will define dynamic checks in the next section. The idea is to build a rewriting tree for each clause, and check whether term trees featured in that derivation tree obey contraction ordering.

## 3 Rewriting Trees: Guardedness Checks for Rewriting Derivations

To properly reason about rewriting derivations in LP, we need to take into account that i) in LP, unlike, e.g., in TRS, we have conjuncts of terms in the bodies of clauses, and ii) a logic program can have overlapping clauses, i.e., clauses whose heads unify. These two facts have been analysed in detail in the LP literature, usually using the notion of andor-trees and, where optimisation has been concerned, and-or-parallel trees. We carry on this tradition and consider a variant of and-or trees for derivations. However, the trees we consider are not formed by general SLD-resolution, but rather by term matching resolution. *Rewriting trees* are so named because each of their edges represents a term matching resolution step, i.e., a matching step as in term rewriting.

**Definition 3 (Rewriting tree).** Let P be a logic program with n clauses, and A be an atomic formula. The rewriting tree for P and A is the possibly infinite tree T satisfying the following properties.

- -A is the root of T
- Each node in T is either an and-node or an or-node
- Each or-node is given by P(i), for some  $i \in \{0, \ldots, n\}$
- Each and-node is an atom seen as a term tree.
- For every and-node A' occurring in T, if there exist exactly k > 0 distinct clauses  $P(j), \ldots, P(m)$  in P (a clause P(i) has the form  $B_i \leftarrow B_1^i, \ldots, B_{n_i}^i$  for some  $n_i$ ), such that  $A' = \theta_j(B_j) = \ldots = \theta_m(B_m)$ , for mgms  $\theta_j, \ldots, \theta_m$ , then A' has exactly k children given by or-nodes  $P(j), \ldots, P(m)$ , such that, every or-node P(i) has  $n_i$  children given by and-nodes  $\theta_i(B_1^i), \ldots, \theta_i(B_{n_i}^i)$ .

When constructing rewriting trees, we assume a suitable algorithm [9] for renaming free variables in clause bodies apart. Figure 1 gives examples of rewriting trees. An and-subtree of a rewriting tree (a subtree in which a derivation always pursues only one or-choice at a time) is a *rewriting derivation*, see [9] for a formal definition.

Because mgms are unique up to variable renaming, given a program P and an atom A, rewriting tree T for P and A is unique. Following the same principle as with definition of term trees, we use suitably defined finitely-branching tree languages for indexing rewriting trees, see [9] for precise definitions. When we need to talk about a node of a rewriting tree T indexed by a word  $w \in L$ , we will use notation T(w).

We can now formally define our notion of universal observability.

**Definition 4 (Universal observability).** A program P is universally observable if, for every atom A, the rewriting tree for A and P is finite.

Programs  $P_1$ ,  $P'_1$ ,  $P_2$ ,  $P_5$ ,  $P_6$  are universally observable, whereas programs  $P_3$ ,  $P_4$  and  $P_7$  are not. An exact analysis of why  $P_7$  is not universally observable is given in Example 9.

We can now apply the contraction ordering we defined in the previous section to analyse termination properties of rewriting trees. A suitable notion of guardedness can be defined by checking for loops in rewriting trees whose terms fail to decrease by any contraction ordering. But note that our notion of a loop is more general than that used in CoLP [7,21] since it does not require the looping terms to be unifiable.

**Definition 5 (Loop in a rewriting tree).** Given a program P and an atom A the rewriting tree T for P and A contains a loop at nodes w and v, denoted loop(T, w, v), if w properly precedes v on some branch of T, T(w) and T(v) are and-nodes whose atoms have the same predicate, and parent or-nodes of T(w) and T(v) are given by the same clause P(i).

Examples of loops in rewriting trees are given (underlined) in Figure 1.

If T has a loop at nodes w and v, and if t is a recursive reducing subterm for  $T(w) \triangleright T(v)$ , then loop(T, w, v) is guarded by (P(i), t), where P(i) is the clause that was resolved against to obtain T(w) and T(v). It is unguarded otherwise. A rewriting tree T is guarded if all of its loops are guarded, and is unguarded otherwise. We write GC2(T) when T is guarded, and say that GC2(T) holds.

*Example 8.* In Figure 1, we have (underlined) loops in the third rewriting tree (for q(s(X''), s(X''), s(Y'), Y'') and q(s(X'), s(X''), Y'', Y'')) and the fourth rewriting tree (for q(s(X''), s(X''), s(Y'), s(Y')) and q(s(X''), s(X''), s(Y''))). Neither is guarded. In the former case, there is a contraction on the third argument, but because s(Y') and Y'' do not share a variable, it is not recursive contraction. In the latter loop, there is no contraction at all.

By Definition 5, each repetition of a clause and predicate in a branch of a rewriting tree triggers a check to see if the loop is guarded by some recursive reducing subterm.

**Proposition 1** (GC2 is decidable). GC2 is a decidable property of rewriting trees.<sup>4</sup>

The proof of Proposition 1 also establishes that every guarded rewriting tree is finite.

The decidable guardedness property GC2 is a property of individual rewriting trees. But our goal is to decide guardedness universally, i.e., for *all* of a program's rewriting trees. The next example shows that extrapolating from existential to universal guardedness is a difficult task.

Example 9 (Existential guardedness does not imply universal guardedness). For program  $P_7$ , the rewriting trees constructed for the two clause heads p(s(X'), X'', Y', Y'') and q(s(X'), X'', s(Y'), Y'') are both guarded since neither contains any loops at all. Nevertheless, there is a rewriting tree for  $P_7$  (the last tree in Figure 1) that is unguarded and infinite. The third tree is not guarded (due to the unguarded loop), but it is finite.

The example above shows that our initial idea of checking rewriting trees generated by clause heads is insufficient to detect all cases of nonterminating rewriting. Since a similar

<sup>&</sup>lt;sup>4</sup> All proofs are in an **Appendix A** supplied as supplementary material online. Corresponding pseudocode algorithms are given in **Appendix B**.

