On H-topological intersection graphs *

Steven Chaplick¹, Martin Töpfer², Jan Voborník², and Peter Zeman^{†2}

¹Department of Data Science and Knowledge Engineering, Maastricht University, The Netherlands, s.chaplick@maastrichtuniversity.nl.

²Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Czech Republic, {topfer,vobornik,zeman}@kam.mff.cuni.cz.

Abstract

Biró, Hujter, and Tuza (1992) introduced the concept of H-graphs, intersection graphs of connected subgraphs of a subdivision of a graph H. They are related to and generalize many important classes of geometric intersection graphs, e.g., interval graphs, circular-arc graphs, split graphs, and chordal graphs. Our paper starts a new line of research in the area of geometric intersection graphs by studying several classical computational problems on H-graphs: recognition, graph isomorphism, dominating set, clique, and colorability.

We negatively answer the 25-year-old question of Biró, Hujter, and Tuza which asks whether H-graphs can be recognized in polynomial time, for a fixed graph H. We prove that it is NP-complete if H contains the diamond graph as a minor. On the positive side, we provide a polynomial-time algorithm recognizing T-graphs, for each fixed tree T. For the special case when T is a star S_d of degree d, we have an $\mathcal{O}(n^{3.5})$ -time algorithm.

We give FPT- and XP-time algorithms solving the minimum dominating set problem on S_d -graphs and H-graphs, parametrized by d and the size of H, respectively. The algorithm for H-graphs adapts to an XP-time algorithm for the independent set and the independent dominating set problems on H-graphs.

If H contains the *double-triangle* as a minor, we prove that the graph isomorphism problem is GI-complete and that the clique problem is APX-hard. On the positive side, we show that the clique problem can be solved in polynomial time if H is a cactus graph. Also, when a graph has a $Helly\ H$ -representation, the clique problem is polynomial-time solvable.

Further, we show that both the k-clique and the list k-coloring problems are solvable in FPT-time on H-graphs, parameterized by k and the treewidth of H. In fact, these results apply to classes of graphs with treewidth bounded by a function of the clique number.

We observe that H-graphs have at most $n^{O(\|H\|)}$ minimal separators which allows us to apply the meta-algorithmic framework of Fomin, Todinca, and Villanger (2015) to show that for each fixed t, finding a maximum induced subgraph of treewidth t can be done in polynomial time. In the case when H is a cactus, we improve the bound to $O(\|H\|n^2)$.

^{*}This paper is the combination and extension of the conference versions which appeared at WG 2017 [CTVZ17] and Eurocomb 2017 [CZ17].

[†]Supported by GAUK 1224120 and by GAČR 19-17314J.

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1 Introduction

An intersection representation \mathcal{R} of a graph G is a collection of sets $\{R_v : v \in V(G)\}$ such that $R_u \cap R_v \neq \emptyset$ if and only if $uv \in E(G)$. Many important classes of graphs arise from restricting the sets R_v to geometric objects (e.g., intervals, circular-arcs, convex sets, planar curves). The study of these geometric representations has been motivated through various application domains. For example, intersection graphs of planar curves relate to circuit layout problems [Sin66, BS90], interval graphs relate to scheduling problems [Rob78] and can be used to model biological problems (see, e.g., [JMT92]), and intersection representations of convex sets relate to the study of wireless networks [HS95].

We study H-graphs, intersection graphs of connected subgraphs of a subdivision of a fixed graph H, introduced by Biró, Hujter, and Tuza [BHT92]. We answer their open question concerning the problem of recognition of H-graphs and further start a new line of research in the area of geometric intersection graphs, by studying H-graphs from the point of view of fundamental computational problems of theoretical computer science: recognition, graph isomorphism, dominating set, clique, and colorability. We begin by discussing several closely related graph classes.

Interval graphs (INT) form one of the most studied and well-understood classes of intersection graphs. In an interval representation, each set R_v is a closed interval of the real line; see Fig. 1a. A primary motivation for studying interval graphs (and related classes) is the fact that many important computational problems can be solved in linear time on them; see for example [BL76, Cha98, LB79].

Chordal graphs (CHOR) were originally defined as the graphs without induced cycles of length greater than three. Equivalently, as shown by Gavril [Gav74b], a graph is chordal if and only if it can be represented as an intersection graph of subtrees of some tree; see Fig. 1b. This immediately implies that INT is a subclass of the chordal graphs.

The recognition problem can be solved in linear time for CHOR [RTL76], and such algorithms can be used to generate an intersection representation by subtrees of a tree. However, asking for special host trees can be more difficult. For example, when the desired tree T is a part of the input, deciding whether G is a T-graph is NP-complete [KKOS15]. Additionally, some other important computational problems, for example the dominating set [BJ82] and graph isomorphism [LB79], are harder on chordal graphs than on interval graphs.

One can ask related questions about having "nice" tree representations of a given chordal graph. For example, for a given graph G, if one would like to find a tree T with the fewest leaves such that G is a T-graph, it can be done in polynomial time [HS12], this is known as the leafage problem. However, for any fixed $d \geq 3$, if one would like to find a tree T where G is a T-graph and, for each vertex v, the subtree representing v has at most d leaves, the problem again

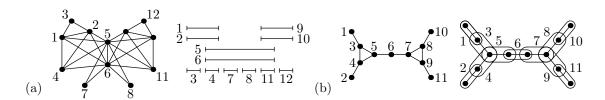


Figure 1: (a) An interval graph and one of its interval representations. (b) A chordal graph and one of its representations as an intersection graph of subtrees of a tree.

becomes NP-complete [CS14], this is known as the *d-vertex leafage* problem. The minimum vertex leafage problem can be solved in $n^{O(\ell)}$ -time via a somewhat elaborate enumeration of $minimal^1$ tree representations of G with exactly ℓ leaves where ℓ is the leafage of G [CS14].

Split graphs (SPLIT) form an important subclass of chordal graphs. These are the graphs that can be partitioned into a clique and an independent set. Note that every split graph can be represented as an intersection graph of subtrees of a $star\ S_d$, where S_d is the complete bipartite graph $K_{1,d}$.

Circular-arc graphs (CARC) naturally generalize interval graphs. Here, each set R_v corresponds to an arc of a circle. The Helly circular-arc graphs form an important subclass of circular-arc graphs. A graph G is a Helly circular-arc graph if the collection of circular-arcs $\mathcal{R} = \{R_v\}_{v \in V(G)}$ satisfies the Helly property, i.e., in each sub-collection of \mathcal{R} whose sets pairwise intersect, the common intersection is non-empty. Interestingly, it is NP-hard to compute a minimum coloring for Helly circular-arc graphs [Gav96].

1.1 H-graphs

Biró, Hujter, and Tuza [BHT92] introduced H-graphs. Let H be a fixed graph. A graph G is an intersection graph of H if it is an intersection graph of connected subgraphs of H, i.e., the assigned subgraphs H_v and H_u of H share a vertex if and only if $uv \in E(G)$.

A subdivision H' of a graph H is obtained when the edges of H are replaced by internally disjoint paths of arbitrary lengths. A graph G is a topological intersection graph of H if G is an intersection graph of a subdivision H' of H. We say that G is an H-graph and the collection $\{H'_v: v \in V(G)\}$ of connected subgraphs of H' is an H-representation of G. The class of all H-graphs is denoted by H-GRAPH. Alternatively, we can view H-graphs geometrically as intersection graphs of connected subregions of a one-dimensional simplicial complex (this is a topological definition of a graph). We have the following relations:

$$\mathsf{INT} = K_2\mathsf{-}\mathsf{GRAPH}, \quad \mathsf{CARC} = K_3\mathsf{-}\mathsf{GRAPH},$$

$$\mathsf{SPLIT} \subsetneq \bigcup_{d=2}^\infty S_d\mathsf{-}\mathsf{GRAPH}, \quad \mathsf{CHOR} = \bigcup_{\mathsf{Tree}\ T} T\mathsf{-}\mathsf{GRAPH}.$$

Motivation. It is easy to see that every graph G is an H-graph for an appropriate choice of H (e.g., by taking H = G). In this sense, the families of H-graphs provide a parameterized view through which we can study all graphs. We also mentioned that several important computational problems are polynomial on interval (the most basic class of H-graphs), but are hard on chordal graphs. This inspires the question of when we can use this parameterization to provide a refined understanding of computational problems. Of course, to approach this problem, we first need to observe some relations among the classes of H-graphs and related well-studied graph classes.

For any pair of (multi-)graphs H_1 and H_2 , if H_1 is a minor of H_2 , then H_1 -GRAPH \subseteq H_2 -GRAPH. Moreover, if H_1 is a subdivision of H_2 , then H_1 -GRAPH = H_2 -GRAPH. Specifically, we have an infinite hierarchy of graph classes between interval and chordal graphs since for every tree T with at least one edge, $\mathsf{INT} \subseteq T$ -GRAPH \subsetneq CHOR. This motivates the study of the above mentioned problems on T-graphs, for a fixed tree T.

We note a dichotomy regarding computing a minimum coloring on H-GRAPH. Namely, if H contains a cycle, then computing a minimum coloring on H-GRAPH is already NP-hard even

 $^{^{1}}$ where each node of T corresponds to a maximal clique of G

for the subclass of Helly H-graphs [Gav96]. On the other hand, when H is acyclic, a minimum coloring can be computed in linear time since H-GRAPH is a subclass of CHOR.

Biró, Hujter, and Tuza originally introduced H-graphs in the context of the (p,k) pre-coloring extension problem (PRCOLEXT(p,k)). In this problem, the input is a graph G together with a p-coloring of $W \subseteq V(G)$, and the goal is to find a proper k-coloring of G extending this pre-coloring. Biró, Hujter, and Tuza [BHT92] provide an XP (in k and ||H||) algorithm to solve PRCOLEXT(k,k) on H-graphs. Biró, Hujter, and Tuza asked the following question which we answer negatively.

[Biró, Hujter, and Tuza [BHT92], 1992] Let H be an arbitrary fixed graph. Is there a polynomial algorithm testing whether a given graph G is an H-graph?

1.2 Our results

We give a comprehensive study of H-graphs from the point of view of several important problems of theoretical computer science: recognition, graph isomorphism, dominating set, clique, and colorability. We focus on five collections of classes of graphs. In particular, S_d -GRAPH, T-GRAPH, C-GRAPH, Helly H-GRAPH, and H-GRAPH, where S_d is the star of degree d, T is a tree, C is a cactus, and H is an arbitrary graph. Our results are displayed in Table 1. The following list provides a summary of our results and should help the reader to navigate through the paper:

- Recognition. In Section 3 we negatively answer the question of Biró, Hujter, and Tuza. We prove that recognizing H-graphs is NP-complete if H is not a cactus (Theorem 1). Equivalently this means that H contains the diamond graph as a minor. We do this by a reduction from the problem of testing whether the interval dimension of a partial order of height 2 is at most 3. On the positive side, in Section 4, we give an $\mathcal{O}(n^{3.5})$ -time algorithm for recognizing S_d -graphs (Theorem 3), and we give a polynomial-time algorithm for recognizing T-graphs (Theorem 4), for a fixed tree T.
- Dominating set. In Section 5, we solve the problem of finding a minimum dominating set for S_d -graphs in time $\mathcal{O}(dn(n+m)) + 2^d(d+2^d)^{\mathcal{O}(1)}$ (Theorem 5) and for H-graphs in $n^{\mathcal{O}(\|H\|)}$ -time (Theorem 6). The latter algorithm can be easily adapted to solve the maximum independent set problem and minimum independent dominating set problem in $n^{\mathcal{O}(\|H\|)}$ -time for H-graphs (Corollary 7).
- Clique. In Section 6, we study the clique problem. We show that if H contains the double-triangle Δ_2 (see Fig. 5a) as a minor, then the clique problem is APX-hard for H-graphs (Theorem 8). On the positive side, we solve the clique problem in polynomial time for Helly H-graphs (Theorem 9), and in the case when H is a cactus (Theorem 10).
- Graph isomorphism. Theorem 8 also gives that if H contains the double-triangle Δ_2 (see Fig. 5a) as a minor, then graph isomorphism problem is GI-complete for H-graphs.
- k-coloring and k-clique. In Section 7, we use treewidth based methods to provide an FPT-time algorithm for finding a k-clique in an H-graphs (Theorem 11) and an FPT-time algorithm for k-coloring of H-graphs (Theorem 12). In fact, these results apply to more general graph classes formalized via the concept of a clique-treewidth property

	S_d -graphs	T-graphs	C-graphs	Helly H -graphs	H-graphs
Recognition	$\mathcal{O}(n^{3.5})$	$n^{\mathcal{O}(\ T\ ^2)}$	Open	Open	$\begin{array}{c} NP\text{-complete \ if} \\ H \neq cactus \end{array}$
Graph isomorphism	FPT in d	Open	Open	Open	GI-complete if $\Delta_2 \preceq H$
Dominating set	FPT in d	$n^{\mathcal{O}(\ T\)}$	$n^{\mathcal{O}(\ C\)}$	$n^{\mathcal{O}(\ H\)}$	$n^{\mathcal{O}(\ H\)}$
Maximum clique	$\mathcal{O}(n+m)$	$\mathcal{O}(n+m)$	Polynomial	Polynomial	$\begin{array}{c} APX\text{-}hard \\ if\ \Delta_2 \preceq H, FPT \end{array}$
Coloring	$\mathcal{O}(n+m)$	$\mathcal{O}(n+m)$	FPT	FPT	FPT
# of minimal separators	$\leq n$	$\leq n$	$\mathcal{O}(\ C\ n^2)$	$n^{\mathcal{O}(\ H\)}$	$n^{\mathcal{O}(\ H\)}$

Table 1: The table of the complexity of different problems for the four considered classes. Our contributions are highlighted. Note: $A \leq B$ denotes that A is a minor of B, and Δ_2 denotes the double-triangle (see Fig. 5).

(which is defined as in the parameter-treewidth properties of bidimensionality theory; see, e.g., [DFHT04]) and may be of independent interest.

• Minimal Separators. Finally, in Section 8, we show that each H-graph has $n^{O(\|H\|)}$ minimal separators (Theorem 13) and, when H is a cactus, we improve this bound to $O(\|H\|n^2)$ (Theorem 15). Thus, by the algorithmic framework of Fomin, Todinca, and Villanger [FTV15], on H-graphs, we obtain a large class of problems (including, e.g., feedback vertex set) which can be solved in XP-time (parameterized by $\|H\|$) and polynomial time (in both $\|H\|$ and the size of the input graph) when H is a cactus.

Open problems. Since all the sections are mostly self-contained, instead of including a separate section for open problems and conclusions, we decided to include the open problems and possible future research directions in the corresponding sections.

