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Classes of Directed Graphs

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Preface

The two editions of our book *Digraphs: Theory, Algorithms and Applications*, which were published by Springer in 2000 and 2009, respectively, remain the only modern books on graph theory covering more than a small fraction of the theory of directed graphs. We are happy to see that the book has been useful, both for students of advanced courses and to researchers from a wide range of areas, some of which are far from mathematics, such as sociology and medicine.

Since we completed the second edition in 2008, the theory of directed graphs has continued to evolve at a high speed; many important results, including solutions some of the conjectures from *Digraphs*, have appeared and new methods have been developed. So we were faced with the choice of either writing a 3rd edition of our book or developing a new book from scratch. We decided to do the latter for the following main reason: By taking a new, somewhat orthogonal, approach of writing chapters on different and important classes of digraphs, we could give a different viewpoint of digraph theory and include a number of authors whose combined expertise would greatly simplify the process and at the same time increase the quality of the book. We are very happy that the following authors agreed to (co)author chapters for the book: César Hernández-Cruz, Hortensia Galeana-Sánchez, Yubao Guo, Richard Hammack, Frédéric Havet, Jing Huang, Stephan Kreutzer, O-joun Kwon, Marcin Pilipczuk, Michał Pilipczuk, Michel Surmacs, Magnus Wahlström and Anders Yeo.

The book contains more than 120 open problems and conjectures, a feature which should help to stimulate lots of new research. Even though this book should not be seen as an encyclopedia on directed graphs, we have included as many important results as possible. The book contains a considerable number of proofs, illustrating various approaches, techniques and algorithms used in digraph theory.

As was the case with 'Digraphs', one of the main features of this book is its strong emphasis on algorithms. Algorithms on (directed) graphs often play an important role in problems arising in several areas, including computer science and operations research. Secondly, many problems on (directed) graphs are inherently algorithmic. Hence, the book contains many constructive proofs from which one can often extract an efficient algorithm for the problem studied.

To facilitate the use of this book as a reference book and as a graduate textbook, we have added comprehensive symbol and subject indexes. The latter includes separate entries for open problems, conjectures and proof-techniques as well as $\mathcal{NP}$-complete problems. It is our hope that the organization of the book, as well as the detailed subject index, will help many readers to find what they are looking for without having to read through whole chapters.

Highlights

The book covers the majority of important topics on some of the most important classes of directed graphs, ranging from quite elementary to very advanced results. By organizing the book so as to single out important classes of digraphs, we hope to make it easy for the readers to find results and problems of their interest.

Below we give a brief outline of some of the main highlights of this book. Readers who are looking for more detailed information are advised to consult the list of contents or the subject index at the end of the book.

Chapter 1, by Bang-Jensen and Gutin, contains most of the terminology and notation used in this book as well as several basic results. These are not only used frequently in other chapters, but also serve as illustrations of digraph concepts. Since the terminology and notation used in this book is similar to that in 'Digraphs' some readers may skip parts of this chapter.

Chapter 2, by Bang-Jensen and Havet, deals with tournaments (orientations of complete graphs) and semicomplete digraphs (digraphs whose underlying graphs are complete). Tournaments form undoubtedly the most well-understood class digraphs and they continue to fascinate researchers due to their beautiful theory and a surprisingly large number of difficult open problems. The literature on tournaments is so extensive that one could write a whole book on these, so the chapter attempts to give a comprehensive overview of the theory, various proof techniques and many challenging open problems on tournaments. The chapter contains a number of classical results, including Rédei's theorem that every tournament has an odd number of Hamiltonian paths and Camion's theorem that every strongly connected tournament has a Hamiltonian cycle. It covers important topics such as arc-disjoint in- and out-branchings, decompositions into arc-disjoint Hamiltonian cycles or strong spanning subdigraphs, feedback sets, Seymour's second neighbourhood conjecture, vertex-partitions with prescribed properties, oriented Hamiltonian paths and cycles, (Hamiltonian)-connectivity, Hamiltonian cycles avoiding prescribed arcs, disjoint cycles and finally linkages (disjoint paths with prescribed end vertices). The chapter also contains a beautiful proof, due to Pokrovskiy, of the result that a linear bound on the vertex connectivity suffices to ensure that a semicomplete digraph is k-linked.