**Fig. 1.** An initial fragment of the derivation tree (comprising four rewriting trees) for the program  $P_7$  of Example 7 and the atom p(s(X'), X'', Y', Y''). Its third and fourth rewriting trees each contain an unguarded loop (underlined), so both are unguarded. The fourth tree is infinite.

situation can obtain for any finite set of rewriting trees, universal observability, and hence observational productivity, of programs cannot be determined by guardedness of rewriting trees for program clauses alone. The next section addresses this problem.

### 4 Derivation Trees: Observational Productivity Checks

The key idea of this section is, given a program P, to identify a finite set S of rewriting trees for P such that checking guardedness of all rewriting trees in S is sufficient for guaranteeing guardedness of *all* rewriting trees for P. One way to identify such sets will be to use the strategy of Example 9 and Figure 1: for every clause P(i) of P, to construct a rewriting tree for the head of P(i), and, if that tree is guarded, explore what kind of mgus the leaves of that tree generate, and see if applications of those mgus may give an unguarded tree. As Figure 1 shows, we may need to apply this method iteratively until we find a nonguarded rewriting tree. But we want the number of such iterations to be finite. This section presents a solution to this problem.

We start with a formal definition of rewriting tree transitions, which we have seen already in Figure 1, see also Figure 2.

**Definition 6 (Rewriting tree transition).** Let P be a program and T be a rewriting tree for P and an atom A. If T(w) is a leaf node of T given by an atom B, and B unifies with a clause P(i) via  $mgu \sigma$ , we define a tree  $T_w$  as follows: we apply  $\sigma$  to every and-node of T, and extend the branches where required, according to Definition 3.

Computation of  $T_w$  from T is denoted  $T \to T_w$ . The operation  $T \to T_w$  is the tree transition for T and w.

If a rewriting tree T is constructed for a program P and an atom A, a (finite or infinite) sequence  $T \to T' \to T'' \to \ldots$  of tree transitions is an *S*-resolution derivation for P and A. For a given rewriting tree T, several different S-resolution derivations are possible from T. This gives rise to the notion of a derivation tree.

**Definition 7 (Derivation tree, guarded derivation tree).** Given a logic program P and an atom A, the derivation tree D for P and A is defined as follows:

- The root of D is given by the rewriting tree for P and A.
- For a rewriting tree T occurring as a node of D, if there exists a transition  $T \to T_w$ , for some leaf node w in T, then the node T has a child given by  $T_w$ .

A derivation tree is guarded if each of its nodes is a guarded rewriting tree, i.e., if GC2(T) holds for each of its nodes T.

Figure 1 shows an initial fragment of the derivation tree for  $P_7$  and p(s(X'), X'', Y', Y'').

Note that we now have three kinds of trees: term trees have signature symbols as nodes, rewriting trees have atoms (term trees) as nodes, and derivation trees have rewriting trees as nodes. For a given P and A, the derivation tree for P and A is unique up to renaming. We use our usual notation D(w) to refer to the node of D at index  $w \in L$ .

**Definition 8 (Existential liveness, observational productivity).** Let P be a universally observable program and let A be an atom. An S-resolution derivation for P and A is live if it constitutes an infinite branch of the derivation tree for P and A. The program P is existentially live if there exists a live S-resolution derivation for P and some atom A. P is observationally productive if it is universally observable and existentially live.

To show that observational productivity is semi-decidable, we first show that universal observability is semi-decidable by means of a finite (i.e., decidable) guardedness check. We started this section by motivating the need to construct a finite set S of rewriting trees checking guardedness of which will guarantee guardedness for *any* rewriting tree for the given program. Our first logical step is to use derivation trees built for clause heads as generators of such a set S. Due to the properties of mgu's used in forming branches of derivation trees, derivation trees constructed for clause heads generate the set of *most general* rewriting trees. The next lemma exposes this fact:

Lemma 1 (Guardedness of derivation trees implies universal observability). Given a program P, if derivation trees for P and each head(P(i)) are guarded, then P is universally observable.

However, derivation trees are infinite, in general. So it still remains to define a method that extracts representative finite subtrees from such derivation trees; we call such subtrees observation subtrees. For this, we need only be able to detect an invariant property guaranteeing guardedness through tree transitions in the given derivation tree. To illustrate, let us check guardedness of the program  $P_6$ . As it consists of just one clause, we take the head of that clause as the goal atom, and start constructing the infinite derivation tree D for  $P_6$  and from(X, scons(X, Y)) as shown in Figure 2. The first rewriting tree in the derivation tree has no loops, so we cannot identify any invariants. We make a transition to the second rewriting tree which has one loop (underlined) involving the recursive reducing subterm [s(X), Y']. This reducing subterm is our first candidate invariant, it is the pattern that is *consumed* from the root of the second rewriting tree to its leaf. We now need to check this pattern is added back, or *produced*, in the next tree transition. The next mgu involves substitution  $Y' \mapsto [s(s(X)), Y'']$ . Because this derivation gradually computes an infinite irrational term (rational terms are terms that can be represented as trees that have a finite number of distinct subtrees), the two terms [s(X), Y']and  $[\mathbf{s}(\mathbf{s}(\mathbf{X})), \mathbf{Y}'']$  we have identified are not unifiable. We need to be able to abstract away

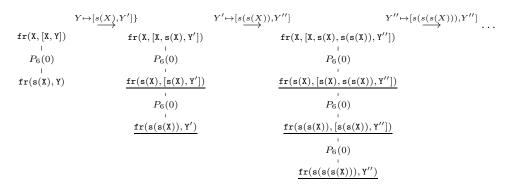


Fig. 2. An initial fragment of the infinite derivation tree D for the program  $P_6$  from Example 4 and its clause head. It is also the observation subtree of D. We abbreviate scons by [,], and from by fr. The guarded loops in each of its rewriting trees are underlined.

from their current shape and identify a common pattern, which is  $[\_,\_]$ . By the properties of mgu's used in transitions, such most general pattern can always be extracted from the clause head itself. Indeed, the subterm of the clause head from(X, scons(X, Y)) has the subterm [X, Y] that is exactly the pattern we look for. Thus, our current *(coinductive)* assumption is: given a rewriting tree T in the derivation tree D, [X, Y] will be consumed by rewriting steps from its root to its leaves, and exactly [X, Y] will be produced *(i.e.,* added back) in the next tree transition. X and Y are seen as placeholders for some terms. Consumption is always finite (by the loop guardedness), and production is potentially infinite.