Recent developments. After the publication of the two conference articles [CTVZ17, CZ17] (which this paper includes and extends), there have already been further developments regarding H-graphs [FGR20, JKT20]. The results contained in these articles complement and build on our work regarding combinatorial optimization problems. For instance, to complement our XP-time algorithms for minimum dominating set and maximum independent set, Fomin, Golovach, and Raymond [FGR20] show that these problems are W[1]-hard, when parameterized by ||H|| and the desired solution size. They additionally tighten our result regarding the fixed parameter tractability of the k-Clique problem on H-graphs by showing that this problem admits a polynomial size kernel in terms of both ||H|| and the solution size. Jaffke, Kwon, and Telle [JKT20] adapt the W[1]-hardness proof from [FGR20] for maximum independent set to additionally show that feedback vertex set is also W[1]-hard. Only recently, the problem of testing isomorphism of S_d graphs was solved in FPT time [AH20].

2 Preliminaries

We assume that the reader is familiar with the following standard and parameterized computational complexity classes: NP, XP, and FPT (see, e.g., [CFK+15] for further details).

Let G be an H-graph. For a subdivision H' certifying $G \in H$ -GRAPH, we use H'_v to denote the subgraph of H' corresponding to $v \in V(G)$. The vertices of H and H' are called *nodes*. By ||H|| we denote the size of H, i.e., ||H|| = |V(H)| + |E(H)|.

We refer to the degree one nodes of H as leaves and the nodes degree at least three as branching points. Note that, while we sometimes speak of degree two nodes in H, they are actually redundant since their presence or absence does not change H-GRAPH. As such, by thinking of H as a multi-graph with loops one can nearly always avoid the need for any nodes of degree two (by contracting edges where one end point has degree two). The exception here is the case of H being a cycle which leads to the true H simply being a single vertex with one loop, i.e., this vertex has degree two. Of course, when H is a tree, this works without the need for H to be a multi-graph.

We have some special notation for the case when H is a tree. Let a, b be two nodes of H'. By $P_{[a,b]}$ we denote the path from a to b. Further, we define $P_{(a,b]} := P_{[a,b]} - a$, and $P_{[a,b)}$, $P_{(a,b)}$ analogously.

Let $S \subseteq G$. Then G[S] is the subgraph of G induced by S, and G - S is the graph obtained from G by deleting the vertices in S (together with the incident edges). For a graph G, we assume G has n vertices and m edges.

In 1965, Fulkerson and Gross proved the following fundamental characterization of interval graphs by orderings of maximal cliques. It is used implicitly in several proofs.

Lemma 2.1 (Fulkerson and Gross [FG65]). A graph G is an interval graph if and only if there exists a linear ordering \leq of the maximal cliques of G such that for every $u \in V(G)$ the maximal cliques containing u appear consecutively in \leq .

A remark on the size of subdivisions and membership in NP. As membership in H-GRAPH is certified through the existence of an appropriate subdivision of H, one might wonder just how large subdivision H' is necessary to ensure that any n-vertex H-graph G has a representation by connected subgraphs of H'. Note that as long as the size of this subdivision is bounded by a polynomial in n, H-graph recognition does indeed belong to NP. We observe that it suffices to subdivide every edge of H 2n times to accommodate an n-vertex H-graph, i.e., without loss of generality the size of H' is at most |V(H)| + 4n|E(H)|.

To see this, we consider an edge ab of H, and its corresponding path $a, c_1, \ldots, c_\ell, b$ in H'. Observe that, for each vertex $v \in V(G)$, H'_v has at most two leaves on this path. Thus, if $\ell > 2n$, there must be a c_i which does not contain any leaf of any H'_v . In particular, this c_i can be contracted into its neighbour on the path while preserving the representation of G. Therefore, it suffices to consider subdivisions of size |V(H)| + 4n|E(H)| and, in particular, for every H, recognition of H-graphs is in NP.

3 Recognition is hard if H is not a cactus

In this section, we negatively answer a question posed by Biró, Hujter, and Tuza [BHT92, Problem 6.3]. Namely, we prove that testing whether a graph is an *H*-graph is NP-complete

when the diamond graph² D is a minor of H. Note that this sharply contrasts the polynomial time solvability of the recognition problem for circular-arc graphs (i.e., when H is a cycle). Before getting to the hardness proof itself, we first establish a technical (though rather straightforward to prove) lemma regarding the essentially unique (up to automorphism) H-representability of the 3-subdivision H_3 of H as an H-graph. Namely, H_3 is obtained from H by subdividing each edge exactly 3 times, that is, in H_3 we have one vertex x_v for each vertex v of H, and for each edge e = uv of H we have the path $x_u, x_{ue}, x_e, x_{ve}, x_v$.

Lemma 3.1. Let H be any multi-graph without vertices of degree 2, and let H_3 be the 3-subdivision of H. The graph H_3 is an H-graph and, for every subdivision H' certifying $H_3 \in H$ -GRAPH (via the representation $\{H'_x : x \in V(H_3)\}$, we have:

- For each non-leaf vertex v of H, the representation H'_{x_v} of the corresponding vertex x_v in H_3 contains exactly one branching point p of H where the degree of v (and x_v) and p coincide.
- For each edge e = uv of H and the corresponding path $x_u x_{ue}, x_e, x_{ve}, x_v$ in H_3 , the representation H'_{x_e} of x_e is strictly contained within the subdivision of a single edge zz' of H such that for distinct edges e, f of H with corresponding "middle" vertices x_e, x_f in H_3 , H'_{x_e} and H'_{x_f} are contained within subdivisions of distinct edges of H.

Moreover, each H-representation of H_3 defines an automorphism of H.

Proof. We first note that this holds trivially for the case when H is K_1 or K_2 .

We now observe that H_3 is indeed an H-graph. Let H' be the 4-subdivision of H, that is, in H' the edge e = uv of H becomes the path $y_u y_{ue}, z_{ue}, z_{ve}, y_{ve}, y_v$. For each vertex v of H with incident edges $\{e_1, \ldots, e_k\}$, we represent x_v by the star $H'_v = H'[\{y_v, y_{ve_1}, y_{ve_2}, \ldots, y_{ve_k}\}]$. For each edge uv of H, we represent:

- x_{ue} by $H'_{x_{ue}}$ = the edge $y_{ue}z_{ue}$,
- x_e by H'_{x_e} = the edge $z_{ue}z_{ve}$, and
- x_{ve} by $H'_{x_{ve}}$ = the edge $z_{ve}y_{ve}$.

It is easy to see that this collection of subgraphs of H' is indeed H-representation of H_3 .

So, we now consider an arbitrary H-representation $\{H'_x : x \in V(H_3)\}$ of H_3 , where H' is the subdivision of H and establish the claimed properties.

Suppose that there is a vertex v of H where v has degree at least three (with incident edges e_1,\ldots,e_k) and H'_{x_v} does not contain a branching point, i.e., all nodes in H'_{x_v} have degree at most two. Now, since the neighborhood $\{x_{ve_1},\ldots,x_{ve_k} \text{ of } x_v \text{ is an independent set (and } k \geq 3),$ this implies that (without loss of generality), $H_{x_{ve_1}}$ is contained within H'_{x_v} . However, this now makes it impossible to represent x_{e_1} since $H'_{x_{e_1}}$ should intersect $H'_{x_{ve_1}}$ but should not intersect H'_{x_v} . Thus, for each vertex v of H with degree at least three, H'_{x_v} contains a branching point. Note that no branching point can occur in two such H'_{x_u} and H'_{x_v} , thus, the vertices of degree at least three are bijectively mapped to the branching points. Finally, since we now know that H'_{x_v} contains exactly one branching point, we remark that the degree of this branching point must match be at least the degree of x_v as otherwise some $H'_{x_{ve_i}}$ would be contained in H'_{x_v}

²The diamond graph is obtained by deleting an edge from a 4-vertex clique.

contradicting the H-representation at hand. Thus, indeed the degree of this branching point must coincide with the degree of x_v (and v).

Now consider any edge e = uv of H. Observe that H'_{x_e} cannot contain any branching points since each branching point is contained in a representation H'_{x_v} where x_v is not a neighbor of x_e . Thus, H'_{x_e} is indeed contained within the subdivision of an edge pq of H, and in particular when p(q) is a branching point, then without loss of generality $H'_{x_u}(H'_{x_v})$ contains u(p). Observe that, when u has degree at least three, $H'_{x_u} \cup H'_{x_{ue}} \cup H'_{x_e}$ consists of a subpath of the subdivision of pq in H' that includes p and as such, for any edge f distinct from e, H'_{x_f} must be contained within the subdivision of a different edge of H. Moreover, this means that for each edge e = uv connecting two vertices of degree at least three, x_e is indeed represented on the subdivision of an edge connecting the corresponding branching points. Also, when one of u or v, say v has degree one (i.e., v is a leaf of H and u has degree at least three), then H'_{x_e} is contained in the subdivision of an edge pq incident to the branching point p contained in H'_{x_u} where q has degree one in H. In particular, here we also have that H'_{x_v} is contained in the subdivision of pq.

Finally, based on these properties, we indeed have an automorphism of H as required. \Box

Our hardness proof stems from the NP-hardness of testing whether a partial order (poset) with *height* one has *interval dimension* at most three; shown by Yannakakis [Yan82]. We denote this problem by INTDIM(1,3). Note that having *height one* means that every element of the poset is either minimal or maximal.

Consider a collection I of closed intervals on the real line. A poset $\mathcal{P}_I = (I, <)$ can be defined on I by considering intervals $x, y \in I$ and setting x < y if and only if the right endpoint of x is strictly to the left of the left endpoint of y. A partial order \mathcal{P} is called an *interval order* when there is an I such that $\mathcal{P} = \mathcal{P}_I$. The *interval dimension* of a poset $\mathcal{P} = (P, <)$, is the minimum number of interval orders whose intersection is \mathcal{P} , i.e., for elements $x, y \in P$, x < y if and only if x is before y in all of the interval orders. Finally, the *incomparability graph* $G_{\mathcal{P}}$ of a poset $\mathcal{P} = (P, <)$ is the graph with V(G) = P and $uv \in E(G_{\mathcal{P}})$ if and only if u and v are not comparable in \mathcal{P} .

Note that if \mathcal{P} has height one, then $G_{\mathcal{P}}$ is the complement of a bipartite graph. The vertices $V(G_{\mathcal{P}})$ naturally partition into two cliques K_{max} and K_{min} , containing the maximal and the minimal elements of \mathcal{P} , respectively. An example depicting a D-representation of a specific \mathcal{P} is provided in Fig. 2, where D is the diamond graph. With these definitions and the prior lemma in place, we now prove the theorem of this section.

Theorem 1. Testing if G is an H-graph is NP-complete if the diamond graph D is a minor of H

Proof. The proof is split into two parts. In the Part 1, we prove the essential case which shows that testing whether G is an D-graph is NP-hard. This argument is generalized in Part 2 to the case when H contains D as a minor.

Part 1: H is the diamond. First, we summarize the idea behind our proof. As stated above, we will encode an instance \mathcal{P} of INTDIM(1,3) as an instance of membership testing in D-GRAPH. For a given height 1 poset \mathcal{P} , we construct its incomparability graph $G_{\mathcal{P}}$, slightly augment $G_{\mathcal{P}}$ to get a graph G, and show that G is in D-GRAPH if and only if the interval dimension of \mathcal{P} is at most 3. In particular, a "middle" part of the three paths connecting the two degree 3 vertices in D will encode the three interval orders whose intersection is \mathcal{P} .

Note that, we consider H as the multi-graph consisting of three parallel edges e_a, e_b, e_c between two nodes v_{\min} and v_{\max} To construct G, we use the graph H_3 which the 3-subdivision

of H. Namely, H_3 has two vertices u_{\min} and u_{\max} of degree three and nine vertices $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ of degree two where $u_{\min}, \aleph_1, \aleph_2, \aleph_3, u_{\max}$ is a path for each $\aleph \in \{a, b, c\}$. Note that, by Lemma 3.1, without loss of generality, H_3 is an H-graph where in every H-representation of H_3 , say on a subdivision H' of H, we have:

- $H'_{u_{\min}}$ contains v_{\min} and $H'_{u_{\max}}$ contains v_{\max} ,
- For each $\aleph \in \{a, b, c\}$, H'_{\aleph_2} is contained in the subdivision of e_{\aleph} .

It is within these H'_{\aleph_2} paths that we will see the interval orders.

We are now ready to construct our graph G from H_3 and the graph $G_{\mathcal{P}}$ of a given height one poset $\mathcal{P} = (P, <)$, recall that K_{\min} and K_{\max} denote cliques on the minima and maxima of \mathcal{P} respectively. Let $V_{\min} = \{u_{\min}, a_1, a_2, b_1, b_2, c_1, c_2\}$ and let $V_{\max} = \{u_{\max}, a_3, a_2, b_3, b_2, c_3, c_2\}$. The graph G is the union of $G_{\mathcal{P}}$ and H_3 where, additionally, each vertex of K_{\min} is adjacent to each vertex of V_{\max} .

Claim 3.1. \mathcal{P} has interval dimension at most 3 if and only if G is a H-graph.

Proof. For the reverse direction, consider an H-representation of G on a subdivision H' of H. As remarked above, by Lemma 3.1, $H'_{u_{\min}}$ contains the node v_{\min} and $H'_{u_{\max}}$ contains the node v_{\max} . The minimal elements of $\mathcal P$ are not adjacent to the vertices of u_{\max} . Therefore, for each $x \in K_{\min}$, H'_x cannot contain v_{\max} , i.e., H'_x is a subtree of $H' - \{v_{\max}\}$. In particular, for each of the three (v_{\min}, v_{\max}) paths A, B, C in H', H'_x defines one (possibly empty) subpath/interval (originating in v_{\min}). Similarly, for each $y \in K_{\max}$, H'_y cannot contain v_{\min} and as such H'_y defines, for each of A, B, C, one subpath (originating in v_{\max}). It is easy to see that these intervals provide the interval orders $\mathcal P_{I_A}$, $\mathcal P_{I_B}$, and $\mathcal P_{I_C}$ such that $\mathcal P_{I_A} \cap \mathcal P_{I_B} \cap \mathcal P_{I_C} = \mathcal P$.

For the forward direction, let I_1 , I_2 , I_3 be sets of intervals such that $\mathcal{P} = \mathcal{P}_{I_1} \cap \mathcal{P}_{I_2} \cap \mathcal{P}_{I_3}$. We assume that each interval in I_i is labelled according to the corresponding element of \mathcal{P} . Further, we assume that the intervals corresponding to the minimal elements have their left endpoints at 0 and their right endpoints are integers in the range [0, n-1]. Similarly, we assume that the intervals corresponding to the maximal elements have their right endpoints at n and their left endpoints are integers in the range [1, n]. With this in mind, for each minimal element x and each $i \in \{1, 2, 3\}$, we use x_i to denote the right endpoint of its interval in I_i , and for each

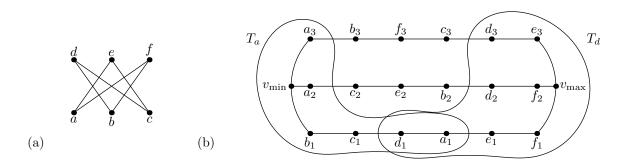


Figure 2: (a) A partially ordered set $\mathcal{P}=(P,<)$ of height 1, interval dimension 3, but not 2. We define the following interval orders: $I_1=l_al_bl_cr_br_cl_dr_al_el_fr_dr_er_f$, $I_2=l_al_bl_cr_ar_cl_er_bl_dl_fr_dr_er_f$, and $I_3=l_al_bl_cr_ar_bl_fr_cl_dl_er_dr_er_f$, where $[l_a,r_a]$ represents an interval corresponding to $a\in P$. Note that $\mathcal{P}_{I_1}\cap\mathcal{P}_{I_2}\cap\mathcal{P}_{I_3}=\mathcal{P}$. (b) An illustration of part of the D-representation. Here, T_a and T_b indicate the subgraphs representing the elements a and b.

maximal element y and each $i \in \{1, 2, 3\}$, we use y_i to denote the left endpoint of its interval in I_i .