Chapter 3, by Gutin, deals with acyclic digraphs, that is, digraphs with no directed cycles. This class of digraphs is often used in digraph applications. Thus, Sections 3.9–3.12 are devoted to four different applications: embedded computing, cryptographic enforcement schemes, project scheduling, and website text analysis. All other sections of the chapter, apart from the last section, consider results on various problems restricted to acyclic digraphs. Most results are on decision problems on subgraphs of acyclic digraphs such as out- and in-branchings, k-linkages, and dicuts, but Section 3.5 describes some enumeration results and algebraic techniques to prove them. The last section is devoted to generalizations of acyclic digraphs to the class of edge-coloured graphs. Somewhat surprisingly there are five such generalizations to edge-coloured graphs. This is partially due to the fact that some notions on directed walks, which are equivalent in digraphs, are no longer equivalent for properly coloured walks in edge-coloured graphs.

Chapter 4, by Wahlström, deals with Euler digraphs, that is, digraphs which are connected and in which every vertex has the same number of in-neighbours and out-neighbours. It is well known that the 1736 result of Euler, probably the first result in graph theory, which says that every connected graph in which all degrees are positive even numbers has a closed walk (called an Euler tour) that uses each edge exactly once, extends directly to Euler digraphs. Euler digraphs are interesting, not only because they have a closed Euler tour but also since they can often be viewed as a "half-way" between undirected and directed graphs. Several problems are tractable for undirected graphs, but intractable for directed graphs. Such problems may be either tractable or intractable for Euler digraphs. Good examples of such problems are linkage problems. However, there are exceptions. One of them is the well-known problem on enumerating Euler tours. While this problem is #P-hard on undirected graphs, it is polynomial-time solvable on Euler digraphs by the so-called BEST theorem proved in the chapter.

Chapter 5, by Pilipczuk and Pilipczuk, deals with planar digraphs, that is, digraphs which can be embedded in the plane with no arc crossings. The main goal of the chapter is to show, from multiple angles, how planarity imposes structure on digraphs and how such structure can be used algorithmically. The main focus of the chapter is to show various techniques used in algorithms on planar digraphs. The chapter is not a survey on planar digraphs, instead the authors concentrate on three topics: the $O(n \log n)$-time algorithm for finding a maximum flow between two vertices by Borradaile and Klein, the polynomial-time algorithm, based on advanced algebraic techniques by Schrijver for the k-Linkage Problem, and the Directed Grid Theorem.

Chapter 6, by Bang-Jensen, deals with locally semicomplete digraphs and some generalizations of these. A digraph is locally semicomplete if both the out-neighbourhood and the in-neighbourhood of each vertex induces a semicomplete digraph. Locally semicomplete digraphs were discovered by Bang-Jensen in 1988 and have since then been the focus of much attention, including several Ph.D. theses. The reason for this is that many results on tournaments and semicomplete digraphs extend to this much larger class of digraphs whose structure is well understood: they consist of three subclasses, namely semicomplete digraphs, round-decomposable digraphs and finally, so-called evil locally semicomplete digraphs. The last class is by far the most complicated of the two non-semicomplete subclasses of locally semicomplete digraphs. The chapter contains a full proof of the above classification of locally semicomplete digraphs as well as several examples on how to use this classification to extend many results on semicomplete digraphs to locally semicomplete digraphs. These include results on pancyclicity, arc-disjoint in- and out-branchings, decompositions into arc-disjoint strong spanning subdigraphs, feedback sets, (Hamiltonian)-connectivity, disjoint cycles, linkages and finally orientations of locally semicomplete digraphs, that is, digraphs that can be obtained by deleting one arcs from every 2-cycle. The chapter also discusses results on superclasses of the class of locally semicomplete digraphs, such as locally in-semicomplete and path-mergeable digraphs.