We now need to check that this coinductive assumption will hold for the next rewriting tree of D. The third rewriting tree indeed has guarded loops with recursive reducing subterm  $[\mathbf{s}(\mathbf{s}(\mathbf{X})), \mathbf{Y}'']$ , and the next mgu it gives rise to is  $\mathbf{Y}'' \mapsto [\mathbf{s}(\mathbf{s}(\mathbf{s}(\mathbf{X}))), \mathbf{Y}'']$ . Again, to abstract away the common pattern, we look for a subterm in the clause head of  $P_6(0)$  that matches with both of these terms, it is the same subterm  $[\mathbf{X}, \mathbf{Y}]$ . Thus, our coinductive assumption holds again, and we conclude by coinduction that the same pattern will hold for any further rewriting tree in D. When implementing this reasoning, we take the observation subtree of D up to the third tree shown in Figure 2 as a sufficient set of rewriting trees to check guardedness of (otherwise infinite) D.

The rest of this section generalises and formalises this approach. In the next definition, we introduce the notion of a *clause projection* to talk about the process of "abstracting away" a pattern from an mgu  $\sigma$  by matching it with a subterm t of a clause head. When t also matches with a recursive reducing subterm of a loop in a rewriting tree, we call t a *coinductive invariant*.

**Definition 9 (Clause projection and coinductive invariant).** Let P be a program and A be an atom, and let D be a derivation tree for P and A in which a tree transition from T to T' is induced by an mgu  $\sigma$  of some P(k) and an atom B given by a leaf node T(u).

The clause projection for T', denoted  $\pi(T')$ , is the set of all triples (P(k), t, v), where t is a subterm of head (P(k)) at position v, such that the following conditions hold:  $\sigma(B) \triangleright B$  with variable reducing subterm t', and t' matches against t (i.e.  $t' = \sigma'(t)$  for some  $\sigma'$ ).

Additionally, the coinductive invariant at T', denoted ci(T'), is a subset of the clause projection for T', satisfying the following condition. An element  $(P(k), t, v) \in \pi(T')$  is also in ci(T'), if T contains a loop in the branch leading from T's root to T(u) that is guarded by (P(k), t'') for some t'' such that t'' matches against t  $(t'' = \theta(t)$  for some  $\theta)$ .

Given a program P, an atom A and a derivation tree D for P and A, the clause projection set for D is  $\operatorname{cproj}(D) = \bigcup_T \pi(T)$  and the coinductive invariant set for D is  $\operatorname{cinv}(D) = \bigcup_T \operatorname{ci}(T)$ , where these unions are taken over all rewriting trees T in D.

Example 10 (Clause projections and coinductive invariants). Coming back to Figure 2, the mgu for the first transition is  $\sigma_1 = \{X' \mapsto s(X), Y \mapsto scons(s(X), Y')\}$  (renaming of variables in  $P_6(0)$  with primes), that for the second is  $\sigma_2 = \{X'' \mapsto s(s(X)), Y' \mapsto scons(s(s(X)), Y'')\}$  (renaming of variables in  $P_6(0)$  with double primes), etc. Clause projections are given by  $\pi(T) = \{(P_6(0), scons(X, Y), 1)\}$  for all trees T in this derivation, and thus cproj(D) is the finite set. Moreover, for the first rewriting tree T,  $ci(T) = \emptyset$ , and  $ci(T') = \{(P_6(0), scons(X, Y), 1)\}$  for all trees T' except for the first one, so  $cinv(D) = \{(P_6(0), scons(X, Y), 1)\}$  is the finite set too.

The clause projections for the derivation of Figure 1 are  $\pi(T') = \pi(T'') = (P(1), \mathbf{s}(\mathtt{Y1}), 2)$ , and  $\pi(T'') = (P(0), \mathbf{s}(\mathtt{X1}), 0)$ , where T', T'', T''' refer to the second, third and fourth rewriting tree of that derivation. All coinductive invariants for that derivation are empty, since none of these rewriting trees contain guarded loops.

Generally, clause projection sets are finite, as the number of subterms in the clause heads of P is finite. This property is crucial for termination of our method:

**Proposition 2** (Finiteness of clause projection sets). Given a program P, an atom A, and a derivation tree D for P and A, the clause projection set cproj(D) is finite.

In particular, this holds for derivation trees induced by clause heads.

We terminate the construction of each branch of a derivation tree when we notice repeating coinductive invariant. A subtree we get as a result is an observation subtree. Formally, given a derivation tree D for a program P and an atom A, with a branch in which nodes D(w) and D(wv) are defined, if  $\operatorname{ci}(D(w)) = \operatorname{ci}(D(wv)) \neq \emptyset$ , then D has a guarded transition from D(w) to D(wv) (denoted  $D(w) \Longrightarrow D(wv)$ ). Every guarded transition thus identifies a repeated "consumer-producer" invariant in the derivation from D(w) to D(wv). This tells us that observation of this branch of D can be concluded. Imposing this condition on all branches of D gives us a general method to construct finite observation subtrees of potentially infinite derivation trees:

**Definition 10 (Observation subtree of a derivation tree).** If D is a derivation tree for a program P and an atom A, the tree D' is the observation subtree of D if

 the roots of D and D' are given by the rewriting tree for P and A, and
 if w is a node in both D and D', then the rewriting trees in D and D' at node w are the same and, for every child w' of w in D, the rewriting tree of D' at node w' exists

and is the same as the rewriting tree of D at w'. unless either

a) GC2 does not hold for D(w'), or

b) there exists a v such that  $D(v) \Longrightarrow D(w)$ .

In either case, D'(w) is a leaf node. We say that D' is unguarded if Condition 2a holds for at least one of D's nodes, and that D' is guarded otherwise.

A branch in an observation subtree is thus truncated when it reaches an unguarded rewriting tree or its coinductive invariant repeats. The observation subtree of any derivation tree is unique. The following proposition and lemma prove the two most crucial properties of observation subtrees: that they are always finite, and that checking their guardedness is sufficient for establishing guardedness of the whole derivation trees.