Let H' be the subdivision of H obtained by subdividing the three $v_{\min}v_{\max}$ edges n+5 times. We label the three (v_{\min}, v_{\max}) -paths in H' as follows:

- $v_{\min}, \alpha_{\min}, \alpha'_{\min}, \alpha_0, \alpha_1, \dots, \alpha_n, \alpha'_{\max}, \alpha_{\max}, v_{\max},$
- $v_{\min}, \beta_{\min}, \beta'_{\min}, \beta_0, \beta_1, \dots, \beta_n, \beta'_{\max}, \beta_{\max}, v_{\max}$ and
- $v_{\min}, \gamma_{\min}, \gamma'_{\min}, \gamma_0, \gamma_1, \dots, \gamma_n, \gamma'_{\max}, \gamma_{\max}, v_{\max}$

We are now ready to describe an H-representation of G on H'. Each minimal element x is represented by the minimal subtree of H' which includes the nodes v_{\min} , α_{x_1} , β_{x_2} , γ_{x_3} . Similarly, each maximal element y is represented by the minimal subtree of H' which includes the nodes v_{\max} , α_{y_1} , β_{y_2} , γ_{y_3} . We can now see that the comparable elements of \mathcal{P} are represented by disjoint subgraphs of H' and that the incomparable elements map to intersecting subgraphs. Finally, the vertices of H_3 are represented as follows:

- u_{\min} is represented by the subtree induced by v_{\min} , α_{\min} , α_{\min} , and γ_{\min} ; analogously, u_{\max} is represented by the subtree induced by v_{\max} , α_{\max} , β_{\max} , and γ_{\max}
- a_1 , b_1 , and c_1 are represented by the edges $\alpha_{\min}\alpha'_{\min}$, $\beta_{\min}\beta'_{\min}$, and $\gamma_{\min}\gamma'_{\min}$, respectively; analogously, a_3 , b_3 , and c_3 are represented by the edges $\alpha_{\max}\alpha'_{\max}$, $\beta_{\max}\beta'_{\max}$, and $\gamma_{\max}\gamma'_{\max}$, respectively, and
- a_2 is represented by the path $\alpha'_{\min}, \alpha_0, \dots, \alpha_n, \alpha'_{\max}$, and
- b_2 is represented by the path $\beta'_{\min}, \beta_0, \dots, \beta_n, \beta_{\min}$, and
- c_2 is represented by the path $\gamma'_{\min}, \gamma_0, \ldots, \gamma_n, \gamma'_{\max}$.

Clearly, in this construction, the graph H_3 is correctly represented. Moreover, the subtree corresponding to every minimal element includes all of the nodes v_{\min} , α_{\min} , α'_{\min} , β'_{\min} , β'_{\min} , γ'_{\min} , but none of the opposite max-nodes. Thus, each minimal element is universal to V_{\min} and non-adjacent to the vertices of $V_{\max} \setminus \{a_2, b_2, c_3\}$, as needed. Symmetrically, each maximal element is universal to V_{\max} and non-adjacent to the vertices of $V_{\min} \setminus \{a_2, b_2, c_3\}$. It follows that G is an H-graph.

This completes the first part of the proof.

Part 2: H contains the diamond graph D as a minor. The argument here follows very similarly to the proof shown in Part 1. We again use the 3-subdivision H_3 of H which, by Lemma 3.1, canonically "covers" H. Again, H_3 will be used as part of the graph G we will construct from $G_{\mathcal{P}}$ so that $G \in H$ -GRAPH if and only if \mathcal{P} has interval dimension at most 3. Importantly, H_3 also allows us to, with a careful choice of V_{\min} and V_{\max} , appropriately restrict the representations of the minima and maxima to only use a chosen diamond minor of H (up to automorphism of course) as before.

Observe that, since H contains D as a minor (and the maximum degree of D is three), a subdivision of D (or D itself) is a subgraph of H. Let D^* be a subgraph of H that is a subdivision of D. In particular, D^* consists of two nodes d_{\min} and d_{\max} of degree 3 and three (d_{\min}, d_{\max}) -paths A, B, C that are edge disjoint and whose internal vertices are of degree 2. Let

 $\alpha = d_{\min} d_{\max}^{\alpha} \in A$, $\beta = d_{\min} d_{\max}^{\beta} \in B$, and $\gamma = d_{\min} d_{\max}^{\gamma} \in C$ denote the three edges incident to the node d_{\min} in D^* . These three edges will be used equivalently to the three $v_{\min} v_{\max}$ edges as in Part 1, i.e., they will the "location" in H where we will see the three intervals certifying that our original poset has interval dimension 3.

Now, let D_3^* be the subgraph of H_3 corresponding to D^* (D_3^* is also a subdivision of D). Let z_{\min}, z_{\max} be the vertices in D_3^* corresponding (via the subdivision of H to H_3) to d_{\min}, d_{\max} in D^* respectively, and further:

- let z_{\min} , a_1 , a_2 , a_3 , a_{\max} be the path in D_3^* corresponding to the subdivision of the edge α of D^* , and
- let $z_{\min}, b_1, b_2, b_3, b_{\max}$ be the path in D_3^* corresponding to the subdivision of the edge β of D^* , and
- let $z_{\min}, c_1, c_2, c_3, c_{\max}$ be the path in D_3^* corresponding to the subdivision of the edge γ of D^* .

We are now ready to construct our graph G from H_3 and the graph $G_{\mathcal{P}}$ of a given height one poset $\mathcal{P}=(P,<)$ so that $G\in H$ -GRAPH if and only if \mathcal{P} has interval dimension three. Recall that K_{\min} and K_{\max} denote cliques on the minima and maxima of \mathcal{P} respectively. As in Part 1, we let $V_{\min}=\{z_{\min},a_1,a_2,b_1,b_2,c_1,c_2\}$. Similarly to Part 1, we let V_{\max} be the vertex set of the minimal subgraph of D_3^* containing $\{z_{\max},a_2,b_2,c_2\}$. In other words $V_{\max}=V(D_3^*)\setminus V_{\min}\cup\{a_2,b_2,c_2\}=V(D_3^*)\setminus\{z_{\min},a_1,b_1,c_1\}$. Now, as in Part 1, the graph G is the union of $G_{\mathcal{P}}$ and H_3 where, additionally, each vertex of K_{\min} is adjacent to each vertex of V_{\min} and each vertex of K_{\max} is adjacent to each vertex of V_{\max} .

The completion of the proof now follows nearly identically to the proof of the claim in Part 1. Namely, by Lemma 3.1, H_3 has a unique up to automorphism H-representation, and the vertices of K_{\min} and K_{\max} can essentially only be represented on the D^* part of H (due to their adjacency with the vertices of H_3)³. Moreover, within the three edges α, β, γ , there will be the representations of a_2, b_2, c_2 and within these representations we will indeed have the (at most) 3 interval models.

The next section gives a positive answer for the following problem in the case when H is a tree. Also, recall that when H is a single cycle, H-GRAPH is the class of circular-arc graphs and as such can be recognized in linear time. This leaves the following problem.

Problem 1. For a non-tree fixed cactus graph H (other than a single cycle), is there a polynomial-time time algorithm testing whether G is an H-graph?

4 Polynomial-time recognition algorithms

We present an $\mathcal{O}(n^{3.5})$ -time algorithm recognizing S_d -graphs and an XP-time algorithm recognizing T-graphs (parametrized by the size of the tree T). We begin with a lemma that motivates our approach. It implies that if G is a T-graph, then there exists a representation of G such that every branching point is "contained" in some maximal clique of G.

³While the representation of a vertex of $G_{\mathcal{P}}$ might "reach out" beyond D^* onto an incident edge, it can never traverse all of such an edge because, by Lemma 3.1, there is a vertex x_e of H_3 occupying the "middle" of that edge and, by construction, x_e is not adjacent to any vertex of $G_{\mathcal{P}}$.

Lemma 4.1. For any T-graph G and T-representation \mathcal{R} of G, \mathcal{R} can be modified such that for every branch node $b \in V(T')$, we have $b \in \bigcap_{v \in C} V(T'_v)$, for some maximal clique C of G.

Proof. For every node x of the subdivision T', let $V_x = \{u \in V(G) : x \in V(T'_u)\}$ be the set of vertices of G corresponding to the subtrees passing through x. Let b be a branching point such that V_b is not a maximal clique.

We pick a maximal clique C with $C \supseteq V_b$. Since \mathcal{R} satisfies the Helly property, there is a node $a \in \bigcap \{V(T'_v) : v \in C\}$. Note that for every node x of $P_{[a,b]}$, we have $V_x \supseteq V_b$. Let x be the node of $P_{(b,a]}$ closest to b such that $V_x \neq V_b$. Then, for each $v \in V_x \setminus V_b$, we update T'_v to be $T'_v \cup P_{[b,x]}$. Thus, we obtain a correct representation of G with $V_b = V_x$.

We repeat the process described in the previous paragraph until V_b is a maximal clique. \Box

Remark on subdivisions. For convenience, we assume throughout the whole section that we already have a sufficiently large subdivision T' of T. At the end, it will be clear that a subdivision T' of T with $|V(T')| \le cn + |V(T)|$, for some constant c, suffices. In fact, it suffices to have c = 3.

General idea. It is well-known that chordal graphs, and therefore also T-graphs, have at most n maximal cliques and that they can be listed in linear time. Let \mathcal{B} be the set of branching points of T and let \mathcal{C} be the set of all maximal cliques of G. The main part of our algorithm attempts, for a given $f: \mathcal{B} \to \mathcal{C}$, to construct a T-representation satisfying $V_b = \bigcap_{v \in f(b)} V(T'_v)$, for every $b \in \mathcal{B}$, where $V_b = \{u \in V(G) : b \in V(T'_u)\}$. By Lemma 4.1, there always exists such a representation.

To this end, we try find interval representations of the connected components of $G - \bigcup_{b \in \mathcal{B}} f(b)$ on the paths $T' - \mathcal{B}$ such that the following conditions hold:

- (i) If interval representations of the connected components X_1, \ldots, X_k are on a path $P_{(b,l]}$, where $b \in \mathcal{B}$ and l is a leaf of T', then the induced subgraph $G[f(b) \cup V(X_1) \cup \cdots \cup V(X_k)]$ has an interval representation on $P_{[b,l]}$ in which f(b) is the leftmost clique.
- (ii) If interval representations of the connected components X_1, \ldots, X_k are on a path $P_{(b,b')}$, where $b, b' \in \mathcal{B}$, then the induced subgraph $G[f(b) \cup V(X_1) \cup \cdots \cup V(X_k) \cup f(b')]$ has an interval representation on $P_{[b,b']}$ in which f(b) and f(b') are the rightmost and leftmost cliques, respectively.

4.1 Recognition of S_d -graphs

In the case when $T = S_d$, we have $\mathcal{B} = \{b\}$ and $V(T) = \{b\} \cup \{l_1, \ldots, l_d\}$. The number of mappings $f : \{b\} \to \mathcal{C}$ is exactly the same as the number of maximal cliques of G, which is at most n (otherwise it is not an S_d -graph). For every maximal clique C of G, we try to construct a T-representation \mathcal{R} such that $b \in \bigcap_{v \in C} V(T'_v)$.

Assume that G has such an S_d -representation, for some maximal clique C. Then the connected components of G-C are interval graphs and each connected component can be represented on one of the paths $P_{(b,l_i]}$, which is a subdivision of the edge bl_i ; see Fig. 3a and 3c. However, some pairs of connected components of G-C cannot be placed on the same path $P_{(b,l_i]}$, since their "neighborhoods" in C are not "compatible". The idea is to define a partial order \triangleright on the components of G-C such that for every linear chain $X_1 \triangleright \cdots \triangleright X_k$, the induced subgraph $G[C,V(X_1),\ldots,V(X_k)]$ can be represented on some path $P_{(b,l_i]}$; see Fig. 3b.

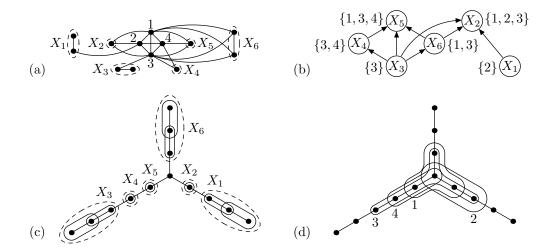


Figure 3: (a) An example of an S_d -graph G with a maximal clique $C = \{1, 2, 3, 4\}$. (b) The partial ordering \triangleright on the connected components of G - C with chain cover of size 3: $X_2 \triangleright X_1$, $X_5 \triangleright X_4 \triangleright X_3$, and X_6 . (c) The connected components placed on the paths $P_{(b,l_1]}$, $P_{(b,l_2]}$, and $P_{(b,l_3]}$, according to the chain cover of \triangleright . (d) The subtrees T'_1, T'_2, T'_3, T'_4 corresponding to the vertices of the maximal clique C give an S_d -representation of G with $b \in \bigcap_{v \in C} V(T'_v)$.

We define $N_C(u)$ and $N_C(X)$ to be the *neighbourhoods* of the vertex u in C and of the components X in C, respectively. Formally,

$$N_C(u) = \{ v \in C : vu \in E(G) \}$$
 and $N_C(X) = \bigcup \{ N_C(u) : u \in V(X) \}.$

Note that, if we have two components X and X' on the same branch where $N_C(X') \subseteq N_C(u)$ for every $u \in V(X)$, then X must be closer to C than X' if they are represented on the same path $P_{(b,l_i]}$.

We say that components X and X' are equivalent, $X \sim X'$, if there is a subset C' of C such that $N_C(u) = C'$ for every $u \in V(X)$ and $N_C(u') = C'$ for every $u' \in V(X')$. Note that equivalent components X and X' can be represented in an interval representation of G[C, V(X), V(X')] in an arbitrary order and they can be treated as one component. We denote the set of the equivalence classes $G - C/ \sim$ by \mathcal{X} . For $X, X' \in \mathcal{X}$, we put:

$$X \triangleright X'$$
 if for every $u \in V(X), N_C(X') \subseteq N_C(u)$ or if $X = X'$. (1)

Lemma 4.2. The relation \triangleright is a partial ordering on \mathcal{X} .

