

Minimal Forbidden Factors of Circular Words[☆]

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Abstract

Minimal forbidden factors are a useful tool for investigating properties of words and languages. Two factorial languages are distinct if and only if they have different (antifactorial) sets of minimal forbidden factors. There exist algorithms for computing the minimal forbidden factors of a word, as well as of a regular factorial language. Conversely, Crochemore et al. [IPL, 1998] gave an algorithm that, given the trie recognizing a finite antifactorial language M , computes a DFA recognizing the language whose set of minimal forbidden factors is M . In the same paper, they showed that the obtained DFA is minimal if the input trie recognizes the minimal forbidden factors of a single word. We generalize this result to the case of a circular word. We discuss several combinatorial properties of the minimal forbidden factors of a circular word. As a byproduct, we obtain a formal definition of the factor automaton of a circular word. Finally, we investigate the case of minimal forbidden factors of the circular Fibonacci words.

Keywords: Minimal forbidden factor; finite automaton; factor automaton; circular word; Fibonacci words.

1. Introduction

Minimal forbidden factors are a useful combinatorial tool in several areas, ranging from symbolic dynamics to string processing. They have many applications, e.g. in text compression (where they are also known as *antidictionaries*) [13], in bioinformatics (where they are also known under the name *minimal absent words*) [7, 1], etc. Given a word w , a word v is called a *minimal forbidden factor* of w if v does not appear as a factor in w but all the proper factors of v do. For example, over the alphabet $A = \{a, b\}$, the word $w = aabbabb$ has the following minimal forbidden factors: $aaa, aba, baa, babba, bbb$.

The theory of minimal forbidden factors is well developed, both from the combinatorial and the algorithmic point of view (see, for instance, [4, 3, 12, 13, 22, 17]). In particular, there exist algorithms for computing the minimal forbidden factors of a single word [25, 19, 1, 2], as well as of a regular factorial language [3]. Conversely, Crochemore et al. [12], gave an algorithm, called L-AUTOMATON that, given a trie (tree-like automaton) recognizing a finite antifactorial set M , builds a deterministic automaton recognizing the language L whose set of minimal forbidden factors is M . The automaton built by the algorithm is not, in general, minimal. However, if M is the set of minimal forbidden factors of a single word w , then the algorithm builds the factor automaton of w , i.e., the minimal deterministic automaton recognizing the language of factors of w (see [12]).

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The notion of a minimal forbidden factor has been recently extended to the case of circular words (a.k.a. necklaces) [10, 23, 24]. A circular word can be seen as a sequence of symbols drawn on a circle, where there is no beginning and no end. Although a circular word can be formally defined as an equivalence class of the free monoid under the relation of conjugacy, the fact that in a circular word there is no beginning and no end leads to a less clear definition of the notions like prefix, suffix and factor. For this reason, we consider the set of factors of a circular word w as the (infinite) set of words that appear as a factor in some power of w . Although this set is infinite, we show that its set of minimal forbidden factors is always finite, as it coincides with the set of minimal forbidden factors of the word ww that have length bounded by the length of w .

As a main result, we prove that if M is the set of minimal forbidden factors of a circular word, then algorithm L-AUTOMATON with input a trie recognizing M builds the minimal automaton accepting the set of factors of the circular word. To this end, we use combinatorial properties of the minimal forbidden factors of a circular word. This also allows us to show that it is possible to retrieve a circular word from its set of minimal forbidden factors in linear time with respect to the length of (any linearization of) the word.

Finally, we explore the case of circular Fibonacci words, and give a combinatorial characterization of their minimal forbidden factors.

2. Preliminaires

Let A be a finite alphabet, and let A^* be the free monoid generated by A under the operation of concatenation. The elements of A^* are called *words* over A . The *length* of a word w is denoted by $|w|$. The *empty word*, denoted by ε , is the unique word of length zero and is the neutral element of A^* . If $x \in A$ and $w \in A^*$, we let $|w|_x$ denote the number of occurrences of x in w .

A *prefix* (resp. a *suffix*) of a word w is any word u such that $w = uz$ (resp. $w = zu$) for some word z . A *factor* of w is a prefix of a suffix (or, equivalently, a suffix of a prefix) of w . From the definitions, we have that ε is a prefix, a suffix and a factor of any word. A prefix/suffix/factor of a word is *proper* if it is nonempty and does not coincide with the word itself. An *occurrence* of a factor u in w is a factorization $w = vuz$. An occurrence of u is *internal* if both v and z are nonempty. The set of factors of a word w is denoted by \mathcal{F}_w .

The word \tilde{w} obtained by reading w from right to left is called the *reversal* (or *mirror image*) of w . A *palindrome* is a word w such that $\tilde{w} = w$. In particular, the empty word is a palindrome.

The *conjugacy* is the equivalence relation over A^* defined by

$$w \sim w' \text{ if and only if } \exists u, v \mid w = uv, w' = vu.$$

When the word w is conjugate to the word w' , we say that w is a *rotation* of w' . An equivalence class $[w]$ of the conjugacy relation is called a *circular word* (or *necklace*). A representative of a conjugacy class $[w]$ is called a *linearization* of the circular word $[w]$. Therefore, a circular word $[w]$ can be viewed as the set of all the rotations of a word w .

A word w is a *power* of a word v if there exists a positive integer $k > 1$ such that $w = v^k$. Conversely, w is *primitive* if $w = v^k$ implies $k = 1$. Notice that a word is primitive if and only if any of its rotations also is. We can therefore extend the definition of primitivity to circular words straightforwardly. Notice that a word w (resp. a circular word $[w]$) is primitive if and only if there are precisely $|w|$ distinct rotations in the conjugacy class of w .

Remark 1. A circular word can be seen as a word drawn on a circle, where there is no beginning and no end. Therefore, the classical definitions of prefix/suffix/factor of a word lose their meaning for a circular

word. In the literature, a factor of a circular word $[w]$ is often defined as a factor of any linearization w of $[w]$. Nevertheless, since there is no beginning and no end, one can define a factor of w as a word that appears as a factor in w^k for some k . We will adopt this point of view in this paper.

2.1. Minimal Forbidden Factors

We now recall some basic facts about minimal forbidden factors. For further details and references, the reader may see [22, 10].