**Proposition 3 (Finiteness of observation subtrees).** If D is a derivation tree for a program P and an atom A then the observation subtree of D is finite.

Lemma 2 (Guardedness of observation subtree implies guardedness of derivation tree). If the observation subtree for a derivation tree D is guarded, then D is guarded.

Example 11 (Finite observation subtree of an infinite derivation tree). The initial fragment D' of the infinite derivation tree D given by the three rewriting trees in Figure 2 is D's observation subtree. The third rewriting tree T'' in D is the last node in the observation tree D' because  $ci(T') = ci(T'') = \{(P_6(0), scons(X, Y), 1)\} \neq \emptyset$ . Since D' is guarded, Lemma 2 above ensures that the whole infinite derivation tree D is guarded.

It now only remains to put the properties of the observation subtrees into practical use, and, given a program P, construct finite observation subtrees for each of its clauses. If none of these observation subtrees detects unguarded rewriting trees, we have guarantees that this program will never give rise to infinite rewriting trees. The next definition, lemmas and a theorem make this intuition precise.

**Definition 11 (Guarded clause, guarded program).** Given a program P, its clause P(i) is guarded if the observation subtree for the derivation tree for P and the atom head(P(i)) is guarded, and P(i) is unguarded otherwise. A program P is guarded if each of its clauses P(i) is guarded, and unguarded otherwise. We write GC3(P(i)) to indicate that P(i) is guarded, and similarly for P.

Lemma 3 uses Proposition 3 to show that GC3 is decidable.

Lemma 3 (GC3 is decidable). GC3 is a decidable property of logic programs.

**Theorem 1** (Universal observability is semi-decidable). If GC3(P) holds, then P is universally observable.

PROOF: If GC3(P) holds, then the observation subtree for each P(i) is guarded. Thus, by Lemma 2, the derivation tree for each P(i) is guarded. But then, by Lemma 1, P is universally observable. Combining this with Lemma 3, we also obtain that universal observability is semi-decidable.

The converse of Theorem 1 does not hold: the program comprising the clause  $p(a) \leftarrow p(X)$  is universally observable but not guarded, hence the above *semi*-decidability result.

From our check for universal observability we obtain the desired check for existential liveness, and thus for observational productivity:

**Corollary 1** (Observational productivity is semi-decidable). Let P be a guarded logic program. If there exists a clause P(i) such that the derivation tree D for P and P(i) has an observation subtree D' one of whose branches was truncated by Condition 2b of Definition 10, then P is existentially live. In this case, since P is also guarded and hence universally observable, P is observationally productive.

## 5 Related Work: Termination Checking in TRS and LP

Because observational productivity is a combination of universal observability and existential liveness, and the former property amounts to termination of all rewriting trees, there is an intersection between this work and termination checking in TRS [22,1,8].

Termination checking via transformation of LP into TRS has been given in [20]. Here we consider termination of restricted form of SLD-resolution (given by rewriting derivations), therefore a much simpler method of translation of LP into TRS can be used for our purposes [6]: Given a logic program P and a clause  $P(i) = A \leftarrow B_1, \ldots, B_n$  containing no existential variables, we define a rewrite rule  $A \rightarrow f_i(B_1, \ldots, B_n)$  for some fresh function symbol  $f_i$ . Performing this translation for all clauses, we get a translation from P to a term-rewriting system  $\mathcal{T}_P$ . Rewriting derivations for P can be shown operationally equivalent to term-rewriting reductions for  $\mathcal{T}_P$ ; see [6] for a proof. Therefore, for logic programs containing no existential variables, any termination method from TRS may be applied to check universal observability (but not existential liveness).

Algorithmically, our guardedness check compares directly with the method of dependency pairs due to Arts and Giesl [1,8]. Consider again  $\mathcal{T}_P$  obtained from a program P. The set R of dependency pairs contains, for each rewrite rule  $A \to f_i(B_1, \ldots, B_n)$ in  $\mathcal{T}_P$ , a pair  $(A, B_i), j = 1, \ldots, n$ ; see [6]. The method of dependency pairs consists of checking whether there exists an infinite chain of dependency pairs  $(s_i, t_i)_{i=1,2,3,...}$  such that  $\sigma_i(t_i) \to^* \sigma_{i+1}(s_{i+1})$ . If there is no such infinite chain, then  $\mathcal{T}_P$  is terminating. Again this translation from LP to dependency pairs in TRS is simpler than in [15], as rewriting derivations are a restricted form of SLD-resolution. Due to the restricted syntax of  $\mathcal{T}_P$ (compared to the general TRS syntax), generating the set of dependency pairs is equivalent to generating a set of rewriting trees for each clause of P and assuming  $\sigma_i = \sigma_{i+1}$  (cf. our GC2). To find infinite chains, a dependency graph is defined, in which dependency pairs are nodes and arcs are defined whenever a substitution that allows a transition from one pair to another can be found. Finding such substitutions is the hardest part algorithmically. Note that every pair of neighboring and-nodes in a rewriting tree corresponds to a node in a dependency graph. Generating arcs in a dependency graph is equivalent to using our GC3 to find a representative set of substitutions. However, the way GC3 generates such substitutions via rewriting tree transitions differs completely from the methods approximating dependency graphs [1,22], and relies on the properties of S-resolution, rather than recursive path orderings. This is because GC3 additionally generates coinductive invariants for checking existential liveness of programs.

Conceptually, observational productivity is a new property that does not amount to either termination or nontermination in LP or TRS. E.g. programs  $P_3$  and  $P_4$  are nonterminating (seen as LP or TRS), and  $P_8: p(X) \leftarrow q(Y)$  is terminating (seen as LP and TRS) but none of them is productive. This is why the existing powerful tools (such as AProVE) and methods [1,8,15,20] that can check termination or nontermination in TRS or LP are not sufficient to serve as productivity checks. To check *termination* of rewriting trees, GC3 can be substituted by existing termination checkers for TRS, but none of the previous approaches can semi-decide existential liveness as GC3 does.