Proof. The relation \triangleright is reflexive by definition. Suppose that $X \triangleright X'$ and $X' \triangleright X$. For every $u \in V(X)$ and $u' \in V(X')$, we have

$$N_C(u') \subseteq N_C(X') \subseteq N_C(u)$$
 and $N_C(u) \subseteq N_C(X) \subseteq N_C(u')$.

Therefore, $N_C(u) = N_C(u')$ for every $u \in V(X)$ and $u' \in V(X')$ and X' are equivalent. We assume that X contains only non-equivalent components. So, X = X' and the relation \triangleright is asymmetric. It can be easily checked that \triangleright is also transitive.

Lemma 4.3. Let $X_1, \ldots, X_k \in \mathcal{X}$. Then the induced subgraph $G[C, V(X_1), \ldots, V(X_k)]$ has an interval representation with C being the leftmost clique if and only if $X_1 \triangleright \cdots \triangleright X_k$ and each $G[C, X_i]$ has an interval representation with C being the leftmost clique.

Proof. Suppose that there is an interval representation \mathcal{R} of $G[C, V(X_1), \ldots, V(X_k)]$ with C being the leftmost maximal clique. Since each X_i is a connected components of G - C, their representations in \mathcal{R} cannot overlap. Without loss of generality, we assume that the components X_1, \ldots, X_k are ordered such that i < j if and only if X_i is placed closer to C in \mathcal{R} than X_j . Let $u \in V(X_i)$ and $v \in N_C(X_j)$. The vertex v is adjacent to at least one vertex of X_j . Therefore, the representation of v covers the whole component X_i in \mathcal{R} , i.e., we have $v \in N_C(u)$ and $X_i \triangleright X_j$.

For the converse, we assume that X_1, \ldots, X_k form a chain in \triangleright and every $G[C, X_i]$ has an interval representation \mathcal{R}_i with C being the leftmost clique. Since $X_i \triangleright X_j$, for i < j, every vertex in $N_C(X_j)$ is adjacent to every vertex of X_i . We now construct an interval representation of $G[C, V(X_1), \ldots, V(X_k)]$. We first place the interval representations of all X_i 's (i.e., we use \mathcal{R}_i restricted to the intervals of $V(X_i)$) on the real line according to \triangleright , with X_1 being the leftmost. Let $x_1, \ldots, x_{k+1} \in \mathbb{R}$ be the points of the real line such that X_i is represented on the interval $(x_i, x_{i+1}) \subseteq \mathbb{R}$.

It remains to construct a representation for every vertex $v \in C$. Let

$$C_k = N_C(X_k)$$
 and $C_i = N_C(X_i) \setminus \bigcup_{j=i+1}^k N_C(X_j), i = 0, \dots, k-1$ where $X_0 = C$.

Let $x_0 \in \mathbb{R}$ be a point left of x_1 . All the vertices in C_0 are represented by the interval $[x_0, y]$, for some $y < x_1$. The intervals representing vertices in C_i are constructed inductively, for $i = k, k - 1, \ldots, 1$. For $i \le k$, we assume that we constructed the representations of vertices in C_{i+1}, \ldots, C_k . Note, if $X_j \triangleright X_i$, then for every $u \in V(X_j)$, we have $N_C(X_i) \subseteq N_C(u)$. Therefore, every vertex in C_i is represented by an interval of the form $[x_0, z]$, where $z \in (x_i, x_{i+1})$ is a suitable point given by the representation \mathcal{R}_i of $G[C, X_i]$.

The following theorem gives a characterization of S_d -graphs. It generalizes the characterization of interval graphs due to Fulkerson and Gross; see Lemma 2.1.

Theorem 2 (Characterization of S_d -graphs). A graph G is an S_d -graph if an only if there is a maximal clique C of G such that the following hold:

- (i) For every connected component X of G-C, the induced subgraph G[C,X] has an interval representation with C being the leftmost clique.
- (ii) The partial order \triangleright on $\mathcal{X} = G C / \sim$ has a chain cover of size at most d.

Proof. Suppose that G is an S_d -graph with a representation satisfying $b \in \bigcap_{v \in C} V(T'_v)$; such a representation always exists by Lemma 4.1. The representation of a connected component $X \in \mathcal{X}$ can not pass through the node b since otherwise C would not be a maximal clique. Clearly, the conditions (i) is satisfied. The representations of every two components in \mathcal{X} have to be placed on non-overlapping parts of the subdivided S_d . By Lemma 4.3, we have that the components placed on some path $P_{(b,l_i]}$ of the subdivided S_d form a linear chain in \triangleright . Therefore, the partial order \triangleright has a chain cover of size at most d and the condition (ii) is satisfied; see Fig. 3b.

Suppose that the conditions (i) and (ii) are satisfied. We put the components in \mathcal{X} on the paths $P_{(b,l_1]},\ldots,P_{(b,l_d]}$ according to the chain cover of the partial order \triangleright which has size at most d, i.e, every chain of \triangleright is placed on one $P_{(b,l_i]}$. By Lemma 4.3, for every chain X_1,\ldots,X_k in \triangleright , we can find an interval representation of the graph $G[C,V(X_1),\ldots,V(X_k)]$ with C being the leftmost maximal clique.

Algorithm. By combining Lemmas 4.3 and Theorem 2 we obtain an algorithm for recognizing S_d -graphs. For a given graph G and its maximal clique C, we do the following:

- 1. We delete the maximal clique C and construct the partial order \triangleright on the set of non-equivalent connected components \mathcal{X} .
- 2. We test whether the partial order \triangleright can be covered by at most d chains.
- 3. For each linear chain $X_1^i \triangleright \cdots \triangleright X_k^i$, $1 \le i \le d$, we construct an interval representation \mathcal{R}_i of the induced subgraph $G[C, V(X_1^i), \ldots, V(X_k^i)]$, with C being the leftmost maximal clique, on one of the paths of the subdivided S_d .
- 4. We complete the whole representation by placing each \mathcal{R}_i on the path $P_{[b,l_i]}$ so that $b \in \bigcap_{v \in C} V(T'_v)$.

Theorem 3. Recognition of S_d -graphs can be solved in $\mathcal{O}(n^{3.5})$ time.

Proof. Every chordal graph has at most n maximal cliques, where n is the number of vertices, and they can be listed in linear time [RTL76]. For every clique C, our algorithm tries to find an S_d -representation with $b \in \bigcap_{v \in C} V(T'_v)$. The partial order \triangleright can be constructed in time $\mathcal{O}(n^2)$. By forgetting the orientation in the partial order \triangleright , we get a comparability graph, and every clique in the comparability graph induces a linear chain in \triangleright . A relatively simple algorithm finds a minimum clique-cover of a comparability graph in time $\mathcal{O}(n^3)$ [Gol77]. An algorithm that runs in time $\mathcal{O}(n^{2.5})$ can by obtained by a combination of [Ful56] and [HK73]. Testing whether $G[C, V(X_1^i), \ldots, V(X_k^i)]$ has an interval representation with C being the leftmost maximal clique can be done in linear time. Thus, the overall time complexity of our algorithm is $\mathcal{O}(n^{3.5})$. \square

Problem 2. Can we recognize S_d -graphs in time $\mathcal{O}(n^{2.5})$? In particular, can we find the clique that can be placed in the center of S_d efficiently?

4.2 Recognition of T-graphs

The algorithm for recognizing T-graphs is a generalization of the algorithm for recognizing S_d -graphs described above. Let $f: \mathcal{B} \to \mathcal{C}$ be an fixed assignment of cliques.

Assumption (connectedness of G). Suppose that G is disconnected. Then it can be written as a disjoint union of some X and \widehat{G} , where X is a connected component of G. Let \mathcal{C}_X and $\widehat{\mathcal{C}}$ be the maximal cliques of X and \widehat{G} , respectively. The sets $f^{-1}(\mathcal{C}_X)$ and $f^{-1}(\widehat{\mathcal{C}})$ induce subtrees T_X and \widehat{T} of T separated by the branch ab, where $a \in V(T_X)$ and $b \in V(\widehat{T})$ (otherwise f is invalid). We subdivide the branch ab by nodes c_1 and c_2 . Then we try to find a representation of X on the tree $T_X \cup ac_1$ and a representation of \widehat{G} on $\widehat{T} \cup c_2b$. Therefore, we may assume that G is connected.

Assumption (injectiveness of f). Suppose that f is not injective, i.e., f(b) = f(b'). Then for every branching point b'' which lies on the path from b to b', we must have f(b) = f(b'') = f(b') (otherwise f is invalid). For $C \in f(\mathcal{B})$, the branching points in $f^{-1}(C)$, together with the paths connecting them, have to form a subtree T_C of T. In this case the whole subtree T_C can be contracted into a single node a. Note that if there is a component X of $G - \bigcup_{b \in \mathcal{B}} f(b)$ where every vertex of X adjacent to every vertex of C, then X can be represented on any branch incident to a by subdividing it appropriately. Thus, we may assume that f is injective.

Step 1 (components between branching points). The first step of our algorithm is to find for $b, b' \in E(T)$, which components have to be represented on the path $P_{(b,b')}$ of T'.

Lemma 4.4. Let X be a connected component of $G - \bigcup_{b \in \mathcal{B}} f(b)$ and $bb' \in E(T)$. If the sets

$$(f(b) \setminus f(b')) \cap N_{f(b)}(X) \neq \emptyset$$
 and $(f(b') \setminus f(b)) \cap N_{f(b')}(X) \neq \emptyset$,

then X has to be represented on $P_{(b,b')}$ of T'.

Proof. Let $v \in (f(b) \setminus f(b')) \cap N_{f(b)}(X)$ and $u \in (f(b') \setminus f(b)) \cap N_{f(b')}(X)$. Since $v \notin f(b')$, we have $b' \notin V(T'_v)$. Similarly we have $b \notin V(T'_u)$. Putting it together, we have that $b \in V(T'_v)$ and $b' \notin V(T'_v)$, and $b \notin V(T'_u)$ and $b' \in V(T'_u)$. Since X is adjacent to both u and v, the only possible path where X can be represented is $P_{(b,b')}$; see Fig. 4a and 4b.

We do the following for each $b, b' \in \mathcal{B}$ such that $bb' \in E(T)$. Let $X_{b,b'}$ be the disjoint union of the components satisfying the conditions of Lemma 4.4. If the induced interval subgraph $G[C \cup V(X_{b,b'}) \cup C']$ has a representation such that the cliques C and C' are the leftmost and the rightmost, respectively, then we can represent $X_{b,b'}$ in the middle of the path $P_{(b,b')}$. If no such representation exists, then G does not have T-representation for this particular $f: \mathcal{B} \to \mathcal{C}$. This means that the representation of $X_{b,b'}$ is constructed on a proper subpath of $P_{(b,b')}$ – recall that we are assuming that the subdivision T' is sufficiently large.

Next, we do the following for every $b \in \mathcal{B}$. Let l_1, \ldots, l_p and b_1, \ldots, b_q be the leaves of T and the branching points of T, respectively, such that $bl_i \in E(T)$, for every $i = 1, \ldots, p$, and $bb_j \in E(T)$, for every $j = 1, \ldots, q$. Let a_1, \ldots, a_q and a'_1, \ldots, a'_q be the points of the paths $P_{[b,b_1]}, \ldots, P_{[b,b_q]}$, respectively, such that X_{b,b_i} is represented on the subpath $P_{(a_i,a'_i)}$. We define S(b) to be the subdivided star consisting of the paths $P_{[b,l_1]}, \ldots, P_{[b,l_p]}, P_{[b,a_1)}, \ldots, P_{[b,a_q)}$. Note that if a vertex $u \in V(X_{b,b_i})$ is adjacent to a vertex v in f(b), then the representation of T'_v of v contains the whole subpath $P_{(b,a_i)}$. This means that a component X, which does not satisfy the condition of Lemma 4.4, can be represented on $P_{(b,a_i)}$ only if $N_{f(b)}(X_{b,b_i}) \subseteq N_{f(b)}(X)$. We remove the subpaths $P_{(a_i,a'_i)}$ (together with the representations of X_{b,b'_i}) and we are left with disjoint subdivided stars with restrictions; see Fig 4c.

Step 2 (disjoint stars with restrictions). We reduced the problem of recognizing T-graphs to the following problem. Let H be a fixed graph formed by the disjoint union of k stars $S(b_1), \ldots, S(b_k)$ with branching points b_1, \ldots, b_k . On the input we have a graph G, an injective mapping $f: \{b_1, \ldots, b_k\} \to \mathcal{C}$, and for every edge of $S(b_i)$ a subset of $f(b_i)$, called restrictions. We want to find a representation of G on H such that $b_i \in \bigcap_{v \in f(b_i)} V(H'_v)$, and for every connected component X of $G - \bigcup_{i=1}^k f(b_i)$, the vertices V(X) have to be adjacent to every vertex in the restrictions corresponding to the path on which X is represented.



Figure 4: (a) A T-graph G, where T is the tree shown on the right. We have $f(b) = C_1$ and $f(b') = C_2$. (b) Component X_4 with $C_1 \setminus C_2 \cap N_{C_1}(X) \neq \emptyset$ and $C_2 \setminus C_1 \cap N_{C_2}(X) \neq \emptyset$. In this case, $X_{b,b'} = X_4$. (c) A segment of the star corresponding to the clique is labeled by $\{1,3,4\} \subseteq C_1$. A component X can be represented on this segment only if $\{1,3,4\} \subseteq N_{C_1}(X)$.

To solve this problem, we define a partial ordering on the connected components of G-C, where $C=\bigcup f(b_i)$. The notions $N_C(u)$ and $N_C(X)$ are defined as in the same way as in the algorithm for recognizing S_d -graphs. We get a partial ordering \triangleright on the set of non-equivalent connected components \mathcal{X} of G-C. Moreover, to each component $X\in\mathcal{X}$, we assign a list of colors L(X) which correspond to the subpaths from a branching point to a leaf in the stars $S(b_1),\ldots,S(b_k)$, on which they can be represented. Each list L(X) has size at most $d=\sum_{i=1}^k d_i$, where d_i is the degree of b_i .

Suppose that there exists a chain cover of \triangleright of size d such that for every chain $X_1 \triangleright \cdots \triangleright X_\ell$ in this cover we can pick a color belonging to every $\bigcap_{j=1}^{\ell} L(X_j)$ such that no two chains get the same color. In that case a representation of G satisfying the restrictions can be constructed analogously as in the proof of Lemma 4.3 and 2.

The partial ordering \triangleright on the components \mathcal{X} defines a comparability graph P with a list of colors L(v) assigned to every vertex $v \in V(P)$. If we find a list coloring c of its complement \overline{P} , i.e., a coloring that for every vertex v uses only colors from its list L(v), then the vertices of the same color in \overline{P} correspond to a chain (clique) in P. Therefore, we have reduced our problem to list coloring co-comparability graphs with lists of bounded size.

Step 3 (bounded list coloring of co-comparability graphs). We showed that to solve the problem of recognizing T-graphs we need to solve the ℓ -list coloring problem for co-comparability graphs where $\ell = 2 \cdot |E(T)|$. In particular, given a co-comparability graph G, a set of colors S such that $|S| \leq \ell$, and a set $L(v) \subseteq S$ for each vertex v, we want to find a proper coloring $c: V(G) \to S$ such that for every vertex v, we have $c(v) \in L(v)$.