A language over the alphabet A is a set of finite words over A , that is, a subset of A^* . A language is *factorial* if it contains all the factors of its words. The *factorial closure* of a language L is the language consisting of all factors of the words in L , that is, the language $\mathcal{F}_L = \cup_{w \in L} \mathcal{F}_w$.

The counterparts of factorial languages are antifactorial languages. A language is called *antifactorial* if no word in the language is a proper factor of another word in the language. Dual to the notion of factorial closure, there also exists the notion of *antifactorial part* of a language, obtained by removing the words that are factors of another word in the language.

Definition 1. Given a factorial language L over the alphabet A , the (antifactorial) language of minimal forbidden factors of L is defined as

$$\mathcal{M}_L = \{a \in A \mid a \notin L\} \cup \{aub \in A^* \mid a, b \in A, aub \notin L, au, ub \in L\}.$$

A minimal forbidden factor of the language L is therefore a word in \mathcal{M}_L .

Every factorial language L is uniquely determined by its (antifactorial) language of minimal forbidden factors \mathcal{M}_L , through the equation

$$L = A^* \setminus A^* \mathcal{M}_L A^*. \quad (1)$$

The converse is also true, since by the definition of a minimal forbidden factor we have

$$\mathcal{M}_L = AL \cap LA \cap (A^* \setminus L). \quad (2)$$

The previous equations define a bijection between factorial and antifactorial languages.

In the case of a single word w over an alphabet A , the set of minimal forbidden factors of w , that we denote by \mathcal{M}_w , is defined as the antifactorial language $\mathcal{M}_{\mathcal{F}_w}$. Indeed, a word $v \in A^*$ is a minimal forbidden factor of the word w if v is a letter of A not appearing in w or $v = aub$, with $a, b \in A$, $aub \notin \mathcal{F}_w$ and $au, ub \in \mathcal{F}_w$.

For example, consider the word $w = aabbabb$ over the alphabet $A = \{a, b\}$. The set of minimal forbidden factors of w is $\mathcal{M}_w = \{aaa, aba, bbb, baa, babba\}$.

Remark 2. Applying (1) and (2) to the language of factors of a single word, we have that, given two words u and v , one has $u = v$ if and only if $\mathcal{M}_u = \mathcal{M}_v$, i.e., every word can be uniquely represented by its set of minimal forbidden factors.

An important property of the minimal forbidden factors of a word w , which plays a crucial role in algorithmic applications, is that their number is linear in the size of w . Let w be a word of length n over an alphabet A of cardinality σ . In [22] it is shown that the total number of minimal forbidden factors of w is smaller than or equal to σn . Actually, $\mathcal{O}(\sigma n)$ is a tight asymptotic bound for the number of minimal forbidden factors of w whenever $2 \leq \sigma \leq n$ [10]. They can therefore be stored on a trie, whose number of nodes is linear in the size of the word. Recall that a *trie* representing a finite language L is a tree-like deterministic automaton recognizing L , where the set of states is the set of prefixes of words in L , the initial state is the empty word ε , the set of final states is a set of *sink* states, and the set of transitions is $\{(u, a, ua) \mid a \in A\}$.

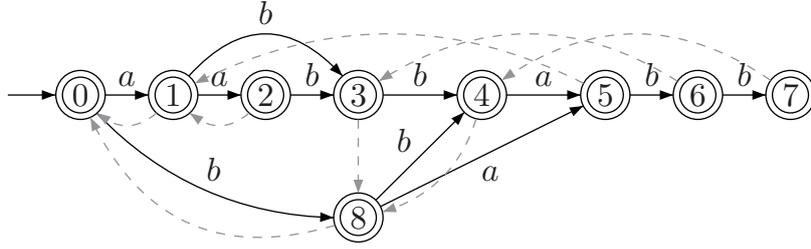


Figure 1: The factor automaton of the word $w = aabbabb$. It is the minimal DFA recognizing the language \mathcal{F}_w of factors of w . Dashed edges correspond to the failure function links.

2.2. Automata for Minimal Forbidden Factors

Recall that a *deterministic finite automaton* (DFA) is a 5-tuple $\mathcal{A} = (Q, A, i, T, \delta)$, where Q is the finite set of states, A is the current alphabet, i is the initial state, T the set of terminal (or final) states, and $\delta : (Q \times A) \mapsto Q$ is the transition function. A word is *recognized* (or *accepted*) by \mathcal{A} if reading w character by character from the initial state leads to a final state. The language recognized (or accepted) by \mathcal{A} is the set of all words recognized by \mathcal{A} . A language is *regular* if it is recognized by some DFA. A DFA \mathcal{A} is *minimal* if it has the least number of states among all the DFA's recognizing the same language as \mathcal{A} . The minimal DFA is unique.

It follows from basic closure properties of regular languages that the bijection between factorial and antifactorial languages expressed by (1) and (2) preserves regularity, i.e., a factorial language is regular if and only if its language of minimal forbidden factors also is.

The *factor automaton* of a word w is the minimal DFA recognizing the (finite) language \mathcal{F}_w . The factor automaton of a word of length $n > 3$ has at least¹ $n + 1$ and at most $2n - 2$ states [9, 6]. It can be built in $\mathcal{O}(n)$ time and space by an algorithm that also constructs the *failure function* of the automaton [11]. The failure function of a state p (different from the initial state) is a link to another state q defined as follows: Let u be a nonempty word and $p = \delta(i, u)$. Then $q = \delta(i, u')$, where u' is the longest suffix of u for which $\delta(i, u) \neq \delta(i, u')$. It can be shown that this definition does not depend on the particular choice of u [12]. An example of a factor automaton of a word is displayed in Figure 1.

In [3], the authors gave a quadratic-time algorithm to compute the set of minimal forbidden factors of a regular factorial language L .