## 6 Implementation and Applications

We implemented the observational productivity checker in parallel Go (golang.org) [19], which allows to experiment with parallelisation of proof search [10]. Loading a logic

program P, one runs a command line to initialise the GC3 check. The algorithm then certifies whether or not the program is guarded (and hence universally observable). If that is the case, it also checks whether GC3 found valid coinductive invariants, i.e. whether P is existentially live and hence admits coinductive interpretations for some predicates. Appendix B (available in online version) gives further details.

In the context of S-resolution [11,9], observational productivity of a program is a pre-condition for (coinductive) soundness of S-resolution derivations. This gives the first application for the productivity checker. But the notion of global productivity (as related to *computations at infinity* [14]) is a general property tracing its roots to the 1980s. A program is productive, if it admits SLD- or S-resolution derivations that compute (or produce) an infinite term at infinity. Thus the productivity checker has more general practical significance for Prolog. In this paper we further exposed its generality by showing that productivity can be seen as a general property of logic programs, rather than property of derivations in some special dialect of Prolog.

Based on this observation, we identify three applications for productivity checks encompassing the S-resolution framework. (1) In the context of CoLP [7,21] or any other similar tool based on loop detection in SLD-derivations, one can run the observational productivity checker for a given program prior to running the usual interpreter of CoLP. If the program is certified as productive, all computations by CoLP for this program will be sound relative to the computations at infinity [14]. It gives a way to characterise a subset of theorems proven by CoLP that describe the process of production of infinite data. I.e., as explained in Introduction, CoLP will return answers for programs  $P_3$ ,  $P_4$  and  $P_5$ . But if we know that only  $P_5$  is productive, we will know that only CoLP's answers for  $P_5$  will correspond to production of infinite terms at infinity.

(2) As our productivity checker also checks *liveness* of programs, it effectively identifies which predicates may be given coinductive semantics. This knowledge can be used to type predicates as inductive or coinductive. We can use these types to mark predicates in CoLP or any other coinductive dialect of logic programming, cf. Appendix B.

(3) Observational productivity is also a guarantee that a sequence of mgus approximating the infinite answer can be constructed *lazily* even if the answer is irregular. E.g. our running example of program  $P_6$  is irrational and hence cannot be handled by CoLP's loop detection. But even if we cannot form a closed-term answer for a query from(0, X), the productivity checker gives us a weaker but more general certificate that lazy approximation of our infinite answer is possible.

These three groups of applications show that the presented productivity checker can be implemented and applied in any dialect of logic programming, irrespective of the fact that it initially arose from S-resolution research [11,9].

## 7 Conclusions

In this paper we have introduced an observational counterpart to the classical notion of global productivity of logic programs. Using the recently introduced formalism of Sresolution, we have defined observational productivity as a combination of two program properties, namely, universal observability and existential liveness. We have introduced an algorithm for semi-deciding observational productivity for any logic program. We did not impose any restrictions on the syntax of logic programs. In particular, our algorithm handles both existential variables and non-linear recursion. The algorithm relies on the observation that rewriting trees for productive and guarded programs must show term reduction relative to a contraction ordering from their roots to their leaves. But S-resolution derivations involving such trees can only proceed by adding term structure back in transitioning to new rewriting trees via mgus. This "producer/consumer" interaction can be formally traced by observing a derivation's coinductive invariants: these record exactly the term patterns that both reduce in the loops of rewriting trees and are added back in transitions between these trees.

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## Appendix A. Proofs

### Proof of Proposition 1: GC2 is decidable

Any rewriting tree T is either finite or infinite. If T is finite, then its guardedness is clearly decidable. So, we may, without loss of generality, assume T is infinite. If T is infinite then it must have an infinite branch B. We now show that an infinite branch B in T must necessarily contain an unguarded loop. Thus, whether T is finite or infinite, its guardedness is decidable.

Assume B has only guarded loops, but it is infinite. Since the number of clauses and the number of function symbols in  $\Sigma$  are finite, B must contain an infinite number of loops. Consider one such infinite sequence  $\mathbf{q}(t_{11}, ..., t_{1j}) \to ... \to \mathbf{q}(t_{k1}, ..., t_{kj}) \to ... \to$  $\mathbf{q}(t_{l1}, ..., t_{lj}) \to ...$ , where  $\mathbf{q}(t_{11}, ..., t_{1j})$ ,  $\mathbf{q}(t_{k1}, ..., t_{kj})$ ,  $\mathbf{q}(t_{l1}, ..., t_{lj})$ , ... are all atoms with the same predicate  $\mathbf{q}$  obtained by rewriting using clause P(i). Because all loops in B are guarded, we have  $\mathbf{q}(t_{11}, ..., t_{1j}) \triangleright \mathbf{q}(t_{k1}, ..., t_{kj})$ ,  $\mathbf{q}(t_{11}, ..., t_{1j}) \triangleright \mathbf{q}(t_{l1}, ..., t_{lj})$ ,... But since  $\mathbf{q}(t_{11}, ..., t_{1j})$  is finite, there are only finitely many ways to construct a reducing subterm on it. Thus, there will be a point when some terms  $\mathbf{q}(t_{m1}, ..., t_{mj})$  and  $\mathbf{q}(t_{n1}, ..., t_{nj})$ in the infinite sequence in B have the same recursive reducing subterm  $t^*$  relative to  $\mathbf{q}(t_{11}, ..., t_{1j})$ .

Now, there are two cases: i)  $\mathbf{q}(t_{11}, ..., t_{1j}) \triangleright \mathbf{q}(t_{m1}, ..., t_{mj})$  and  $\mathbf{q}(t_{11}, ..., t_{1j}) \triangleright \mathbf{q}(t_{n1}, ..., t_{nj})$ hold, but  $\mathbf{q}(t_{m1}, ..., t_{mj}) \triangleright \mathbf{q}(t_{n1}, ..., t_{nj})$  does not, and ii) the negation of this case.