In [BMO11], the authors consider the capacitated coloring problem for co-comparability graphs. Namely, given a graph G, an integer $s \ge 1$ of colors, and positive integers $\alpha_1^*, \ldots, \alpha_s^*$, a capacitated s-coloring c of G is a proper s-coloring such that the number of vertices assigned color i is bounded by α_i^* , i.e., $|c^{-1}(i)| \le \alpha_i^*$. The authors prove that the capacitated coloring of co-comparability graphs can be solved in polynomial time for fixed s. In the next section, we modify their approach to solve the s-list coloring problem on co-comparability graphs in $\mathcal{O}(n^{s^2+1}s^3)$ time. This provides the following theorem.

Theorem 4. Recognition of T-graphs can be solved in $n^{\mathcal{O}(\|T\|^2)}$.

Problem 3. Is there an FPT algorithm for recognizing T-graphs?

4.3 Bounded list coloring of co-comparability graphs

Here, we provide a polynomial time algorithm for the problem of bounded list coloring of cocomparability graphs. Our result can be seen as a generalization of the polynomial time algorithm of Enright, Stewart, and Tardos [EST14] for bounded list coloring on a class which includes both interval graphs and permutation graphs. However, they [EST14] explicitly state that their approach does not extend to co-comparability graphs. To prove this also for co-comparability graphs, we slightly modify the approach in [BMO11].

In [BMO11], the problem of capacitated coloring is solved for a more general class of graphs, so called k-thin graphs. A graph G is k-thin if there exists an ordering v_1, \ldots, v_n of V(G) and a partition of V(G) into k classes V^1, \ldots, V^k such that, for each triple p, q, r with p < q < r, if v_p, v_q belong to the same class and $v_r v_p \in E(G)$, then $v_r v_q \in E(G)$. Such ordering and partition are called *consistent*. The minimum k such that G is k-thin is called the *thinness* of G. Graphs with bounded thinness were introduced in [MORC07] as a generalization of interval graphs. Note that interval graphs are exactly the 1-thin graphs.

Recall that a graph G is a comparability graph if there exits an ordering v_1, \ldots, v_n of V(G) such that, for each triple p, q, r with p < q < r, if $v_p v_q$ and $v_q v_r$ are edges of G, then so is $v_p v_r$. Such an ordering is a comparability ordering.

Lemma 4.5 (Theorem 8, [BMO11]). Let G be a co-comparability graph. Then the thinness of G is at most $\chi(G)$, where χ is the chromatic number. Moreover, any vertex partition given by a coloring of G and any comparability ordering for its complement are consistent.

Let G be k-thin graph, and let v_1, \ldots, v_n and V^1, \ldots, V^k be an ordering and a partition of V(G) which are consistent. Note that the ordering induces an order on each class V^j . For each vertex v_r and class V^j , let $N(v_r, j)_{<}$ be the set of neighbors of v_r in V^j that are smaller than v_r , i.e., $N(v_r, j)_{<} = V^j \cap \{v_1, \ldots, v_{r-1}\} \cap N(v_r)$. For each class V^j let $\Delta(j)_{<}$ be the maximum size of $N(v_r, j)_{<}$ over all vertices v_r . The following lemma gives an alternative definition of k-thin graphs.

Lemma 4.6 (Fact 7, [BMO11]). For each vertex $v_r \in \{v_1, \ldots, v_n\}$ and each $j \in \{1, \ldots, k\}$, the set $N(v_r, j)_{<}$ is such that:

- the vertices in $N(v_r, j)_{<}$ are consecutive, with respect to the order induced on V^j .
- if $N(v_r, j) < \neq \emptyset$, then it includes the vertex with largest index in $V^j \cap \{v_1, \dots, v_{r-1}\}$.

Bounded List Coloring On k-thin Graphs. In [BMO11], the problem of capacitated coloring is reduced to a reachability problem on an auxiliary acyclic digraph. We obtain an algorithm for bounded list coloring on k-thin graphs by slightly modifying the algorithm for capacitated coloring in [BMO11]. The only difference is that we do not have a restriction on how many times we can use a particular color and for every vertex we can only use the colors from the list assigned to it. Otherwise, everything is the same as in [BMO11]. We include it here for the sake of completeness.

Let G be a k-thin graph with an ordering v_1, \ldots, v_n and a partition V^1, V^2, \ldots, V^k of V(G). Let S be a set of colors, s = |S|, and $L : V(G) \to \mathcal{P}(S)$ be a function that assigns a list of allowed colors to a vertex. Consider an instance (G, L) of list coloring. We reduce the problem

⁴Note that this is just a representational convenience for dynamic programming.

to a reachability problem on an auxiliary acyclic digraph D(N, A). We will refer to the elements of N and A as nodes and arcs while the elements of V(G) and E(G) will be referred to as vertices and edges (as we did so far).

The digraph D will be *layered*, i.e., the set N is the disjoint union of subsets (layers) N_0, N_1, \ldots, N_n and all arcs in A have the form (u, w) with $u \in N_r$ and $w \in N_{r+1}$, for some $0 \le r \le n-1$. Note for each vertex $v_r \in V$, there is a layer N_r with $r \ne 0$. We denote by j(r) the class index q such that $v_r \in V^q$.

We first describe the set of nodes in each layer. The first layer consists of colors which can be assigned to the first vertex, i.e., $N_0 = L(v_1)$. For the layers N_1, \ldots, N_{n-1} , there is a one-to-one correspondence between nodes at layer N_r and (sk+1)-tuples $(r, \{\beta_i^j\}_{i=1,\ldots,s,j=1,\ldots,k})$ with $0 \le \beta_i^j \le \Delta(j)_{<}$, for each i, j. The last layer N_n has only one node t corresponding to the tuple $(n, 0, \ldots, 0)$.

We associate with each node $u \notin N_0$ a suitable list coloring problem with additional constraints, that we call the constrained sub-problem associated with u. As we show in the following, u is reachable from a node $z \in N_0$ if and only if this constrained sub-problem has a solution. Namely, we will show that the following property holds:

(*) a node $(r, \{\beta_i^j\}_{i=1,\dots,s,j=1,\dots,k})$ is reachable from a node $z \in N_0$ if and only if the induced subgraph $G[\{v_1,\dots,v_r\}]$ admits a list coloring with the lists given by L and with additional constraint that, for each $i=1,\dots,s$ and $j=1,\dots,k$, color i is forbidden for the last β_i^j vertices in $V^j \cap \{v_1,\dots,v_r\}$.

In this case, G admits a list coloring if and only if the node t is reachable from a node $z \in N_0$. Property (*) will follow from the definition of the set of arcs A given as follows. Let $u = (r, \{\beta_i^j\}_{i=1,\dots,s,j=1,\dots,k})$. Note that the problem associated with u has a solution where the vertex v_r gets color i only if $\beta_i^{j(r)} = 0$. Let $C(u) = \{i \in L(r) : \beta_i^{j(r)} = 0\}$. We will make exactly |C(u)| arcs entering into u, and give each such arc a color $i \in C(u)$ (exactly one color from C(u) per arc). Each arc $(u', u) \in A$, with $u' \in N_{r-1}$ and $i \in C(u)$, will then have the following meaning: if the constrained sub-problem associated with u' has a solution, i.e., a coloring φ' , then we can extend φ' into a solution φ to the constrained sub-problem associated with u by giving color i to vertex v_r .

We now give the formal definition of the set A. We start with the arcs from N_0 to N_1 . Let $u=(1,\{\beta_i^j\}_{i=1,\ldots,s,j=1,\ldots,k})\in N_1$. There is an arc from z_i (where $i\in L(1)$), to u if and only if $i\in C(u)$; moreover, the color of its arc is i. We now deal with the arcs from N_{r-1} to N_r , with $2\leq r\leq n$. Let $u=(r,\{\beta_i^j\}_{i=1,\ldots,s,j=1,\ldots,k})\in N_r$. As we discussed above, for each $i^*\in C(u)$, there will be an arc from a node $u_{i^*}\in N_{r-1}$ to u, with color i^* . Namely, $u_{i^*}=(r-1,\{\beta_i^j\}_{i=1,\ldots,s,j=1,\ldots,k})$, where:

$$\tilde{\beta}_{i}^{j} = \begin{cases} \max\{|N(v_{r}, j)_{<}|, \beta_{i}^{j}\} & i = i^{*} \\ \max\{0, \beta_{i}^{j} - 1\} & i \neq i^{*}, j = j(r) \\ \beta_{i}^{j} & i \neq i^{*}, j \neq j(r) \end{cases}$$
(2)

Note that u_{i^*} is indeed a node of N_{r-1} , as the (sk+1)-tuple $(r-1, \{\tilde{\beta}_i^j\}_{i=1,\dots,s,j=1,\dots,k})$ is such that $0 \leq \tilde{\beta}_i^j \leq \Delta(j)_{<}$, for each i, j (in fact, $\beta_i^j \leq \Delta(j)_{<}$, since u is a node of N_r).

Lemma 4.7. G admits an L list coloring if and only if D contains a directed path from a node $z \in N_0$ to t. Moreover, if such a path exists, then a list coloring of G can be obtained by assigning each node v_r $(r \in \{1, ..., n\})$ the color of the arc of the path entering into layer N_r .

Proof. The proof is analogous to the proof of Lemma 10 in [BMO11] and we omit it here. \Box

Lemma 4.8. Suppose that for a (k-thin) graph G with n vertices we are given an ordering and a partition of V(G) into k classes that are consistent. Further consider an instance (G,L) of the list coloring problem. Let $s = |\bigcup_{v \in V(G)} L(v)|$. Then (G,L) can be solved in $\mathcal{O}(ns^2k\prod_{j=1,\ldots,k}\Delta(j)^s_{<})$ -time, i.e., $\mathcal{O}(n^{ks+1}s^2k)$ -time.

Proof. By definition, for $r=1,\ldots,n-1$, $|N_r|=\prod_{i=1,\ldots,k}(\Delta(j)_<+1)^s$. Note that each node of D has at most s incoming arcs, and each arc can be built in $\mathcal{O}(sk)$ -time. Therefore, D can be built in $\mathcal{O}(ns^2k\prod_{i=1..k}(\Delta(j)_<+1)^s)$ -time. Since D is acyclic, the reachability problem on D can be solved in linear time. Therefore the list coloring problem on G can be solved in $\mathcal{O}(ns^2k\prod_{i=1..k}\Delta(j)^s_<)$ -time, that is $\mathcal{O}(n^{ks+1}s^2k)$ -time.

Lemma 4.9. Let G be a co-comparability graph and (G, L) an instance of the list coloring problem with the total number of colors $s \geq 2$. Then (G, L) can be solved in $\mathcal{O}(n^{s^2+1}s^3)$ -time, i.e., polynomial time when s is fixed.

Proof. By Lemma 4.5, the graph G is k-thin. It can be tested in $\mathcal{O}(n^3)$ time whether G is s-colorable [Gol77]. If it is s-colorable, then by Lemma 4.5 we get a comparability ordering and a k-partition of V(G). Moreover, by Lemma 4.5 we know that $k \leq s$. Thus, by Lemma 4.8, we can solve the problem in time $\mathcal{O}(n^3 + n^{s^2+1}s^3) = \mathcal{O}(n^{s^2+1}s^3)$.

5 Minimum dominating Set

In this section, we discuss the minimum dominating set problem on H-GRAPH. The basic idea behind our algorithms is to reduce the minimum dominating set problem for H-graphs to several minimum dominating set problems on interval graphs, obtained as induced subgraphs of the original graph.

We start with a useful tool (Lemma 5.1) which states that that one can compute a dominating set of an interval graph G which is minimum subject to including one or two of certain special vertices of G. This lemma is an essential tool for both of our dominating set algorithms presented in the subsequent subsections.

Lemma 5.1. Let G = (V, E) be an interval graph and let C_1, \ldots, C_k be the left-to-right ordering of the maximal cliques in an interval representation of G.

- 1. For every $x \in C_1$, a dominating set of G which is minimum subject to including x can be found in linear time.
- 2. For every $x \in C_1$ and $y \in C_k$, a dominating set of G which is minimum subject to including both x and y can be found in linear time.

Proof. We provide the proof for the part 1 (the proof of the part 2 follows analogously). We construct a new graph G' = (V', E') where $V' = V \cup \{u, u'\}$ and $E' = E \cup \{ux, u'x\}$. Clearly, G' is an interval graph as certified by the following linear order of its maximal cliques $\{u, x\} = C_0, C'_0 = \{u', x\}, C_1, \ldots, C_k$. Furthermore, to dominate both u and u' without using x, we would need to include both u and u'. Thus, every minimum dominating set of G' includes x, i.e., we can find such a dominating set in linear time using the standard greedy algorithm [Gol04]. \square

5.1 Dominating sets in S_d -graphs

Here, we solve the minimum dominating set problem on S_d -GRAPH in FPT-time, parameterized by d.

Theorem 5. For an S_d -graph G, a minimum dominating set of G can be found in $\mathcal{O}(dn(n+m)) + 2^d(d+2^d)^{\mathcal{O}(1)}$ time when an S_d -representation is given. (If such a representation is not given, we can compute one in $\mathcal{O}(n^{3.5})$ time by Theorem 3.)

Proof. Let G be an S_d -graph and let S' be a subdivision of the star S_d such that G has an S'-representation. Let b be the central branching point of S' and let l_1, \ldots, l_d be the leaves of S'. Recall that, by Lemma 4.1, we may assume $b \in \bigcap \{S'_v : v \in C\}$, for some maximal clique C of G. Let $C_{i,1}, \ldots, C_{i,k_i}$ be the maximal cliques of G as they appear on the branch $P_{(b,l_i]}$, for $i = 1, \ldots, d$.

For each $G_i = G[C_{i,1}, \ldots, C_{i,k_i}]$, we use an interval graph greedy algorithm [Gol04] to find the size d_i of a minimum dominating set in G_i . Let B_i be the set of vertices of C that can appear in a minimum dominating set of G_i . By Lemma 5.1, a minimum dominating set D_i^x containing a vertex $x \in C$ can be found in linear time. Note that $x \in B_i$ if and only if $|D_i^x| = d_i$. Therefore, every B_1, \ldots, B_d can be found in $\mathcal{O}(d \cdot n \cdot (n+m))$ time. Let $\mathcal{B} = \{B_1, \ldots, B_d\}$.

If B_i is empty, then no minimum dominating set of G_i contains a vertex from C. So for G_i , we pick an arbitrary minimum dominating set D_i . Note that D_i dominates $C \cap C_{i,1}$ regardless of the choice of D_i . Thus, if $\bigcup_{i=1}^d D_i$ dominates C, then it is a minimum dominating set of G. Otherwise, $\{x\} \cup \bigcup_{i=1}^d D_i$ is a minimum dominating set of G where X is an arbitrary vertex of G.