However, computing the minimal forbidden factors of a single word can be done in linear time in the length of the word. Algorithm MF-TRIE, presented in [12], builds the trie of the set \mathcal{M}_w having as input the factor automaton of w , together with its failure function. Moreover, the states of the output trie recognizing the set \mathcal{M}_w are the same as those of the factor automaton of w , plus some sink states, which are the terminal states with no outgoing edges, corresponding to the minimal forbidden factors. This in particular proves that the size of the trie recognizing the set \mathcal{M}_w is linear in the length of w , since, as already mentioned, the size of the factor automaton of w is linear in the length of w . This property is not an immediate consequence of the fact that the *number* of minimal forbidden factors of w is linear in the length of w , since in fact the *sum of the lengths* of the minimal forbidden factors of w can be quadratic in the length of w (for example, if w is of the form $ab^n a$, then $ab^i a$ is a minimal forbidden factor of w for every $i = 0, 1, \dots, n - 1$).

¹We do not require here and in the remainder of this paper that an automaton be complete. However, to make an automaton complete it is sufficient to add one sink state towards which all missing transitions go.

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L-AUTOMATON (trie  $\mathcal{T} = (Q, A, i, T, \delta')$ )
1. for each  $a \in A$ 
2.   if  $\delta'(i, a)$  is defined
3.      $\delta(i, a) \leftarrow \delta'(i, a)$ 
4.      $f(\delta(i, a)) \leftarrow i$ 
5.   else
6.      $\delta(i, a) \leftarrow i$ 
7. for each state  $p \in Q \setminus \{i\}$  in breadth-first search and each  $a \in A$ 
8.   if  $\delta'(p, a)$  is defined
9.      $\delta(p, a) \leftarrow \delta'(p, a)$ 
10.     $f(\delta(p, a)) \leftarrow \delta(f(p), a)$ 
11.   else if  $p \notin T$ 
12.      $\delta(p, a) \leftarrow \delta(f(p), a)$ 
13.   else
14.      $\delta(p, a) \leftarrow p$ 
15. return  $(Q, A, i, Q \setminus T, \delta)$ 

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Figure 2: Algorithm L-AUTOMATON. It builds an automaton recognizing the language $L(M)$ of words avoiding an antifactorial language M on the input trie \mathcal{T} accepting M .

Despite these theoretical advantages, algorithm MF-TRIE may not be the best algorithm to be used in applications. In recent years, other algorithms have been introduced to compute the minimal forbidden factors of a word. The computation of minimal forbidden factors based on the construction of suffix arrays was considered in [25]; although this algorithm has a linear-time performance in practice, the worst-case time complexity is $\mathcal{O}(n^2)$. New $\mathcal{O}(n)$ -time and $\mathcal{O}(n)$ -space suffix-array-based algorithms were presented in [19, 1, 2]. A more space-efficient solution to compute all minimal forbidden factors in time $\mathcal{O}(n)$ was also presented in [5].

We have discussed algorithms for computing the set of minimal forbidden factors of a given factorial language. We are now describing an algorithm performing the reverse operation. Let M be an antifactorial language. We let $L(M)$ denote the (factorial) language avoiding M , that is, the language of all the words that do not contain any word of M as a factor. Clearly, from equations (1) and (2), we have that $L(M)$ is the unique language whose set of minimal forbidden factors is M , i.e., the unique language L such that $\mathcal{M}_L = M$.

For a finite antifactorial language M , algorithm L-AUTOMATON [12] builds a DFA recognizing $L(M)$. It is presented in Figure 2. The algorithm runs in linear time in the size of the trie storing the words of M . It uses a failure function f defined in a way analogous to the one used for building the factor automaton.

The algorithm can be applied for retrieving a word from its set of minimal forbidden factors, and this can be done in linear time in the length of the word, since, as already mentioned, the size of the trie of minimal forbidden factors of a word is linear in the length of the word. Notice that, even if M is finite, the language $L(M)$ can be finite or infinite. Moreover, even in the case that $L(M)$ is also finite, it can be the language of factors of a single word or of a finite set of words.

Algorithm L-AUTOMATON builds an automaton recognizing the language $L(M)$ of words avoiding a given antifactorial language M , but this automaton is not, in general, minimal. However, the following result holds [12]:

Theorem 1. *If M is the set of the minimal forbidden factors of a finite word w , then the automaton output from algorithm L-AUTOMATON on the input trie recognizing M , after removing sink states, is the factor*

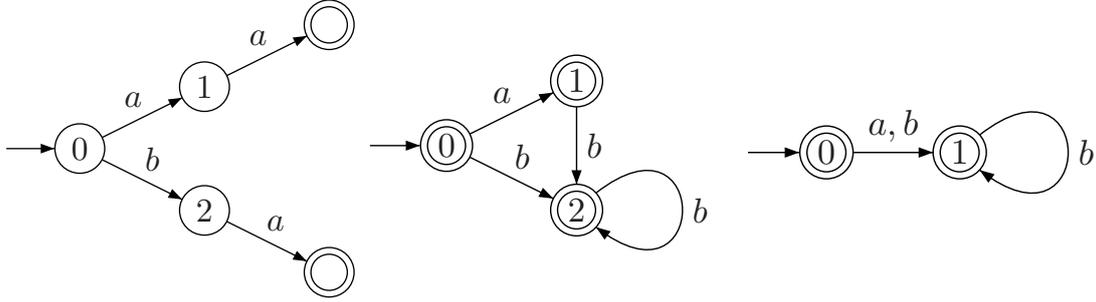


Figure 3: The trie \mathcal{T} recognizing $M = \{aa, ba\}$ (left), the automaton output from algorithm L-AUTOMATON on input \mathcal{T} after removing sink states (center) and the minimal automaton recognizing the language $L(M) = \{b^n \mid n \geq 0\} \cup \{ab^n \mid n \geq 0\}$ (right).

automaton of w , i.e., it is minimal.

To see that the minimality described in the previous theorem does not hold in general, consider for instance the antifactorial language $M = \{aa, ba\}$. It can be easily checked that, taking as input a trie recognizing M , algorithm L-AUTOMATON outputs an automaton which, after removing sink states, has 3 states, while the minimal automaton of the language $L(M) = \{b^n \mid n \geq 0\} \cup \{ab^n \mid n \geq 0\}$ has 2 states (see Figure 3).

We will prove in the next section that this minimality property still holds true in the case of minimal forbidden factors of a circular word.