- If  $\mathbf{q}(t_{11}, ..., t_{1j}) \triangleright \mathbf{q}(t_{m1}...t_{mj})$  and  $\mathbf{q}(t_{11}, ..., t_{1j}) \triangleright \mathbf{q}(t_{n1}, ..., t_{nj})$  hold, but not  $\mathbf{q}(t_{m1}, ..., t_{mj}) \triangleright \mathbf{q}(t_{n1}, ..., t_{nj})$ , then  $\mathbf{q}(t_{m1}, ..., t_{mj}) \triangleright \mathbf{q}(t_{n1}, ..., t_{nj})$  is an unguarded loop, which contradicts the assumption that all loops in B are guarded.
- If the negation holds i.e., if  $\mathbf{q}(t_{m1}, ..., t_{mj}) \triangleright \mathbf{q}(t_{n1}, ..., t_{nj})$  holds or  $\mathbf{q}(t_{11}, ..., t_{1j}) \triangleright$  $\mathbf{q}(t_{m1},...,t_{mj})$  does not hold or  $\mathbf{q}(t_{11},...,t_{1j}) \triangleright \mathbf{q}(t_{n1},...,t_{nj})$  does not hold — then there are three cases. If either  $q(t_{11}, ..., t_{1j}) \triangleright q(t_{m1}, ..., t_{mj})$  or  $q(t_{11}, ..., t_{1j}) \triangleright q(t_{n1}, ..., t_{nj})$ does not hold, then the existence of this unguarded loop in B gives a contradiction. So we need only consider the case when  $\mathbf{q}(t_{11},...,t_{1j}) \triangleright \mathbf{q}(t_{m1},...,t_{mj}), \mathbf{q}(t_{11},...,t_{1j}) \triangleright$  $q(t_{n1},...,t_{nj})$ , and  $q(t_{m1},...,t_{mj}) \triangleright q(t_{n1},...,t_{nj})$  are all guarded loops in B. Let  $t^{**}$ be the recursive reducing subterm for the loop  $q(t_{m1},...,t_{mj}) \triangleright q(t_{n1},...,t_{nj})$ . Since the same recursive reducing subterm cannot be contracted twice along the same path from  $q(t_{11}, ..., t_{1j})$ , we must have that  $t^* \neq t^{**}$  and, moreover, contracting  $t^{**}$ must somehow "restore"  $t^*$  to  $q(t_{11}, ..., t_{1j})$ . And this means that  $t^*$  and  $t^{**}$  must be "independent", in the sense of being on independent paths in  $q(t_{11}, ..., t_{1i})$ . But then there will be cycles of terms in B in which one argument of q decreases in one step and another independent one grows, and then the first argument grows while the other one decreases. So  $q(t_{11}, ..., t_{ij})$  will appear in the infinite branch infinitely many times (so definitely more than once!), and B will thus contain an unguarded loop in this case as well. This is again a contradiction.

# Proof of Lemma 1: Guardedness of derivation trees implies universal observability

If P is not universally observable, then there exists an atom A such that the rewriting tree T for P and A is infinite. Moreover, A must match some clause head(P(i)) via a mgm  $\theta$ ,

so in fact T is an infinite rewriting tree for P and  $\theta(head(P(i)))$ , with additional condition that  $\theta$  is also applied to all atoms of this tree. Then, as in the proof of Proposition 1, there must exist an unguarded loop L on an infinite branch B of T. We claim that, if we construct a derivation tree  $D_i$  for the program P and head(P(i)), then some rewriting tree in  $D_i$  will contain an unguarded loop. Let us consider the construction of  $D_i$ .

If the first rewriting tree of  $D_i$ , i.e. the tree T' for P and head(P(i)) does not itself contain an unguarded loop, then the branch in T' corresponding to B in T must have a leaf node T'(w) given by an atom that unifies with a clause  $P(k_1)$  via mgu  $\sigma_1$ , say. Moreover,  $P(k_1)$  is exactly the clause used to construct a node T(wi) of B in T via its mgm with T(w). Now, consider the rewriting tree transition determined by the mgu  $\sigma_1$ , i.e. consider  $T' \to T'_w$ . If the branch corresponding to B in  $T'_w$  does not contain an unguarded loop, then it too must have a leaf node  $T'_w(u)$  that unifies with  $P(k_2)$  via mgu  $\sigma_2$ , say, and  $P(k_2)$  is exactly the clause used to construct a node  $T(v_j)$  of B in T via its mgm with T(v). And so on. After some finite number n of tree transitions, of "growing" the branch corresponding to B in T' by taking further mgu's on its leaves, we must come to a rewriting tree  $T^*$  for P and the root atom  $\sigma(head(P(i)))$  in  $D_i$  that contains an unguarded loop corresponding to L, and where  $\sigma = \sigma_n \circ \dots \circ \sigma_1$  for the mgu's  $\sigma_1, ..., \sigma_n$  involved in the tree transitions in  $D_i$ . Indeed, since branches B of T and  $T^*$ are constructed using mgm's with exactly the same clauses at each step, and since the mgu's  $\sigma_1, \ldots, \sigma_n$  are all most general unifiers, we must have that  $\sigma$  is more general than  $\theta$ , and thus  $T^*$  is a more general version of T, and so contains an unguarded loop that is a more general version of L.

#### **Proof of Proposition 3: Finiteness of observation subtrees**

Let D' be the observation subtree of the derivation tree D for a program P and an atom A. If D is finite, then D' will necessarily be finite, so we may, without loss of generality, suppose D is infinite.

If there exists a rewriting tree in D that is unguarded, then, by Condition 2a of Definition 10, the branch of D on which that tree appears will end at that tree in D' and will thus be finite. For D' to be infinite, there must exist an infinite branch of D containing only guarded rewriting trees such that coinductive invariants computed in that branch never repeat. In fact, every infinite branch of D' must satisfy these two conditions.

Let T be any guarded rewriting tree on any infinite branch of D. We first note that the coinductive invariant ci(T) must be non-empty. In addition, T must itself be finite. Indeed, if T were infinite then, by the completeness of breadth-first search, an unguarded rewriting tree would have to exist at some finite depth on T's branch of D. Then, by the argument of the preceding paragraph, T's branch of D would have to be finite. But this is not the case.