Let us assume now that the B_i 's are nonempty (every branch with an empty B_i can be simply ignored). Let H be a subset of C such that $H \cap B_i$ is not empty, for every $i = 1, \ldots, d$, and |H| is smallest possible. For every branch $P_{(b,b_i]}$, we pick a minimum dominating set D_i of G_i containing an arbitrary vertex $x_i \in H \cap B_i$. Now, the union $D_1 \cup \cdots \cup D_d$ is a minimum dominating set of G. It remains to show how to find the set H in time depending only on d.

Finding the set H can be seen as a set cover problem where \mathcal{B} is the ground set. Namely, we have one set for each vertex x in C where the set of x is simply its subset of \mathcal{B} , and our goal is to cover \mathcal{B} . Note, if two vertices cover the same subset of \mathcal{B} it suffices to keep just one of them for our set cover instance, i.e., giving us at most 2^d sets over a ground set of size d. Such a set cover instance can be solved in $2^d(d+2^d)^{\mathcal{O}(1)}$ time (see Theorem 6.1 [CFK+15]).

Thus, we spend $\mathcal{O}(dn(n+m)) + 2^d(d+2^d)^{\mathcal{O}(1)}$ time in total.

5.2 Dominating sets in *H*-graphs

We turn to H-GRAPH, for general fixed H. There we solve the problem in XP-time, parameterized by ||H||. This latter result can be easily adapted to also obtain XP-time algorithms to find a maximum independent set and minimum independent dominating set on H-GRAPH (these algorithms are also parameterized by ||H||); see Corollary 7.

Theorem 6. For an H-graph G the minimum dominating set problem can be solved in $n^{O(\|H\|)}$ time when an H-representation is given as part of the input.

Proof. Recall that, when H is a cycle, H-GRAPH = CARC, i.e., minimum dominating sets can be found efficiently [Cha98]. Thus, we assume H is not a cycle.

To introduce our main idea, we need some notation. Consider $G \in H$ -GRAPH and let H' be a subdivision of H such that G has an H'-representation $\{H'_v : v \in V(G)\}$. We distinguish two important types of nodes in H'; namely, $x \in V(H')$ is called *high degree* when it has at least three neighbors and x is *low degree* otherwise. As usual, the high degree nodes play a key role. In particular, if we know the sub-solution which *dominates* the high degree nodes of H', then the remaining part of the solution must be strictly contained in the low degree part of H'. Moreover, since H is not a cycle, the subgraph $H'_{\leq 2}$ of H' induced by its low degree nodes is a collection of paths. In particular, the vertices v of G where H'_v only contains low degree nodes, induce an interval graph $G_{\leq 2}$ and, as such, we can efficiently find minimum dominating sets on them. Thus, the general idea here is to first enumerate the possible sub-solutions on the high degree nodes, then efficiently (and optimally) extend each sub-solution to a complete solution. In particular, one can show that in any minimum dominating set these sub-solutions consist of at most $2 \cdot |E(H)|$ vertices (as in Claim 5.1 below), and from this property it is not difficult to produce the claimed $n^{\mathcal{O}(\|H\|)}$ -time algorithm. These ideas are formalized as follows.

We observe that the size of these sub-solutions is "small". Let $D \subseteq V(G)$ be a minimum dominating set of G. For each node x of H', let $V_x = \{v : v \in V(G), x \in H'_v\}$ and $D_x = \{v : v \in D, x \in H'_v\}$. We further let $D_{\geq 3} = \bigcup \{D_x : \delta_H(x) \geq 3\}$. We now bound the size of $D_{\geq 3}$ in terms of H.

Claim 5.1. If D is a minimum dominating set in an H-graph G, then $|D_{>3}| \le 2|E(H)|$.

Proof. Consider a high degree node x of H such that $x \in D_{\geq 3}$. For each edge xx' in H, let $x = x_1, \ldots, x_k = x'$ be the corresponding path in H'. We assign a single vertex a in D to the ordered pair (x, x') such that H'_a contains the longest subpath of x_1, \ldots, x_k including $x = x_1$. Notice that each ordered pair receives precisely one element of D. However, if some element v of $D_{\geq 3}$ was not assigned to an ordered pair, then it is easy to see that D is not a minimum dominating set (since all adjacencies achieved by this element are already achieved by the elements we have charged to ordered pairs).

By Claim 5.1, there are at most $n^{2\cdot |E(H)|}$ possible sets $D_{\geq 3}$. We now fix one such $D_{\geq 3}$ and describe how to compute a minimum dominating set of G containing it. Notice that, there can be some difficult decisions we might need to make in this process. In particular, suppose there is a high degree node x of H' where no vertex from V_x is in $D_{\geq 3}$. It is not clear how we might be able to efficiently choose from "nearby" x to dominate these vertices. To get around this case, we simply enumerate more vertices. Specifically, for each path $P_{[x,y]} = (x,x_1,\ldots,x_k,y)$ in H' where x and y are high degree nodes (or where x is high degree and y is a leaf), and the x_i 's are low degree, we will pick a "first" and "last" vertex among the vertices v of v where v is contained in the subpath v in v of v in v i

We now have our candidate sub-solutions $D^* = D_{\geq 3} \cup D_{\leq 2}$. There are just some simple sanity checks we must make on D^* to test if it is a good candidate to be extended to a dominating set. First, by the definition of $D_{\geq 3}$, it must already dominate every vertex of $G_{\geq 3}$. Second, if there is some path $P_{[x,y]}$ where $D_{\leq 2}$ contains fewer than two vertices from $P_{[x,y]}$, then D^* must already dominate every vertex contained in this path. And finally, for every path $P_{[x,y]}$, for every v with H'_v contained strictly between x and the "left-end" of the "first" chosen vertex, then v must be dominated by $D_{\geq 3}$. If one of these conditions is violated, we discard this candidate D^* and go to the next one.

Finally, what remains to be dominated consists of a collection of disjoint interval graphs where possibly some sequence of "left-most" and "right-most" maximal cliques have already been dominated by D^* . Observe that the partially constructed dominating set will consist of one vertex which reaches the farthest in from the right and one which does the same from the left. Namely, we can apply Lemma 5.1, to construct a minimum dominating set for each such interval graph subject to the inclusion of these two special vertices and as such compute a minimum dominating set of G which contains our candidate partial dominating set.

This completes the description of the algorithm. From the discussion, we can see that the algorithm is correct and that the total running time is dominated by the enumeration of the possible sets D^* plus some additional polynomial factors. In particular, the algorithm runs in $n^{\mathcal{O}(\|H\|)}$ time.

We further remark that the above approach can also be applied to solve the maximum independent set and minimum independent dominating set problems in $n^{O(\|H\|)}$ time. This approach is successful since these problem can be solved efficiently on interval graphs.

Corollary 7. For an H-graph G, the maximum independent set problem and minimum independent dominating set problem can both be solved in $n^{\mathcal{O}(\|H\|)}$ time.

Finally, as we have stated in Section 1.2, in a recent manuscript [FGR20], W[1]-hardness has been shown for both the minimum dominating set problem and the maximum independent set problem. Moreover, both of these results concern parameterization by both ||H|| and the solution size. Thus, this classifies the computational complexity for both of these problems. It would be interesting to also have W[1]-hardness for the minimum independent dominating set problem. Additionally, one could make a more fine-grained examination of the running time and look for lower bounds via ETH.

Problem 4. Is the minimum independent dominating set problem W[1]-hard on H-graphs (parametrized by ||H|| and the solution size)?

Problem 5. Can we obtain some interesting lower bounds using ETH?

6 Finding cliques in *H*-graphs

We discuss computational aspects of the maximum clique problem for H-graphs, parametrized by ||H||. Let Δ_2 be the double-triangle (see Fig. 5a). First, we show that the maximum clique problem is APX-hard for H-graphs if H contains Δ_2 as a minor (Theorem 8). In other words, the maximum clique problem is para-NP-hard when parameterized only by ||H||. As a consequence of our reduction, we also show that if $\Delta_2 \leq H$, then H-GRAPH is GI-complete (the graph isomorphism on H-GRAPH is as hard as the general graph isomorphism problem). We then turn to cases where the clique problem can be solved efficiently. Namely, we consider two cases: one where we have a "nice" representation but H is arbitrary, and the other where we restrict H to be a cactus.

6.1 Clique (and isomorphism) hardness results

To obtain our hardness results we show that there are graphs H such that the complement of a 2-subdivision of every graph is an H-graph. The 2-subdivision G_2 of a graph G is the result of

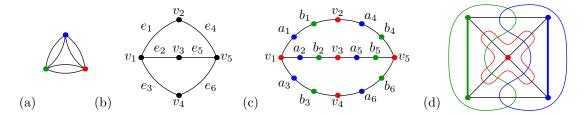


Figure 5: (a) The double-triangle graph. (b) A graph G. (c) The 2-subdivision G_2 of G. A three-clique cover of $\overline{G_2}$ is indicated by colors. (d) The 4-wheel graph (which contains the double-triangle as a minor) and a sketch of our H-representation of $\overline{G^*}$. For example, the edges between the green clique and the blue clique are represented where the green and blue regions intersect.

subdividing every edge of G exactly two times. The complement of a graph G is denoted by \overline{G} . We define

$$\overline{\mathsf{SUBD}_2} = \{ \overline{G_2} : G \text{ is a graph} \}.$$

In other words, $\overline{\mathsf{SUBD}_2}$ is the class of complements of 2-subdivisions of all graphs.

This seemingly esoteric family of graphs is interesting for two reasons. Firstly, the isomorphism relation on graphs is closed under k-subdivision and complement operations. This implies that $G \cong H$ if and only if $\overline{G_2} \cong \overline{H_2}$. So, the class $\overline{\mathsf{SUBD_2}}$ is GI -complete. Secondly, the clique problem is APX -hard on $\overline{\mathsf{SUBD_2}}$. More specifically, Chlebík and Chlebíková [CC07] proved that the maximum independent set problem is APX -hard on the class of 2k-subdivisions of 3-regular graphs for any fixed integer $k \geq 0$; in particular, for 2-subdivisions. Thus, showing that $\overline{\mathsf{SUBD_2}} \subseteq H$ -GRAPH, for a fixed H, implies that the maximum clique problem is APX -hard on H-GRAPH and that H-GRAPH is GI -complete.

Theorem 8. If $\Delta_2 \leq H$, then the maximum clique problem is APX-hard for H-graphs and H-GRAPH is GI-complete.

Proof. As already mentioned, we prove the theorem by showing $\overline{\mathsf{SUBD}_2} \subseteq H\text{-}\mathsf{GRAPH}$. Since $\Delta_2 \preceq H$, the graph H can be partitioned into three connected subgraphs H_1, H_2, H_3 such that there are at least two edges connecting H_i and H_j , for each $i \neq j$. For every graph G, we show that the complement of its 2-subdivision has an H-representation.

The construction proceeds similarly to the constructions used by Francis et al. [FGO13], and we borrow their convenient notation. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{e_1, \ldots, e_m\}$. If $e_k \in E(G)$ and $e_k = v_i v_j$ where i < j, we define l(k) = i and r(k) = j (as if v_i and v_j were respectively the *left* and *right* ends of e_k). In the 2-subdivision G_2 of G, the edge e_k of G is replaced by the path $(v_{l(k)}, a_k, b_k, v_{r(k)})$; see Fig. 5a and Fig. 5b.

Note that G_2 can be covered by three cliques, i.e., $C_v = \{v_1, \ldots, v_n\}$, $C_a = \{a_1, \ldots, a_m\}$, and $C_b = \{b_1, \ldots, b_m\}$. We now describe a subdivision H' of H which admits an H-representation $\{H'_v : v \in V(\overline{G_2})\}$ of $\overline{G_2}$. We obtain H' by subdividing the six edges connecting H_1 , H_2 , and H_3 . Specifically:

- We *n*-subdivide the edges connecting H_1 to H_2 to obtain two paths $P_{12} = (\alpha_0, \alpha_1, \ldots, \alpha_n, \alpha_{n+1}), Q_{12} = (\beta_0, \beta_1, \ldots, \beta_n, \beta_{n+1})$ where $\alpha_0, \beta_0 \in H_1$ and $\alpha_{n+1}, \beta_{n+1} \in H_2$.
- We *n*-subdivide the edges connecting H_1 to H_3 to obtain two paths $P_{13} = (\gamma_0, \gamma_1, \dots, \gamma_n, \gamma_{n+1}), Q_{13} = (\eta_0, \eta_1, \dots, \eta_n, \eta_{n+1})$ where $\gamma_0, \eta_0 \in H_1$ and $\gamma_{n+1}, \eta_{n+1} \in H_2$.

• We *m*-subdivide the edges connecting H_2 and H_3 to obtain two paths $P_{23} = (\mu_0, \mu_1, \dots, \mu_m, \mu_{m+1}), Q_{23} = (\nu_0, \nu_1, \dots, \nu_m, \nu_{m+1})$ where $\mu_0, \nu_0, \mu_{m+1}, \eta_{m+1} \in H_2$.

We now describe each H_{v_i} , H_{a_j} and H_{b_j} . The idea is that H'_{v_i} will contain H_1 and extend from the "start" of P_{12} up to the position i, and from the "start" of Q_{12} up to position (n-i). From the other side, each H'_{a_j} will contain H_2 and extend from the "end" of P_{12} down to position (l(j)+1), and from the end of Q_{12} down to position (n-l(j)+1); an example is sketched in Fig. 5d. In this way, we ensure that H'_{a_j} does not intersect $H'_{v_{l(j)}}$ while H'_{a_j} does intersect every H'_{v_i} for $i \neq l(j)$. The other pairs proceed similarly, and we describe the subgraphs $H_{v_i}, H_{a_j}, H_{b_j}$ for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ as follows:

- $H'_{v_i} = H_1 \cup \{\alpha_1, \dots, \alpha_i\} \cup \{\beta_1, \dots, \beta_{n-i}\} \cup \{\gamma_1, \dots, \gamma_i\} \cup \{\eta_1, \dots, \eta_{n-i}\}.$
- $H'_{a_j} = H_2 \cup \{\alpha_n, \dots, \alpha_{l(j)+1}\} \cup \{\beta_n, \dots, \beta_{n-l(j)+1}\} \cup \{\mu_1, \dots, \mu_j\} \cup \{\nu_1, \dots, \nu_{m-j}\}.$
- $H'_{b_j} = H_3 \cup \{\gamma_n, \dots, \gamma_{r(j)+1}\} \cup \{\eta_n, \dots, \eta_{n-r(j)+1}\} \cup \{\mu_m, \dots, \mu_{j+1}\} \cup \{\nu_m, \dots, \nu_{m-j+1}\}.$

Some interesting cases remain concerning the maximum clique problem. In the next subsection we will prove Theorem 10 which states that, for any cactus C, the clique problem can be solved polynomial time on any C-graph. Thus, the open cases which remain are when H is not a cactus (i.e., H contains a diamond as a minor), but H does not satisfy the conditions of Theorem 8 (i.e., H does not contain the double-triangle as a minor).

Problem 6. What is the complexity of the maximum clique problem on H-graphs in the case when H is not a cactus and $\Delta_2 \npreceq H$?