From the set \mathcal{M}_w of the minimal forbidden factors of a finite word w , one can reconstruct the word w in linear time. To this end, one can apply algorithm L-AUTOMATON on the trie recognizing \mathcal{M}_w . After deleting the sink states of the obtained automaton, one can retrieve the longest path starting from the initial state by using a classical topological sort procedure. This path corresponds to the word w .

3. Minimal Forbidden Factors of a Circular Word

Given a word w , the language *generated* by w is the language $w^* = \{w^k \mid k \geq 0\} = \{\varepsilon, w, ww, www, \dots\}$. Analogously, the language L^* generated by $L \subset A^*$ is the set of all possible concatenations of words in L , i.e., $L^* = \{\varepsilon\} \cup \{w_1 w_2 \dots w_n \mid w_i \in L \text{ for } i = 1, 2, \dots, n\}$.

Let w be a word of length at least 2. The language w^* generated by w is not a factorial language, nor is the language generated by all the rotations of w . Nevertheless, if we take the factorial closure of the language generated by w , then of course we get a factorial language \mathcal{F}_{w^*} . Now, if z is conjugate to w , then although w and z generate different languages, the factorial closures of the languages they generate coincide, i.e., $\mathcal{F}_{w^*} = \mathcal{F}_{z^*}$. Moreover, for any power w^k of w , $k > 0$, one clearly has $\mathcal{F}_{w^*} = \mathcal{F}_{(w^k)^*}$.

Based on the previous discussion, and on Remark 1, we give the following definition: We let the set of *factors of a circular word* $[w]$ be the (factorial) language \mathcal{F}_{w^*} , where w is any linearization of $[w]$. By the previous observation, this definition is independent of the particular choice of the linearization. Moreover, we can suppose that $[w]$ is a primitive circular word.

The set of minimal forbidden factors of the circular word $[w]$ is defined as the set $\mathcal{M}_{\mathcal{F}_{w^*}}$ of minimal forbidden factors of the language \mathcal{F}_{w^*} , where w is any linearization of $[w]$. To simplify the notation, in the

remainder of this paper we will let $\mathcal{M}_{[w]}$ denote the set of minimal forbidden factors of the circular word $[w]$.

For instance, if $[w] = [aabbabb]$, then we have

$$\mathcal{M}_{[w]} = \{aaa, aba, bbb, aabbaa, babbab\}.$$

Notice that $\mathcal{M}_{[w]}$ does not coincide with the set of minimal forbidden factors of the factorial closure of the language of all the rotations of w .

Although \mathcal{F}_{w^*} is an infinite language, the set $\mathcal{M}_{[w]} = \mathcal{M}_{\mathcal{F}_{w^*}}$ of minimal forbidden factors of $[w]$ is always finite. More precisely, we have the following structural lemma.

Lemma 2. *Let $[w]$ be a circular word and w any linearization of $[w]$. Then*

$$\mathcal{M}_{[w]} = \mathcal{M}_{ww} \cap A^{\leq |w|}. \quad (3)$$

Proof. If v is an element of $\mathcal{M}_{ww} \cap A^{\leq |w|}$, then clearly $v \in \mathcal{M}_{\mathcal{F}_{w^*}} = \mathcal{M}_{[w]}$.

Conversely, let aub , with $a, b \in A$ and $u \in A^*$, be an element in $\mathcal{M}_{[w]} = \mathcal{M}_{\mathcal{F}_{w^*}}$ (the case of minimal forbidden factors of length 1 is straightforward). Then $aub \notin \mathcal{F}_{w^*}$, while $au, ub \in \mathcal{F}_{w^*}$. So, there exists some letter \bar{b} different from b such that $au\bar{b} \in \mathcal{F}_{w^*}$ and a letter \bar{a} different from a such that $\bar{a}ub \in \mathcal{F}_{w^*}$. Therefore, $au, \bar{a}u, ub, u\bar{b} \in \mathcal{F}_{w^*}$. It is readily verified that any word of length at least $|w| - 1$ cannot be extended to the right nor to the left by different letters in \mathcal{F}_{w^*} . Hence $|aub| \leq |w|$. Since au and ub are factors of some rotation of w , we have $au, ub \in \mathcal{F}_{ww}$, whence $aub \in \mathcal{M}_{ww}$. \square

The equality (3) was first introduced as a definition for the set of minimal forbidden factors of a circular word in [24]. In fact, it can be efficiently exploited in applications of minimal forbidden factors of circular words [24, 8].

It should be noticed here that the fact that the set $\mathcal{M}_{[w]} = \mathcal{M}_{\mathcal{F}_{w^*}}$ is finite although the set \mathcal{F}_{w^*} is not, is a property that cannot be extended to sets of words of cardinality greater than 1. As an example, let $L = \{b, aa\}$; then the set $\mathcal{M}_{\mathcal{F}_{L^*}}$ is infinite, as it contains the words $ba^{2n+1}b$ for all $n \geq 1$.

About the number of minimal forbidden factors of a circular word, we have the following bounds.

Lemma 3. *Let $[w]$ be a circular word of length n over the alphabet A and let $A(w)$ be the set of letters of A that occur in w . Then*

$$|A| - 1 \leq |\mathcal{M}_{[w]}| \leq |A| + (n - 1)|A(w)| - n. \quad (4)$$

In particular, if $|A(w)| = |A|$, then $|\mathcal{M}_{[w]}| \leq n(|A| - 1)$.