Chapter 7, by Yeo, deals with semicomplete multipartite digraphs, that is, digraphs whose underlying undirected graphs are complete multipartite. Clearly semicomplete digraphs form a subclass of this class so it is natural to ask how much of the structure of semicomplete digraphs carries over to semicomplete multipartite digraphs. Moon's book on tournaments from 1968 already contains some results along these lines and in 1976 Bondy initiated the study of cycles intersecting each partite set at least once. In 1988 Gutin solved the Hamiltonian path problem by giving a simple necessary and sufficient condition, and he, Häggkvist and Manoussakis characterized Hamiltonian semicomplete bipartite digraphs. To this date no necessary and sufficient condition for a semicomplete multipartite digraph to be Hamiltonian is known. One of the main results on semicomplete multipartite digraphs is Yeo's irreducible cycle factor theorem from 1997. Using this theorem, many deep results on semicomplete multipartite digraphs have been obtained, e.g. in 1997 Yeo proved a long standing conjecture that every regular semicomplete multipartite digraph is Hamiltonian and Bang-Jensen, Gutin and Yeo used Yeo's theorem to prove the existence of a polynomial algorithm to decide the existence of a Hamiltonian cycle in semicomplete multipartite digraphs. In this chapter Yeo, one of the main contributors to the area, gives a detailed account of the state of the art of results on this important class of digraphs. Besides results on the full class of semicomplete multipartite digraphs and on (almost) regular semicomplete multipartite digraphs, the chapter also contains a number of results on two subclasses, extended semicomplete digraphs and semicomplete bipartite digraphs. For these two classes there is a simple characterization of the length of a longest cycle, leading to a polynomial algorithm to find such a cycle. For the full class of semicomplete multipartite digraphs it is still open whether a polynomial algorithm exists.

Chapter 8, by Galeana-Sánchez and Hernández-Cruz, deals with transitive and quasi-transitive digraphs as well as generalizations of these. A digraph is transitive, respectively, quasi-transitive if it satisfies that whenever x, y, z are distinct vertices so that xy and yz are arcs, then there is also an arc from x to z, respectively, between x and z. In 1962 Ghouila-Houri proved that a graph G has a quasi-transitive orientation if and only if it has a transitive orientation and hence G is a comparability graph. It was only after 1993, when Bang-Jensen and Huang gave a very useful structural characterization of quasi-transitive digraphs, that research into structural and algorithmic aspects of this class of digraphs flourished. They showed that quasi-transitive digraphs have a recursive structure which allows one to decompose them into smaller pieces, each of which is either a transitive oriented graph or a strong semicomplete digraph. The first non-trivial algorithmic application of the characterization was due to Gutin. The characterization and his approach have led to the study of totally Φ-decomposable digraphs for different choices of digraph classes Φ. These are digraphs which can be decomposed into smaller pieces, each of which belong to the class Φ. This research has revealed that many problems, including the Hamiltonian path and cycle problems and linkage problems, can be solved efficiently for quasi-transitive digraphs and much more general classes of totally Φ-decomposable digraphs. The chapter gives a detailed account of these results as well as results on kings, kernels and the path-partition conjecture by Laborde et al. from 1983. The chapter also contains a number of results on k-transitive and k-quasi-transitive digraphs. These are classes where the definition of transitive and quasi-transitive digraphs is relaxed.