So T must be a finite, guarded rewriting tree appearing on an infinite branch of D. Now, although D itself is infinite, Proposition 2 ensures that D's coinductive invariant set still contains only finitely many clause projections, so any branch of D can add only finitely many distinct elements to D's coinductive invariant set. In particular, the coinductive invariants for nodes on T's infinite branch of D must eventually all be equal. Moreover, since  $ci(T) \neq \emptyset$ , these coinductive invariants must eventually all be non-empty. Thus Condition 2b of Definition 10 must eventually be satisfied and the branch of D' corresponding to T's branch in D must thus be finite. Having argued that the branch of D' corresponding to any infinite branch of D is finite, we have that D' is itself finite.

# Proof of Lemma 2: Guardedness of observation subtree implies guardedness of derivation tree

The proof proceeds by induction-coinduction. We assume the observation subtree D' for D is guarded and inductively examine every branch B' of D'. This is possible because the number of such branches and their lengths are all finite by Proposition 3. For any such B', either no parent of any leaf in the last coinductive tree of B' can be resolved with any clause of P, or B' was terminated by Condition 2b of Definition 10. In the former case, the entire branch B' will also appear in D, and each rewriting tree on the corresponding branch B of D will be guarded. In the latter case, we can proceed coinductively.

If B' was terminated because it contains a guarded transition  $T \Longrightarrow T'$  for T = D(w)and T' = D(wv), then both T and T' were formed by resolving with some clause P(k). In this case, we apply the following coinductive argument. Coinductive Hypothesis (CH): The process of resolving with clause P(k) to produce a new guarded rewriting tree whose coinductive invariant has first component P(k) can be repeated infinitely many times in transition sequences originating from T. By computing that CH is again satisfied for T', we can make the following Coinductive Conclusion (CC): For any tree T in any branch B containing B', the process of resolving with clause P(k) to produce a new guarded rewriting tree whose coinductive invariant has first component P(k) can be repeated infinitely many times in transition sequences originating from T. So each of the rewriting trees in B must be guarded.

Unfortunately, CC does not guarantee that no unguarded loop can possibly occur in D by resolving with other clauses in the sequence of transitions from T' that occur in B but not in B'. But if it is possible to compute a sequence of rewriting tree transitions in D from T' involving mgus  $\theta_1, \ldots, \theta_n$  computed by resolving with clauses  $P(k_1), \ldots, P(k_n)$  that lead to an unguarded rewriting tree in B, then, by completeness of the breadth-first construction of the derivation tree D, there must be a rewriting tree  $T^*$  occurring in the sequence of rewriting trees in B' from T to T' that leads to a sequence of rewriting tree transitions in another branch B'' of D involving exactly the same sequence  $P(k_1), \ldots, P(k_n)$  of clauses and mgus  $\theta'_1, \ldots, \theta'_n$  such that, for each  $i \in \{1, \ldots, n\}, \theta'_i = \sigma_i \circ \theta_i$  for some  $\sigma_i$ . This holds because rewriting tree transitions only lead to further instantiations of variables, and the rewriting tree  $T^*$  appears earlier on B than T' does, and hence is more general. But then an unguarded loop induced by the mgus  $\theta'_1, \ldots, \theta'_n$  obtained by resolving with  $P(k_1), \ldots, P(k_n)$  will be found in one of the branches of D' to which  $T^*$  leads.

By inducting on all branches of the observation subtree D' of D, and coinductively terminating each, we conclude that if all branches of D' are terminated by the above coinductive argument with no unguarded rewriting tree being found, then no unguarded loop can exist in any of the rewriting trees of D.

### Proof of Lemma 3: GC3 is decidable

To decide guardedness of logic programs, we must let P be given and construct a set of derivation trees, one derivation tree for each clause head of P, i.e. every such  $D_i$ is a derivation tree for P and head(P(i)). Moreover, we build these trees only until we construct the observation subtree  $D'_i$  for each  $D_i$ . We next check whether or not each observation subtree  $D'_i$  is guarded. That is, we must check whether or not every rewriting tree in  $D'_i$  is guarded and whether or not condition 2.a of Definition 10 was used to construct  $D'_i$ . Since guardedness of rewriting trees is decidable by Proposition 1, and there are only finitely many rewriting trees in any observation tree  $D'_i$ , guardedness of all observation subtrees  $D'_i$  for this program is decidable. Since, by Lemma 2, guardedness of observation subtrees implies guardedness of derivation trees, guardedness of P is also decidable.

## Appendix B. Implementation of Observational Productivity Checks

### Algorithmic overview of observational productivity checking

Definitions of contraction ordering, guarded rewriting trees, and observation subtrees translate naturally into algorithmic forms that give rise to the implementation of our observational productivity checks [19]. Below we give a high-level pseudocode representation of the formal definitions of this paper.

Algorithm 1 below captures the essence of our check that GC3(P) holds for a logic program P. It depends on the definition of the observation subtree (Definition 10), which in turn depends on two conditions:

- finiteness of observation subtrees, as proven in Proposition 3, and
- guardedness of every rewriting tree in a program's observation subtree.

These two conditions ensure termination of Algorithm 1, as expressed formally in Lemma 3. In the main body of this paper we have written GC2(T) to indicate that a rewriting tree T is guarded. A pseudocode description of our check that GC2(T) holds for a rewriting tree T is given in Algorithm 2.

Termination of Algorithm 2 depends crucially on Proposition 1, i.e., on the fact that it is impossible to construct an infinite rewriting tree without finding unguarded loops. Algorithm 2 in turn relies on an algorithmic check that two terms are related via a contraction ordering, but we omit specifying this in pseudocode since it is entirely straightforward.