Proof. The inequality $|A| - 1 \leq |\mathcal{M}_{[w]}|$ follows from the fact that for each letter $a \in A$, except at most one, there exists an integer $n_a > 0$ such that $a^{n_a} \in \mathcal{M}_{[w]}$. For the upper bound, we first observe that the minimal forbidden factors of length 1 of $[w]$ are precisely the elements of $A \setminus A(w)$. We now count the minimal forbidden factors of length greater than 1. By Lemma 2, we know that $\mathcal{M}_{[w]} = \mathcal{M}_{ww} \cap A^{\leq |w|}$. Let $ww = w_1w_2 \cdots w_{2n}$. Consider a position i in ww such that $n \leq i < 2n$. We claim that there are at most $|A|$ distinct elements of $\mathcal{M}_{[w]}$ of length greater than 1 whose longest proper prefixes have an occurrence ending in position i . Indeed, by contradiction, let $b \in A$ such that there exist $ub, vb \in \mathcal{M}_{[w]}$ and both u and v occur in ww ending in position i . This implies that ub and vb are one suffix of another, against the minimality of the minimal forbidden factors. Since the letter b must be different from the letter of ww occurring in position $i + 1$, we therefore have that the number of minimal forbidden factors obtained for i ranging from

n to $2n - 1$ is at most $n(|A(w)| - 1)$. For i such that $1 \leq i < n$ (resp. $i = 2n$), if an element $ub \in \mathcal{M}_{[w]}$, $b \in A$, is such that u has an occurrence in w ending in position i , then u has also an occurrence ending in position $i + n$ (resp. n), so it has already been counted. Hence,

$$|\mathcal{M}_{[w]}| \leq |A| - |A(w)| + n(|A(w)| - 1) = |A| + (n - 1)|A(w)| - n.$$

□

The bounds in the previous lemma are tight. For the lower bound, we have for example that the set of minimal forbidden factors of the circular word $[w] = [a^n]$ is $A \setminus \{a\}$; for the upper bound, the word $[w] = [a^{n-1}b]$ over the alphabet $A = \{a, b\}$ has n distinct minimal forbidden factors, namely a^n and $ba^i b$ for every $i = 0, 1, \dots, n - 2$. As another example, if $[w]$ is a binary de Bruijn word of order k (and hence length 2^k), then for every binary word v of length k there exists exactly one letter a such that va is a minimal forbidden factor of $[w]$, hence $|\mathcal{M}_{[w]}| = |w|$. For instance, if $[w] = [aaababbbb]$, then one has $\mathcal{M}_{[w]} = \{aaaa, aabb, abaa, abba, baab, baba, bbab, bbbb\}$. Over the alphabet $A = \{a, b, c\}$, for any odd n , the circular word $[w] = [a^{\lfloor n/2 \rfloor} b a^{\lfloor n/2 \rfloor - 1} c]$, of length n , has $2n$ distinct minimal forbidden factors. Moreover, over the alphabet $A = \{a_1, a_2, \dots, a_n\}$, the circular word $[w] = [a_1 a_2 \cdots a_n]$ has $n(n - 1)$ distinct minimal forbidden factors.

We now give a result analogous to Theorem 1 in the case of circular words.

Theorem 4. *If M is the set of the minimal forbidden factors of a primitive circular word $[w]$, then the automaton output from algorithm L-AUTOMATON on the input trie \mathcal{T} recognizing M , after removing sink states, is the minimal automaton recognizing the language \mathcal{F}_{w^*} of factors of $[w]$.*

Proof. Let $\mathcal{A} = (Q, A, i, Q \setminus T, \delta)$ be the automaton output by algorithm L-AUTOMATON with input the trie \mathcal{T} recognizing the set of the minimal forbidden factors of a circular word $[w]$. Let $w = w_1 w_2 \cdots w_n$ be a linearization of $[w]$. By the property of algorithm L-AUTOMATON, the automaton \mathcal{A} recognizes the language \mathcal{F}_{w^*} , since its input is the trie that recognizes the language $\mathcal{M}_{[w]} = \mathcal{M}_{\mathcal{F}_{w^*}}$. To prove that \mathcal{A} is minimal, we have to prove that any two states are distinguishable. Suppose by contradiction that there are two nondistinguishable states $p, q \in Q$. By construction, p and q are respectively associated with two proper prefixes, v_p and v_q , of words in $\mathcal{M}_{\mathcal{F}_{w^*}}$, which, by Lemma 2, is equal to $\mathcal{M}_{ww} \cap A^{\leq |w|}$. Therefore, v_p and v_q are factors of w^* of length $\leq |w|$. Hence, they are both factors of w^2 . Let us then write $w^2 = x v_p y = x' v_q y'$, with x and x' of minimal length.

Suppose first that there exists i such that $x v_p$ and $x' v_q$ both end in $w_1 w_2 \cdots w_i$. Then v_p and v_q are one suffix of another. Since p and q are nondistinguishable, there exists a word z such that $v_p z$ and $v_q z$ end in a sink state, that is, are elements of $\mathcal{M}_{[w]}$. This is a contradiction since $\mathcal{M}_{[w]}$ is an antifactorial set and $v_p z$ and $v_q z$ are one suffix of another.

Suppose now that $x v_p$ ends in $w_1 w_2 \cdots w_i$ and $x' v_q$ ends in $w_1 w_2 \cdots w_j$ for $i \neq j$. Since p and q are nondistinguishable, for any word u one has that $v_p u \in \mathcal{F}_{w^*}$ if and only if $v_q u \in \mathcal{F}_{w^*}$. Since \mathcal{F}_{w^*} is a factorial language, we therefore have that there exists a word z of length $|w|$ such that $v_p z$ and $v_q z$ are both in \mathcal{F}_{w^*} . But this implies that $z = w_{i+1} w_{i+2} \cdots w_i = w_{j+1} w_{j+2} \cdots w_j$, and this leads to a contradiction since w is primitive and therefore all its rotations are distinct. □

If one is interested in retrieving the circular word $[w]$ from the minimal automaton recognizing the language \mathcal{F}_{w^*} , this can be done with a simple Depth-First-Search procedure in linear time and space with respect to the size of the automaton. Indeed, the circular word $[w]$ corresponds to a cycle in the (multi-)graph of the automaton, and it can be proved that the size of the minimal automaton recognizing the language \mathcal{F}_{w^*}

is linear in the length of w . This follows from the fact that the size of the output of L-AUTOMATON is, by construction, the same as the size of its input (except for the sink states), and we have that the size of the trie recognizing the minimal forbidden factors of $[w]$ is linear in the size of w — as this is a subtrie of the trie recognizing the minimal forbidden factors of ww (Lemma 2), and this latter has a size that is linear in the length of w , as observed in the previous section.

However, it is possible to give more precise bounds on the size of the minimal automaton recognizing the language \mathcal{F}_{w^*} in terms of the length of w , as shown below.