Chapter 9, by Kreutzer and Kwon, considers structural parameters for digraphs. For undirected graphs, tree-width played a key role in developing an undirected graph structure theory and in designing efficient algorithms for intractable problems restricted to graphs of bounded tree-width. In the chapter, Kreutzer and Kwon classify digraph structural parameter approaches into three categories: tree-width inspired, rank-width inspired and density-based. Each of the approaches has its advantages and disadvantages described in numerous results obtained by various authors. While the great success of tree-width on undirected graphs has not been replicated on directed graphs (in fact, some negative results explain this situation), a number of important positive results and approaches have been obtained recently, such as the Directed Grid Theorem of Kawarabayashi and Kreutzer, Kanté's rank-width concepts, and the directed bounded expansion and nowhere density approaches which generalize their undirected counterparts introduced by Nešetřil and Ossona de Mendez.

Chapter 10, by Hammack, is devoted to products of digraphs. Hammack considers results on four standard digraph products: Cartesian product, direct product, strong product, and lexicographic product. The products have many common properties and several differences. For example, all four products are associative, but only the first three are commutative, and unlike for the three other products, K_1 is not a unit for the direct product. While many results on undirected graph products are well known, those on digraph products are less known outside the community of researchers who study the area. We hope this chapter will change the situation.

Chapter 11, by Guo and Surmacs, covers a number of classes of digraphs for which we could not devote a separate chapter, because there are not so many results on the class and also due to space limitations. Several of these classes have applications in interconnection networks and other areas. The classes considered include line digraphs, iterated line digraphs, de Bruijn digraphs, Kautz digraphs, directed cographs, perfect digraphs, arc-locally semicomplete digraphs and finally some generalizations of the latter class. Line digraphs as defined by Harary and Norman in 1960 naturally generalize line graphs of undirected graphs and they have applications in interconnection networks. The chapter contains a number of structural results on line digraphs and two other classes of digraphs, closely related to line digraphs, namely the de Bruijn and Kautz digraphs. These play an important role in network design as they combine the properties of having low out-degree, high connectivity and low diameter, something which is very important in communication networks. The chapter also contains results on directed cographs, in particular on linkages, and on perfect digraphs. The latter is a recent generalization of perfect graphs to digraphs due to Andres and Hochstättler. In contrast to perfect graphs, which can be recognized in polynomial time, recognizing perfect digraphs is $\mathcal{NP}$-complete. In the last sections of the chapter the authors discuss results on arc-locally semicomplete digraphs and some related classes. A digraph is arc-locally semicomplete if, for any choice of 4 distinct vertices x, y, u, v the presence of the arcs xu, yv implies that either none of the pairs x, y and u, v are adjacent or both pairs are adjacent. This class contains all semicomplete and all semicomplete bipartite digraphs and several characterizations, such as those for having a Hamiltonian path or cycle, carry over from semicomplete bipartite digraphs to arc-locally semicomplete digraphs.

Chapter 12, by Huang, deals with orientations of undirected graphs and mixed graphs (which may have both arcs and edges), that is, assigning for each edge xy one of the two possible orientations $x \to y, y \to x$. The central topic is deciding whether the given graph has an orientation as an oriented graph which has a certain prescribed property Π. This could be the property of belonging to a certain class of digraphs, e.g. quasi-transitive digraphs or locally semicomplete digraphs, being strongly connected, or being acyclic and not containing a prescribed set of digraphs as an induced subdigraph. The chapter illustrates a general technique, the lexicographic orientation method, due to Hell and Huang, for achieving such orientations for graphs that are comparability graphs or proper circular arc graphs. When instead the input is a mixed graph $M = (V, E, A)$ with edge set E and arc set A we must orient the edges of E but leave the arcs of A untouched and again the goal is that the final oriented graph has a prescribed property Π. This is called the Π-orientation completion problem. It is shown in this chapter that even for semicomplete digraphs there are natural $\mathcal{NP}$-hard Π-orientation completion problems.

Technical Remarks

We have tried to unify the book by using common terminology and notation for all chapters. We also used special environments for algorithms and problems, and used a special script for problem names as customary in the modern literature on algorithms. All the above should facilitate the reading of this book.

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