On the other hand, while the isomorphism problem can be solved in linear time on interval graphs and Helly circular-arc graphs [CLM⁺13], split graphs [LB79] are Gl-complete.

Problem 7. Let H be a fixed graph such that $\Delta_2 \npreceq H$. What is the complexity of the graph isomorphism problem on H-graphs?

6.2 Tractable cases

Here, we consider two restrictions which allow polynomial-time algorithms for the maximum clique problem. First, we discuss the case when the H-representation satisfies the Helly property. This is followed by a discussion of the case when H is a cactus. In both situations, we obtain polynomial-time algorithms.

Helly H-graphs. A Helly H-graph G has an H-representation $\{H'_v : v \in V(G)\}$ such that the collection $\mathcal{H} = \{V(H'_v) : v \in V(G)\}$ satisfies the Helly property, i.e., for each sub-collection of \mathcal{H} whose sets pairwise intersect, their common intersection is non-empty. Notice that, when H is a tree, every H-representation satisfies the Helly property. When a graph G has a Helly H-representation, we obtain the following relationship between the size of H and the number of maximal cliques in G.

Lemma 6.1. Each Helly H-graph G has at most $|V(H)| + |E(H)| \cdot |V(G)|$ maximal cliques.

Proof. Let H' be a subdivision of H such that G has a Helly H-representation $\{H'_v : v \in V(G)\}$. Note that, for each maximal clique C of G, $\bigcap_{v \in C} V(H'_v) \neq \emptyset$, i.e., C corresponds to a node x_C of H'.

For every edge $xy \in E(H)$, we consider the path $P = P_{[x,y]} = (x, x_1, \dots, x_k, y)$ in H'. Let G_P be the subgraph of G formed by the union of the maximal cliques C of G such that $x_C \in V(P)$.

Claim 6.1. The graph G_P is a Helly circular-arc graph.

Proof. Note that if a restriction of H'_v , for $v \in V(G_P)$, to P is disconnected, then it is a disjoint union of two paths containing the end-vertices x and y, respectively. Let C by cycle obtained from P by adding the edge xy. We construct a C-representation of G_P . If the restriction of H'_v to P is a subpath of P, then we let C_v to be this subpath. Otherwise, we let C_v to be the restriction of H'_v to P together with the edge xy. Clearly, this is a Helly C-representation. \square

Now, since Helly circular-arc graphs have at most linearly many maximal cliques [Gav74a], G has at most $|V(H)| + |E(H)| \cdot |V(G)|$ maximal cliques.

We can now use Lemma 6.1 to find the largest clique in G in polynomial time. In fact, we can do this without needing to compute a representation of G. In particular, the maximal cliques of a graph can be enumerated with polynomial delay [MU04]. Thus, since G has at most linearly many maximal cliques, we can simply list them all in polynomial time and report the largest, i.e., if the enumeration process produces too many maximal cliques, we know that G has no Helly H-representation. This provides the following theorem.

Theorem 9. The clique problem is solvable in polynomial time on Helly H-graphs.

Note that some co-bipartite circular-arc graphs have exponentially many maximal cliques and these graphs are not contained in Helly H-GRAPH, for any fixed H. However, the clique problem is solvable for circular-arc graphs in polynomial time [Hsu85].

Cactus-graphs. The clique problem is efficiently solvable on chordal graphs [Gol04] and circular-arc graphs [Hsu85]. In particular, when H is either a tree or a cycle, the clique problem can be solved in polynomial-time, independent of ||H||. In Theorem 10, we observe that these results easily generalize to the case when G is a C-graph, for some cactus graph C. We define,

$$\mathsf{CACTUS}\text{-}\mathsf{GRAPH} = \bigcup_{\mathsf{Cactus}\ C} C\text{-}\mathsf{GRAPH}.$$

To prove the result we will use the clique-cutset decomposition, which is defined as follows. A clique-cutset of a graph G is a clique K in G such that G-K has more connected components than G. An atom is a graph without a clique-cutset. An atom of a graph G is a maximal induced subgraph A of G which is an atom. A clique-cutset decomposition of G is a set $\{A_1, \ldots, A_k\}$ of atoms of G such that $G = \bigcup_{i=1}^k A_i$ and for every $i, j, V(A_i) \cap V(A_j)$ is either empty, or induces a clique in G. Algorithmic aspects of clique-cutset decompositions were studied by Whitesides [Whi84] and Tarjan [Tar85]. In particular, if $k \leq n$, then for any graph G a clique-cutset decomposition $\{A_1, \ldots, A_k\}$ of G can be computed in $O(n^2 + nm)$ [Tar85]. Additionally, to solve the clique problem on a graph G it suffices to solve it for each atom of G from a clique-cutset decomposition [Whi84, Tar85]. Theorem 10 now follows from the following easy lemma and the fact that the clique problem can be solved in polynomial time for circular-arc graphs [Hsu85].

Lemma 6.2. Let C be cactus and let $G \in C$ -GRAPH. Then each atom A of G is a circular-arc graph.

Proof. Consider an C-representation $\{C'_v : v \in V(G)\}$ of G. Now, let $C|_A = \bigcup_{v \in V(A)} C'_v$. Clearly, if $C|_A$ is a path or a cycle, then we are done. Otherwise, $C|_A$ must contain a cut-node x. Let X_1, \ldots, X_t be the components of $H|_A - \{x\}$, and let S be the vertices of A whose representations contain x. Note that S is a clique in A. Moreover, since A is an atom, S is not a clique-cutset. Thus, there is a component X_j such that the subgraph C^* of C induced by $V(X_j) \cup \{x\}$ provides a representation of A. In particular, if C^* is either a cycle, or a path we are again done. Moreover, when C^* is neither a path, nor a cycle, repeating this argument on C^* provides a smaller subgraph of C, on which A can be represented, i.e., this eventually produces either a path, or cycle.

Theorem 10. The clique problem can be solved in polynomial time on CACTUS-GRAPH.

7 FPT results via clique-treewidth graph classes

The concept of treewidth was introduced by Robertson and Seymour [RS84]. A tree decomposition of a graph G is a pair (X,T), where T is a tree and $X = \{X_i \mid i \in V(T)\}$ is a family of subsets of V(G), called bags, such that (1) for all $v \in V(G)$, the set of nodes $T_v = \{i \in V(T) \mid v \in X_i\}$ induces a non-empty connected subtree of T, and (2) for each edge $uv \in E(G)$ there exists $i \in V(T)$ such that both u and v are in X_i . The maximum of $|X_i| - 1$, $i \in V(T)$, is called the width of the tree decomposition. The treewidth, tw(G), of a graph G is the minimum width over all tree decompositions of G.

An easy lower bound on the treewidth of a graph G is the size of the largest clique in G, i.e., its clique number $\omega(G)$. This follows from the fact that each edge of G belongs to some bag of T and that a collection of pairwise intersecting subtrees of a tree must have a common intersection (i.e., they satisfy the Helly property). With this in mind, we say that a graph class \mathcal{G} has the clique-treewidth property⁵ if there is a function $f: \mathbb{N} \to \mathbb{N}$ such that for every $G \in \mathcal{G}$, $\mathrm{tw}(G) \leq f(\omega(G))$. This concept generalizes the idea of \mathcal{G} being χ -bounded, namely, that the chromatic number $\chi(G)$ of every graph $G \in \mathcal{G}$ is bounded by a function of the clique number of G. In particular, the chromatic number of a graph G is bounded by its treewidth since a tree decomposition (X,T) of G is a T-representation of a chordal supergraph G' of G where $\omega(G') = \mathrm{tw}(G) + 1$, i.e., $\chi(G') = \mathrm{tw}(G) + 1$ since chordal graphs are perfect. It was recently shown that the graphs which do not contain even holes (i.e., cycles of length 2k for any $k \geq 2$) and pans (i.e., cycles with a single pendent vertex attached) as induced subgraphs have their treewidth bounded by $f(\omega) = 3\omega/2 - 1$ [CCH18]. For a function $f: \mathbb{N} \to \mathbb{N}$, we use \mathcal{G}_f to denote the class of graphs G where $\mathrm{tw}(G) \leq f(\omega(G))$. Each class H-GRAPH is known to be a subclass of \mathcal{G}_f for certain linear functions f, as in the following lemma.

Lemma 7.1 (Biró, Hujter, and Tuza [BHT92]). For every $G \in H$ -GRAPH, $\operatorname{tw}(G) \leq (\operatorname{tw}(H) + 1) \cdot \omega(G) - 1$, i.e., H-GRAPH is a subclass of \mathcal{G}_{f_H} , where $f_H(\omega) = (\operatorname{tw}(H) + 1) \cdot \omega - 1$.

We leverage any clique-treewidth-property (e.g., as in Lemma 7.1) together with some existing algorithms to to classify the k-coloring and k-clique problems as FPT on the \mathcal{G}_f classes (e.g., on H-GRAPH classes as well). We first consider the k-clique problem.

⁵In our prior work [CZ17], we referred to this as being *treewidth-bounded*, but have changed the name to be consistent with other parameter-treewidth bounds given in *bidimensionality theory* [DFHT04].

Theorem 11. For any monotone computable function $f: \mathbb{N} \to \mathbb{N}$, the k-clique problem can be solved in $2^{\mathcal{O}(f(k))} \cdot n$ time for $G \in \mathcal{G}_f$. Thus, for H-GRAPH, the k-clique problem can be solved in $2^{\mathcal{O}(\operatorname{tw}(H) \cdot k)} \cdot n$ time.

Proof. To test if G contains a k-clique, we first try to generate a tree decomposition of G with width roughly f(k) via a recent algorithm [BDD⁺16], which, for any given graph G and number t, provides a tree decomposition of width at most $5 \cdot t$ or states that the treewidth of G is larger than t. This algorithm runs in $2^{\mathcal{O}(t)} \cdot n$ time. If this algorithm does not produce a tree decomposition, then G must contain a k-clique, and we are done. Otherwise, we obtain a tree decomposition (X,T) of G of width $5 \cdot f(k)$. Note that, an easy property of tree decompositions is that, for every clique K, there is a bag which contains the vertices of K. In particular, to check if G has a k-clique it suffices to check whether each the subgraph induced by a bag of G contains a k-clique. This can obviously be done in $2^{\mathcal{O}(f(k))} \cdot n$ time by brute-force. Thus, we have $2^{\mathcal{O}(f(k))} \cdot n$ time in total as needed.

For each fixed $k \geq 3$, it is known that testing (k,k)-pre-colouring extension (see Section 1.1 for a definition) for $G \in H$ -GRAPH can be done in XP time [BHT92]. The authors combine Lemma 7.1 together with a simple argument to obtain the result. We use a similar argument together with a more recent result regarding bounded treewidth graphs to observe that an even more general problem, list k-coloring (where each list is a subset of $\{1,\ldots,k\}$), is FPT on graph class satisfying the clique-treewidth property, and therefore, also on H-GRAPH.

Theorem 12. For any monotone computable function $f: \mathbb{N} \to \mathbb{N}$, the list-k-coloring problem can be solved in $k^{\mathcal{O}(f(k))} \cdot n$ time for $G \in \mathcal{G}_f$. Thus, for H-GRAPH, the list-k-coloring problem can be solved in $k^{\mathcal{O}(\operatorname{tw}(H) \cdot k)} \cdot n$ time.

Proof. For fixed k, clearly, if G contains a clique of size k+1 then G has no k-coloring, i.e., no list-k-coloring, regardless of the lists. We use Theorem 11 to test for such a clique, and reject if one is found. Otherwise, we have a $5 \cdot f(k)$ -width tree decomposition, and this time we use it to solve the list-k-coloring problem via the known $\mathcal{O}(k^{t+2} \cdot n)$ -time algorithm when given a width t tree decomposition [JS97]. Thus, list-k-coloring can be solved in $(2^{\mathcal{O}(f(k))} + k^{\mathcal{O}(f(k))}) \cdot n$ time on \mathcal{G}_f .

Some further natural open questions remain regarding these results. For example, what other problems can be approached on graph classes with the clique-treewidth property? Can we obtain polynomial-size kernels for the k-clique or list-k-coloring problems on H-GRAPH or more generally on graph classes with the clique-treewidth property? The kernelization question has already been partially answered for the k-clique problem. Namely, on H-graphs, it was recently shown [FGR20] that the k-clique admits a polynomial kernel in terms of $\|H\|$ and k, but the kernelization requires an H-representation to be given as part of the input. In contrast, our FPT algorithm for k-clique (while also parameterized by both $\|H\|$ and k) does not need an H-representation.

Problem 8. Can the kernelization for k-clique be done without an H-representation a part of the input?

Problem 9. Can we obtain polynomial-size kernel for the list-k-coloring problem on H-graphs?

8 Minimal separators

For a connected graph G, a subset S of V(G) is a minimal separator when G has vertices u and v belonging to distinct components of G-S such that no proper subset of S disconnects u and v – here, we say that S is minimal (u,v)-separator. We denote the set of all minimal separators in G by S(G). Minimal separators are a commonly studied aspect of many graph classes [BT01, GM18, Gol04, KKW98]. Two particularly relevant cases include the fact that chordal graphs have at most n minimal separators [Gol04], and that circular-arc graphs have at most $2n^2 - 3n$ minimal separators [KKW98].

Recently, several algorithmic results have been developed, where the runtime depends on the number of minimal separators in the input graph. The main result in this direction is the one by Fomin, Todinca, and Villanger [FTV15], which is phrased in terms of potential maximal cliques, but can also be phrased in terms of minimal separators since the number of potential maximal cliques in a graph G is bounded by $n|\mathcal{S}(G)|^2$ (see Proposition 2.8 in [FTV15]). Roughly, in [FTV15] the authors show that a large class of problems can be solved in time polynomial in the number of minimal separators of the input graph. These problems include several standard combinatorial optimization problems, e.g., maximum independent set and maximum induced forest⁶.

The class of problems considered in [FTV15] is formalized as follows. Consider a fixed integer $t \geq 0$, and a formula φ expressed in counting extended monadic second order logic (CMSO)⁷. For an input graph G, the goal is to find a maximum size subset $X \subseteq V(G)$ satisfying: there is $F \subseteq V(G)$ such that $X \subseteq F$, the subgraph G[F] has treewidth at most t, and the structure (G[F], X) models φ . The graph G[X] is called maximum induced subgraph of treewidth $\leq t$ satisfying φ . The main result of [FTV15] is that this problem can be solved in time $O(|S|^2 n^{t+5} f(t, |\varphi|))$ where f is a computable function.

Now, we prove that each H-graph has $n^{O(\|H\|)}$ minimal separators; see Theorem 13. We obtain Corollary 14 by applying the meta-algorithmic result of Fomin, Todinca, and Villanger. Subsequently, we consider the case of H-graphs when H is a cactus and observe a much smaller bound on the number of minimal separators, in particular, $O(\|H\|n^2)$; see Theorem 15. Similarly, by applying the meta-algorithmic result we obtain Corollary 16: for cactus-graphs, the maximum induced subgraph of treewidth t modelling φ can be solved in polynomial time.