Theorem 5. *Let w be a word of length n . The minimal automaton recognizing the language \mathcal{F}_{w^*} has at most $2n - 1$ states.*

Proof. Let $w = a_1a_2 \cdots a_n$. Let \mathcal{S}_{w^*} be the set of suffixes of words in the language w^* . For every $x \in \mathcal{F}_{w^*}$ different from the empty word, let $E_w(x) = \{i \in \{1, \dots, n\} \mid x \in \mathcal{S}_{w^*}a_1a_2 \cdots a_i\}$, and let $E_w(\varepsilon) = \{1, 2, \dots, n\}$. We define the equivalence $x \equiv_w y$ if and only if $E_w(x) = E_w(y)$. This equivalence clearly being right-invariant ($x \equiv_w y$ implies $xa \equiv_w ya$ for every letter a), there exists a DFA \mathcal{A} recognizing \mathcal{F}_{w^*} whose states are identified with the equivalence classes of \equiv_w . It is readily verified that for every $x, y \in \mathcal{F}_{w^*}$, either $E_w(x)$ and $E_w(y)$ are disjoint or one is contained in the other. They therefore form a non-overlapping family of nonempty subsets of $\{1, 2, \dots, n\}$, which implies that there are at most $2n - 1$ of them (see for example Lemma 1 in [6], where the value $2n - 1$ is replaced by $2n$ because the authors consider the equivalence in Σ^* rather than in \mathcal{F}_{w^*} thus allowing the empty set as an extra class corresponding to the additional sink state). This shows that there exists an automaton recognizing \mathcal{F}_{w^*} with at most $2n - 1$ states. \square

The bound in the previous theorem is tight for $n > 2$. As an example, consider $w = ab^{n-1}$. The states of the minimal automaton recognizing the language \mathcal{F}_{w^*} are identified with the classes of the equivalence relation defined on \mathcal{F}_{w^*} by: $x \equiv_M y$ if and only if for every word z one has $xz \in \mathcal{F}_{w^*} \Leftrightarrow yz \in \mathcal{F}_{w^*}$. Now, it is easy to see that the factors b^i and ab^i , $0 \leq i < n - 1$, are each in a distinct class, while b^{n-1} and ab^{n-1} are together in another class, whence there are at least $2n - 1$ states in the minimal automaton recognizing \mathcal{F}_{w^*} .

In view of these considerations, we define *the factor automaton of a circular word $[w]$* as the minimal automaton recognizing the language \mathcal{F}_{w^*} .

4. Circular Fibonacci Words and Minimal Forbidden Factors

In this section, we illustrate the combinatorial results discussed in the previous section in the special case of the circular Fibonacci words. The Fibonacci words are a paradigmatic example that often represents the limit case for some property. For example, it is well known the worst-case running time of some pattern matching algorithms is realized by the Fibonacci words (see, e.g., [20]).

We fix the alphabet $A = \{a, b\}$. The sequence $(f_n)_{n \geq 1}$ of Fibonacci words is defined recursively by: $f_1 = b$, $f_2 = a$ and $f_n = f_{n-1}f_{n-2}$ for $n > 2$. The length of the word f_n is the Fibonacci number F_n . The limit of this sequence is the infinite Fibonacci word $f = \lim_{n \rightarrow \infty} f_n = abaababaabaab \cdots$.

Every Fibonacci word f_n has a factor automaton with $|f_n| + 1$ states (thus attaining the lower bound on the number of states that the factor automaton of a word can have) and the structure of the factor automaton of the Fibonacci words allows one to derive several combinatorial properties of these words (cf. [26, 16]).

We will now describe the structure of the sets $\mathcal{M}_{[f_n]}$ of minimal forbidden factors of the circular Fibonacci words $[f_n]$.

$f_1 = b$	
$f_2 = a$	
$f_3 = ab$	$u_3 = \varepsilon$
$f_4 = aba$	$u_4 = a$
$f_5 = abaab$	$u_5 = aba$
$f_6 = abaababa$	$u_6 = abaaba$
$f_7 = abaababaabaab$	$u_7 = abaababaaba$
$f_8 = abaababaabaababa$	$u_8 = abaababaabaababa$

Table 1: The first few Fibonacci words f_n and the first few words u_n .

Let us recall some well-known properties of the Fibonacci words (the reader may also see [21, Chap. 2]). For every $n \geq 3$, one can write $f_n = u_n ab$ if n is odd or $f_n = u_n ba$ if n is even, where u_n is a palindrome. Moreover, since $f_n = f_{n-1} f_{n-2}$ and the words u_n are palindromes, one has that for every $n \geq 5$

$$f_n = u_n xy = u_{n-1} y x u_{n-2} xy = u_{n-2} xy u_{n-1} xy \quad (5)$$

for letters x, y such that $\{x, y\} = \{a, b\}$. Indeed, since u_n is a palindrome, one has that $u_n = u_{n-1} y x u_{n-2}$ is equal to its mirror image, $\tilde{u}_{n-2} x y \tilde{u}_{n-1}$, which is equal to $u_{n-2} x y u_{n-1}$ since u_{n-1} and u_{n-2} are palindromes. The first few Fibonacci words f_n and the first few words u_n are shown in Table 1.

The words f_n (as well as the words $f_n f_n$) are *balanced*, that is, for every pair of factors u and v of the same length, one has $||u|_a - |v|_a| \leq 1$ (and therefore also $||u|_b - |v|_b| \leq 1$).

A *bispecial factor* of a word w over the alphabet $A = \{a, b\}$ is a word v such that av, bv, va, vb are all factors of w .

Proposition 6. *For every $n \geq 3$, the set of bispecial factors of the word $f_n f_n$ is $\{u_3, u_4, \dots, u_n\}$.*

Proof. As it is well known (cf. [15, Proposition 10]), the bispecial factors of the infinite Fibonacci word f are the central words u_n , $n \geq 3$. Since $f_n f_n$ is a factor of f (the prefix f_{n+3} of f can be written as $f_{n+3} = f_{n+2} f_{n+1} = f_{n+1} f_n f_n f_{n-1}$) we have that the set of bispecial factors of the word $f_n f_n$ is contained in $\bigcup_{n \geq 3} u_n$. Since by construction u_{n-1} appears in u_n both as a prefix and as a suffix, we are left to prove that u_n is a bispecial factor of $f_n f_n$, since this will imply that also the words u_m , $3 \leq m \leq n$, are bispecial factors of $f_n f_n$.