### Implementation

Our observational productivity checker is implemented in Go (golang.org) as a command line program and is part of the general implementation of structural resolution and coalgebraic logic programming (CoALP) [19]. Go was chosen as implementation language because it provides easy primitives for parallelization, which has been explored

Algorithm 1 Observational productivity check for a logic program

```
Require: P – a logic program over signature \Sigma
Require: LC – an empty list
  n = number of clauses in P
 for i = 0, \ldots, n do
     if observation subtree D' of the derivation tree D for P and head(P(i)) is unguarded then
          P(i) is not guarded.
     \mathbf{else}
          P(i) is guarded.
         if D' contains transition D(v) \Longrightarrow D(w) with coinductive invariant c then
            LC := append(LC, c)
         end if
     end if
  end for
  if all P(i) are guarded then
     Result 1 := "P is guarded"
  else
     Result1 := "P is not guarded"
  end if
  if LC is not empty then
     Result2 := "P is existentially live with LC"
  else
     Result2 := "P has finite derivations only"
  end if
  return (Result1, Result2)
```

### Algorithm 2 Guardedness check in a rewriting tree

<b>Require:</b> $T$ – the rewriting tree for a logic program $P$ and an atom $A$
for $i = 0, \dots, depth(T)$ do
for nodes $w_1, \ldots w_m$ at depth $i$ do
if a node $w_j$ forms a loop with some node $v$ above it then
if $loop(T, v, w_j)$ is not guarded <b>then</b>
<b>return</b> " $T$ is not guarded"
end if
end if
end for
end for
<b>return</b> " $T$ is guarded"

to optimize proof search [10]. To compile and install the productivity checker follow the instructions in the README file supplied in the program distribution available at [19].

CoALP can be used not only to check the productivity of logic programs, but to make queries to guarded such programs as well. The checker takes Prolog-style programs saved in text files as input. The format of programs corresponds exactly to that of Prolog. For example, program  $P_6$  is represented as

```
from(X, scons(X, Y)) :- from(s(X), Y).
```

Unlike Prolog, our checker does not support built-in predicates or arithmetic functions.

To check a logic program for observational productivity, the path to the program file has to be given as the first parameter:

### guardcheck somefile.logic

The above command initialises the GC3(P) check for a given logic program P in somefile.logic, and, as GC3 involves computations of coinductive invariants, it simultaneously uses them to detect existential liveness, as detailed in Algorithm 1. Many example files and tests as well as the programs used in this paper can be found in the directory named "examples" in [19].

The output for the observationally productive program  $P_6$  is:

```
Program is guarded.
Program is existentially live with coinductive invariants:
in clause 0 of "from": [{0 | scons(v3,v5) | [1]}]
```

Note that the first 0 in [0 | scons(v3,v5) | [1]] points to the clause  $P_6(0)$ , and would suggest that the predicate from in the head of this clause is a good candidate to be given coinductive semantics and hence coinductive typing. We believe that this general information can be used by CoLP or CoALP to determine typing for coinductive predicates in their programs.

The output for the unguarded program  $P_7$  is:

```
Program is not guarded.
Goal q(s(v34),s(v34),s(v42),v36) results in unguarded loop
in path [(p:0), (q:0), (p:0)].
```

### A more complex example

In this section, we consider a more challenging example of *Sieve of Eratosthenes*, known for its difficulty in the literature on coinductive definitions [7]. The following program  $P_9$  is an observationally productive reformulation of the original *Sieve of Eratothenes* program from [7]:

 $0. prime(X) \leftarrow inflist(I), sieve(I,L), member(X,L)$ 

1.  $sieve(cons(H,T), cons(H,R)) \leftarrow filter(H,T,F), sieve(F,R)$ 

2.  $filter(H, cons(K, T), cons(K, T1)) \leftarrow mod(X, K, H), less(0, X), filter(H, T, T1)$ 

3.  $filter(H, cons(K, T), T1) \leftarrow mod(0, K, H), filter(H, T, T1)$ 

4.  $int(X, cons(X, Y)) \leftarrow int(s(X), Y, Z1)$ 

5.  $\inf$ list(I)  $\leftarrow \inf(s(s(0)), I)$ 6. member(X,  $cons(X, L)) \leftarrow$ 7. member(X,  $cons(Y, L)) \leftarrow$  member(X, L) 8.  $less(0, s(X)) \leftarrow$ 9.  $less(s(X), s(Y)) \leftarrow less(X, Y)$ 

The original program [7] does not use Prolog-style list notations and uses a structural representation of numbers, which we avoid. We also assume a suitable implementation of the modulo operator as the predicate mod above.

If we run the observational productivity check on this program, we obtain the following output:

```
Program is guarded.
Program is existentially live with coinductive invariants:
in clause 0 of "filter": [{0 | cons(v16,v17) | [1]} {0 | cons(v16,v18) | [2]}]
in clause 1 of "filter": [{1 | cons(v24,v25) | [1]}]
in clause 0 of "sieve": [{0 | cons(v10,v11) | [0]} {0 | cons(v10,v12) | [1]}]
in clause 1 of "member": [{1 | cons(v37,v38) | [1]}]
in clause 1 of "less": [{1 | s(v40) | [0]} {1 | s(v41) | [1]}]
```

Above, four predicates have been identified as potentially having a coinductive semantics: filter, sieve, member and less. Generally, most inductive definitions admit coinductive interpretation, and predicates that we intuitively consider as inductive may be identified as potentially coinductive. This situation was analysed in Example 2. Among the four predicates, sieve, that admits only coinductive interpretation, was identified.

We note that the original formulation of [7] is not observationally productive, since it does not possess universal observability property. If we run our checker on the formulation in [7], failure of observational productivity is detected and reported as follows:

```
Program is not guarded.
Goal comember(v136,v140) results in unguarded loop
in path [(primes:0), (comember:0), (comember:0)].
```

As indicated by the checker output, the reason is that the following definition of *comember* used in [7] is not universally observable and hence is not guarded:

- 0. comember(X,L)  $\leftarrow$  drop(X,L,L1), comember(X,L1)
- 1.  $drop(H, cons(H, T), T) \leftarrow$
- 2.  $drop(H, cons(H, T), T) \leftarrow drop(H, T, T1)$

Indeed, the definition of comember in Clause 0 above is not guarded by any constructors.

In our reformulation as program  $P_9$  above, we use a guarded definition of member instead of the definitions of comember and drop used in [7]. The definition of member is guarded by the constructor cons in Clause 7. Thus, in the case of the Sieve of Eratothsenes, the transition from an unproductive to a productive coinductive definition was a simple matter of applying a program transformation that clearly preserves the intended coinductive meaning of the coinductive definition of seive in Clause 1.