Theorem 13. Let G be a connected H-graph. Then G has $n^{O(|E(H)|)}$ minimal separators. ⁸

Proof. We show that each minimal separator arises from vertices of G such that their representations contain a small number of edges of the subdivision H'. Then we count all such subsets of edges of H'.

Let H' be a subdivision of H certifying that G is an H-graph. Let H^* be the subgraph of H' formed by the union of the representations of the vertices of G, i.e.,

$$H^* = \bigcup_{x \in V(G)} H'_x.$$

 $^{^{6}}$ i.e., minimum feedback vertex set

⁷Informally, CMSO consists of all logic formulas with quantifiers over vertices, edges, edge sets and vertex sets, and counting modulo constants. For more information on this logic see, e.g., [CE12]. Note: in [CE12], this logic is abbreviated by CMS₂ instead of CMSO as in [FTV15].

⁸A similar result with a slightly better bound is given in a recent manuscript, see [FGR20]. Our proof and theirs seem to follow similar reasoning, but have been obtained independently, as also noted in [FGR20].

Observe that, since G is connected, H^* must be also connected. Moreover, for any minimal (u,v)-separator S, the graph $H_S^* = \bigcup_{x \in V(G) \setminus S} H_x'$ is not connected. Now, since S is an (u,v)-separator, there are distinct components Z_u^* and Z_v^* of H_S^* such that H_u' is a subgraph of Z_u^* and H_v' is a subgraph of Z_v^* .

Observe that, since S is minimal, then if $x \in S$, then the representation H'_x contains an edge ab of H^* such that either $a \in V(Z_u^*)$ and $b \notin V(Z_u^*)$, or $a \in V(Z_v^*)$ and $b \notin V(Z_v^*)$. Namely, there is a set E_S of edges of H^* such that S is precisely the set of vertices x of G where H'_x contains an edge of E_S . Moreover, for each edge of H, at most two edges from its path in H' occur in E_S .

To bound the number of all possible minimal separators in G, it suffices to enumerate all possible subsets E of E(H') where, for each edge of H, we pick at most two edges from its path in H'. Here, the candidate separator S would simply be all vertices x of G for which H'_x contains an edge of E. Thus, since each edge of H will be subdivided at most 2n-1 times (since 2n nodes are sufficient to accommodate any circular-arc representation), we obtain that the number of minimal separators in G is at most

$$\left(\binom{2n}{2} + \binom{2n}{1} + \binom{2n}{0} \right)^{|E(H)|} = n^{\mathcal{O}(|E(H)|)}.$$

Corollary 14. Let H be a fixed graph. For every H-graph G, $t \geq 0$, and every CMSO formula φ , a maximum induced subgraph of treewidth $\leq t$ modelling φ can be found in time $O(n^{c|E(H)|}n^{t+5}f(t,\varphi))$, where c is a constant and f is a computable function.

Theorem 15. Let G be a connected C-graph, where C is a cactus graph. Then G has at most $|E(C)|(2n^2+n)$ minimal separators.

Proof. The reasoning here follows similarly to the proof of Theorem 13. Namely, if we consider a minimal (u, v)-separator, we again find the components Z_u^* and Z_v^* in the subdivision H'. However, since H' is a cactus, we can now look more closely at the edges which are incident to Z_u^* and Z_v^* but contained in neither. In particular, it is easy to see that among all such edges incident to Z_u^* , there are at most two edges e_1, e_2 which are actually important to ensure that there is no path from H'_u from H'_v . In other words, our set E_S consists of at most two edges of H'. Moreover, these two edges must belong to the same cycle of H'. Finally, since each cycle of H' forms a circular-arc graph, it never needs to contain more than 2n nodes, i.e., also 2n edges. Thus, since H contains at most |E(H)| cycles, the number of minimal separators in G is at most |E(H)| ($\binom{2n}{2} + \binom{2n}{1}$) $\leq |E(H)|(2n^2 + n)$.

Corollary 16. Let C be a cactus. For every $G \in C$ -GRAPH, $t \geq 0$ and every CMSO formula φ , a maximum induced subgraph of treewidth $\leq t$ modelling φ can be found in time $O(|E(C)|^2 n^{t+9} f(t,\varphi))$, where f is a computable function.

As we have mentioned, two recent manuscripts [FGR20, JKT20] have obtained W[1]-hardness results for both the maximum independent set problem and the minimum feedback vertex set problem (respectively) when parameterized by ||H|| and the solution size. In both results, the graphs H which are used have progressively larger clique minors. These indicate that the XP-time results of Corollary 14 are extremely unlikely to be improved to FPT-time, even when adding the solution size as an additional parameter. On the other hand, as in Corollary 16,

when H is a cactus (i.e., diamond-minor free), these problems (and many more) can be solved in polynomial time in both ||H|| and the size of the input graph.

Problem 10. For which classes \mathcal{H} (besides the cacti), can one similarly bound the number of minimal separators by a polynomial in terms of ||H|| and ||G|| where $H \in \mathcal{H}$ and G is an H-graph?

Acknowledgements

We would like to thank Pavel Klavík for suggesting to study of H-graphs and for several helpfull discussions. We would also like to thank the DIMACS REU 2015 program, held at the Rutgers University, where the whole project started.

References

- [AH20] Deniz Agaoglu and Petr Hlinený. Isomorphism problem for s_d-graphs. In Javier Esparza and Daniel Král', editors, 45th International Symposium on Mathematical Foundations of Computer Science, MFCS 2020, August 24-28, 2020, Prague, Czech Republic, volume 170 of LIPIcs, pages 4:1–4:14. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2020.
- [BDD⁺16] Hans L. Bodlaender, Pål Grønås Drange, Markus S. Dregi, Fedor V. Fomin, Daniel Lokshtanov, and Michal Pilipczuk. A c^k n 5-approximation algorithm for treewidth. SIAM J. Comput., 45(2):317–378, 2016.
- [BHT92] Miklós Biró, Mihály Hujter, and Zsolt Tuza. Precoloring extension. I. Interval graphs. *Discret. Math.*, 100(1-3):267–279, 1992.
- [BJ82] Kellogg S. Booth and J. Howard Johnson. Dominating sets in chordal graphs. SIAM J. Comput., 11(1):191–199, 1982.
- [BL76] Kellogg S. Booth and George S. Lueker. Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms. *J. Comput. Syst. Sci.*, 13(3):335–379, 1976.
- [BMO11] Flavia Bonomo, Sara Mattia, and Gianpaolo Oriolo. Bounded coloring of cocomparability graphs and the pickup and delivery tour combination problem. *Theor.* Comput. Sci., 412(45):6261–6268, 2011.
- [BS90] Martin L. Brady and Majid Sarrafzadeh. Stretching a knock-knee layout for multi-layer wiring. *IEEE Trans. Computers*, 39(1):148–151, 1990.
- [BT01] Vincent Bouchitté and Ioan Todinca. Treewidth and minimum fill-in: Grouping the minimal separators. SIAM J. Comput., 31(1):212–232, 2001.
- [CC07] Miroslav Chlebík and Janka Chlebíková. The complexity of combinatorial optimization problems on d-dimensional boxes. SIAM J. Discrete Math., 21(1):158–169, 2007.

- [CCH18] Kathie Cameron, Steven Chaplick, and Chính T. Hoàng. On the structure of (pan, even hole)-free graphs. *Journal of Graph Theory*, 87(1):108–129, 2018.
- [CE12] Bruno Courcelle and Joost Engelfriet. Graph Structure and Monadic Second-Order Logic A Language-Theoretic Approach, volume 138 of Encyclopedia of mathematics and its applications. Cambridge University Press, 2012.
- [CFK+15] Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. Parameterized Algorithms. Springer, 2015.
- [Cha98] Maw-Shang Chang. Efficient algorithms for the domination problems on interval and circular-arc graphs. SIAM J. Comput., 27(6):1671–1694, 1998.
- [CLM⁺13] Andrew R. Curtis, Min Chih Lin, Ross M. McConnell, Yahav Nussbaum, Francisco J. Soulignac, Jeremy P. Spinrad, and Jayme Luiz Szwarcfiter. Isomorphism of graph classes related to the circular-ones property. *Discrete Mathematics & Theoretical Computer Science*, 15(1):157–182, 2013. arXiv:1203.4822.
- [CS14] Steven Chaplick and Juraj Stacho. The vertex leafage of chordal graphs. *Discret.* Appl. Math., 168:14–25, 2014.
- [CTVZ17] Steven Chaplick, Martin Toepfer, Jan Voborník, and Peter Zeman. On H-Topological Intersection Graphs. In Hans L. Bodlaender and Gerhard J. Woeginger, editors, Graph-Theoretic Concepts in Computer Science 43rd International Workshop, WG 2017, Eindhoven, The Netherlands, June 21-23, 2017, Revised Selected Papers, volume 10520 of Lecture Notes in Computer Science, pages 167–179. Springer, 2017.
- [CZ17] Steven Chaplick and Peter Zeman. Combinatorial problems on H-graphs. *Electron.* Notes Discret. Math., 61:223–229, 2017.
- [DFHT04] Erik D. Demaine, Fedor V. Fomin, Mohammad Taghi Hajiaghayi, and Dimitrios M. Thilikos. Bidimensional parameters and local treewidth. *SIAM J. Discret. Math.*, 18(3):501–511, 2004.
- [EST14] Jessica Enright, Lorna Stewart, and Gábor Tardos. On list coloring and list homomorphism of permutation and interval graphs. SIAM J. Discrete Math., 28(4):1675–1685, 2014.
- [FG65] Delbert R. Fulkerson and Oliver A. Gross. Incidence matrices and interval graphs. *Pac. J. Math.*, 15:835–855, 1965.
- [FGO13] Mathew C. Francis, Daniel Gonçalves, and Pascal Ochem. The maximum clique problem in multiple interval graphs. *Algorithmica*, 71(4):812–836, 2013.
- [FGR20] Fedor V. Fomin, Petr A. Golovach, and Jean-Florent Raymond. On the tractability of optimization problems on *H*-graphs. *Algorithmica*, 82(9):2432–2473, 2020.
- [FTV15] Fedor V. Fomin, Ioan Todinca, and Yngve Villanger. Large induced subgraphs via triangulations and CMSO. SIAM J. Comput., 44(1):54–87, 2015.

- [Ful56] Delbert R. Fulkerson. Note on Dilworth's decomposition theorem for partially ordered sets. In *Proc. Amer. Math. Soc.*, volume 7, pages 701–702, 1956.
- [Gav74a] Fanica Gavril. Algorithms on circular-arc graphs. Networks, 4(4):357–369, 1974.
- [Gav74b] Fănică Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *Journal of Combinatorial Theory, Series B*, 16(1):47–56, 1974.
- [Gav96] Fănică Gavril. Intersection graphs of Helly families of subtrees. *Discrete Appl. Math.*, 66(1):45–56, 1996.
- [GM18] Serge Gaspers and Simon Mackenzie. On the number of minimal separators in graphs. J. Graph Theory, 87(4):653–659, 2018.
- [Gol77] Martin Charles Golumbic. The complexity of comparability graph recognition and coloring. *Computing*, 18(3):199–208, 1977.
- [Gol04] Martin Charles Golumbic. Algorithmic graph theory and perfect graphs, volume 57. Elsevier, 2004.
- [HK73] John E. Hopcroft and Richard M. Karp. An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. SIAM J. Comput., 2(4):225–231, 1973.
- [HS95] M. L. Huson and A. Sen. Broadcast scheduling algorithms for radio networks. In *Proceedings of MILCOM '95*, volume 2, pages 647–651 vol.2, 1995.
- [HS12] Michel Habib and Juraj Stacho. Reduced clique graphs of chordal graphs. Eur. J. Comb., 33(5):712-735, 2012.
- [Hsu85] Wen-Lian Hsu. Maximum weight clique algorithms for circular-arc graphs and circle graphs. SIAM J. Comput., 14(1):224–231, 1985.
- [JKT20] Lars Jaffke, O-joung Kwon, and Jan Arne Telle. Mim-width II. the feedback vertex set problem. *Algorithmica*, 82(1):118–145, 2020.
- [JMT92] Deborah Joseph, Joao Meidanis, and Prasoon Tiwari. Determining DNA sequence similarity using maximum independent set algorithms for interval graphs. In Otto Nurmi and Esko Ukkonen, editors, Algorithm Theory SWAT '92, Third Scandinavian Workshop on Algorithm Theory, Helsinki, Finland, July 8-10, 1992, Proceedings, volume 621 of Lecture Notes in Computer Science, pages 326–337. Springer, 1992.
- [JS97] Klaus Jansen and Petra Scheffler. Generalized coloring for tree-like graphs. *Discret.* Appl. Math., 75(2):135–155, 1997.
- [KKOS15] Pavel Klavík, Jan Kratochvíl, Yota Otachi, and Toshiki Saitoh. Extending partial representations of subclasses of chordal graphs. *Theor. Comput. Sci.*, 576:85–101, 2015.
- [KKW98] Ton Kloks, Dieter Kratsch, and C. K. Wong. Minimum fill-in on circle and circular-arc graphs. *J. Algorithms*, 28(2):272–289, 1998.

- [LB79] George S. Lueker and Kellogg S. Booth. A linear time algorithm for deciding interval graph isomorphism. J. ACM, 26(2):183–195, 1979.
- [MORC07] Carlo Mannino, Gianpaolo Oriolo, Federico Ricci-Tersenghi, and L. Sunil Chandran. The stable set problem and the thinness of a graph. *Oper. Res. Lett.*, 35(1):1–9, 2007.
- [MU04] Kazuhisa Makino and Takeaki Uno. New algorithms for enumerating all maximal cliques. In Torben Hagerup and Jyrki Katajainen, editors, Algorithm Theory SWAT 2004, 9th Scandinavian Workshop on Algorithm Theory, Humlebaek, Denmark, July 8-10, 2004, Proceedings, volume 3111 of Lecture Notes in Computer Science, pages 260–272. Springer, 2004.
- [Rob78] Fred S Roberts. Graph theory and its applications to problems of society. SIAM, 1978.
- [RS84] Neil Robertson and Paul D. Seymour. Graph minors. III. planar tree-width. *J. Comb. Theory, Ser. B*, 36(1):49–64, 1984.
- [RTL76] Donald J. Rose, Robert Endre Tarjan, and George S. Lueker. Algorithmic aspects of vertex elimination on graphs. *SIAM J. Comput.*, 5(2):266–283, 1976.
- [Sin66] F. W. Sinden. Topology of thin film rc circuits. Bell System Technical Journal, 45(9):1639–1662, 1966.
- [Tar85] Robert Endre Tarjan. Decomposition by clique separators. *Discret. Math.*, 55(2):221–232, 1985.
- [Whi84] S. H. Whitesides. A method for solving certain graph recognition and optimization problems, with applications to perfect graphs. In *Topics on perfect graphs.*, pages 281–298, 1984. Annals of Discrete Mathematics 21.
- [Yan82] Mihalis Yannakakis. The complexity of the partial order dimension problem. SIAM Journal on Algebraic Discrete Methods, 3(3):351–358, 1982.