The claim can be easily checked for $n = 3, 4$. Let us then suppose $n \geq 5$. We know that $f_n f_n = u_n x y u_n x y$ for letters x, y such that $\{x, y\} = \{a, b\}$. Hence, $u_n x$ and $y u_n$ appear as factors in $f_n f_n$. We will now show that also $x u_n$ and $u_n y$ appear as factors in $f_n f_n$. Indeed, using (5), we have

$$\begin{aligned} f_n f_n &= u_n x y \cdot u_n x y \\ &= u_{n-1} y x u_{n-2} \cdot x y \cdot u_{n-1} y x u_{n-2} \cdot x y \\ &= u_{n-1} y \cdot x u_{n-2} x y u_{n-1} y \cdot x u_{n-2} x y \\ &= u_{n-1} y \cdot x u_n y \cdot x u_{n-2} x y \end{aligned}$$

that gives the desired occurrences of $x u_n$ and $u_n y$. □

$\hat{f}_3 = aa$	$\hat{g}_3 = bb$
$\hat{f}_4 = bab$	$\hat{g}_4 = aaa$
$\hat{f}_5 = aabaa$	$\hat{g}_5 = babab$
$\hat{f}_6 = babaabab$	$\hat{g}_6 = aabaabaa$
$\hat{f}_7 = aabaabababaa$	$\hat{g}_7 = babaabababab$
$\hat{f}_8 = babaababababababab$	$\hat{g}_8 = aabaabababababababaa$

Table 2: The first few elements of the sequences \hat{f}_n and \hat{g}_n .

$\mathcal{M}_{[f_1]} = \{a\}$
$\mathcal{M}_{[f_2]} = \{b\}$
$\mathcal{M}_{[f_3]} = \{aa, bb\}$
$\mathcal{M}_{[f_4]} = \{bb, aaa, bab\}$
$\mathcal{M}_{[f_5]} = \{bb, aaa, aabaa, babab\}$
$\mathcal{M}_{[f_6]} = \{bb, aaa, babab, aabaabaa, babaabab\}$
$\mathcal{M}_{[f_7]} = \{bb, aaa, babab, aabaabaa, aabaabababaa, babaabababab\}$
$\mathcal{M}_{[f_8]} = \{bb, aaa, babab, aabaabaa, babaabababab, aabaabababababababaa, babaabababababababab\}$

Table 3: The first few sets of minimal forbidden factors of the circular Fibonacci words.

Let us now define the sequence of words $(\hat{f}_n)_{n \geq 3}$ by $\hat{f}_n = au_n a$ if n is odd, $\hat{f}_n = bu_n b$ if n is even. These words are known as *singular words*. Analogously, we can define the sequence of words $(\hat{g}_n)_{n \geq 3}$ by $\hat{g}_n = bu_n b$ if n is odd, $\hat{g}_n = au_n a$ if n is even. For every n , the word \hat{g}_n is obtained from the word \hat{f}_n by changing the first and the last letter. The elements of the sequence \hat{g}_n are indeed the minimal forbidden factors of the infinite Fibonacci word f (see [22]). The first few values of the sequences \hat{f}_n and \hat{g}_n are shown in Table 2.

The structure of the sets of minimal forbidden factors of circular Fibonacci words can be described in terms of the words \hat{f}_n and \hat{g}_n , as follows. The first few sets $\mathcal{M}_{[f_n]}$ are displayed in Table 3. We have $\mathcal{M}_{[f_1]} = \mathcal{M}_{[b]} = \{a\}$, $\mathcal{M}_{[f_2]} = \mathcal{M}_{[a]} = \{b\}$ and $\mathcal{M}_{[f_3]} = \mathcal{M}_{[ab]} = \{aa, bb\}$. The following theorem gives a characterization of the sets $\mathcal{M}_{[f_n]}$ for $n \geq 4$.

Theorem 7. For every $n \geq 4$, $\mathcal{M}_{[f_n]} = \{\hat{g}_3, \hat{g}_4, \dots, \hat{g}_n, \hat{f}_n\}$.

Proof. By Lemma 2, $\mathcal{M}_{[f_n]} = \mathcal{M}_{f_n f_n} \cap A^{\leq |f_n|}$. Let xuy , $u \in A^*$, $x, y \in A$, be in $\mathcal{M}_{f_n f_n} \cap A^{\leq |f_n|}$. Then, xu has an occurrence in $f_n f_n$ followed by letter \bar{y} , the letter different from y , and uy has an occurrence in $f_n f_n$ preceded by letter \bar{x} , the letter different from x . Therefore, u is a bispecial factor of the word $f_n f_n$, hence, by Proposition 6, $u \in \{u_3, u_4, \dots, u_n\}$. Thus, an element in $\mathcal{M}_{[f_n]}$ is of the form $\alpha u_i \beta$ for some

$3 \leq i \leq n$ and $\alpha, \beta \in A$.

We first prove that for every $3 \leq i \leq n$, if $\alpha \neq \beta$, then the words $\alpha u_i \beta$ and $\beta u_i \alpha$ occur as factors in $f_n f_n$. Let us write $f_n f_n = u_n x y u_n x y$, for letters x, y such that $\{x, y\} = \{a, b\}$.

As observed in the proof of Proposition 6, $f_n f_n$ contains an internal occurrence of $x u_n y = x u_{n-1} y x u_{n-2} y$, hence it contains an occurrence of $x u_{n-1} y$ and an occurrence of $x u_{n-2} y$. On the other hand, $f_n f_n = u_n x y u_n x y = u_n x y u_{n-2} x y u_{n-1} x y$, hence $f_n f_n$ contains an occurrence of $y u_n x$, an occurrence of $y u_{n-1} x$ and an occurrence of $y u_{n-2} x$. Continuing this way, recursively, we see that $f_n f_n$ contains an occurrence of $x u_i y$ and one of $y u_i x$ for every $3 \leq i \leq n$.

Thus, a word in $\mathcal{M}_{[f_n]}$ can only be of the form \hat{f}_i or \hat{g}_i , for some $3 \leq i \leq n$.

Claim: \hat{f}_n is a minimal forbidden factor of $f_n f_n$.

Proof: Let $\hat{f}_n = x u_n x$, $x \in A$. By Proposition 6, we know that $x u_n$ and $u_n x$ are factors of $f_n f_n$. It remains to show that \hat{f}_n does not occur in $f_n f_n = u_n x y u_n x y$, $y \neq x$. If $x u_n x$ occurs in $f_n f_n$, then it occurs in $u_n x y u_n$. But it is known that the longest repeated prefix of $u_n x y u_n$ is u_n (cf. [14]), so $u_n x$ cannot appear in $u_n x y u_n$.

Claim: \hat{f}_n is a factor of $f_{n+1} f_{n+1}$.

Proof: The first letter of \hat{f}_n is equal to the last letter of f_{n+1} and, by removing the first letter from \hat{f}_n , one obtains a prefix of f_{n+1} . Hence, \hat{f}_n is a factor of $f_{n+1} f_{n+1}$.

Claim: For every $3 \leq i < n$, \hat{g}_i is a minimal forbidden factor of $f_n f_n$.

Proof: By the previous claim, it follows that for every $3 \leq i < n$, the word \hat{f}_i is factor of the word $f_n f_n$. Therefore \hat{g}_i cannot be a factor of $f_n f_n$ otherwise the word $f_n f_n$ would not be balanced. Since removing the first or the last letter from the word \hat{g}_i one obtains a factor of the word $f_n f_n$, the claim is proved.

Claim: \hat{g}_n is a minimal forbidden factor of $f_n f_n$.

Proof: $\hat{g}_n = y u_n y$ is a minimal forbidden factor of the infinite Fibonacci word f , so it cannot be a factor of $f_n f_n$. On the other hand, we proved in Proposition 6 that u_n is a bispecial factor of $f_n f_n$, hence $y u_n$ and $u_n y$ occur in $f_n f_n$. \square

Notice that, by Lemma 2, for any circular word $[w]$, one has that $|w|$ is an upper bound on the length of the minimal forbidden factors of $[w]$. The previous theorem shows that this bound is indeed tight. However, the maximum length of a minimal forbidden factor of a circular word $[w]$ is not always equal to $|w|$. For example, for $w = aabbab$ one has $\mathcal{M}_{[w]} = \{aaa, bbb, aaba, abab, babb, bbaa\}$.

About the cardinality of the set of minimal forbidden factors, however, Fibonacci words are not extremal. Indeed, by Lemma 3, we know that a binary word can have a number of distinct minimal forbidden factors equal to its length, while in the case of the Fibonacci words this number is only logarithmic in the length of the word, as we have, by Theorem 7, that the cardinality of $\mathcal{M}_{[f_n]}$ is $n - 1$ and the length of f_n is exponential in n .

By Theorem 4, if \mathcal{T} is the trie recognizing the set $\{\hat{g}_3, \hat{g}_4, \dots, \hat{g}_n, \hat{f}_n\}$, then algorithm L-AUTOMATON on the input trie \mathcal{T} builds the minimal deterministic automaton recognizing $\mathcal{F}_{f_n^*}$. Since the automaton output by algorithm L-AUTOMATON has the same set of states of the input trie \mathcal{T} after removing sink states, and since removing the last letter from each word \hat{g}_i results in a prefix of \hat{f}_{i+1} , we have that the factor automaton of the circular Fibonacci word $[f_n]$ (that is, the minimal automaton recognizing $\mathcal{F}_{f_n^*}$) has exactly $2F_n - 1$ states (see Figure 4 for an example). In view of Theorem 5, circular Fibonacci words have a factor automaton with the largest possible number of states, while in the linear case it is well known that Fibonacci words have a factor automaton with the smallest possible number of states.

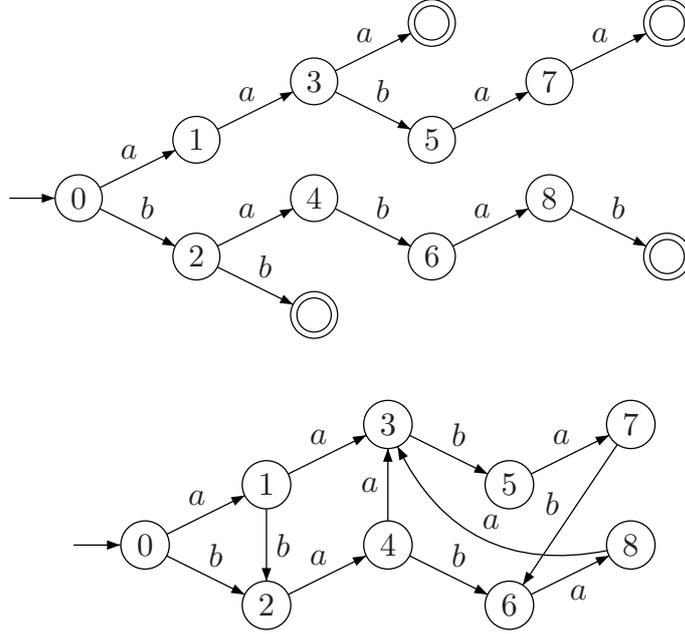


Figure 4: The trie \mathcal{T} recognizing the set $\mathcal{M}_{\{f_5\}}$ (top), and the automaton output by algorithm L-AUTOMATON on the input trie \mathcal{T} after removing sink states (bottom), which is the minimal automaton recognizing $\mathcal{F}_{f_5}^*$. It has $9 = 2F_5 - 1$ states.

5. Conclusions and Open Problems

We investigated combinatorial properties of minimal forbidden factors of circular words.

We proved that the automaton built by algorithm L-AUTOMATON on the input trie recognizing the set of minimal forbidden factors of a circular word is minimal. More generally, it would be interesting to characterize those antifactorial languages for which algorithm L-AUTOMATON builds a minimal automaton.

In addition to being interesting from the point of view of formal language theory, we believe the study of minimal forbidden factors of circular words will also lead to new applications in sequence analysis. An example on this direction is given in [8].

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