Nivat's conjecture holds for sums of two periodic configurations

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Abstract. Nivat's conjecture is a long-standing open combinatorial problem. It concerns two-dimensional configurations, that is, maps $\mathbb{Z}^2 \to \mathcal{A}$ where \mathcal{A} is a finite set of symbols. Such configurations are often understood as colorings of a two-dimensional square grid. Let $P_c(m,n)$ denote the number of distinct $m \times n$ block patterns occurring in a configuration c. Configurations satisfying $P_c(m,n) \leq mn$ for some $m,n \in \mathbb{N}$ are said to have low rectangular complexity. Nivat conjectured that such configurations are necessarily periodic.

Recently, Kari and the author showed that low complexity configurations can be decomposed into a sum of periodic configurations. In this paper we show that if there are at most two components, Nivat's conjecture holds. As a corollary we obtain an alternative proof of a result of Cyr and Kra: If there exist $m, n \in \mathbb{N}$ such that $P_c(m, n) \leq mn/2$, then c is periodic. The technique used in this paper combines the algebraic approach of Kari and the author with balanced sets of Cyr and Kra.

1 Introduction

Let \mathcal{A} be a finite set of symbols and d a positive integer, the dimension. A d-dimensional symbolic configuration c is an element of $\mathcal{A}^{\mathbb{Z}^d}$, that is, a map assigning a symbol to every vertex of the lattice \mathbb{Z}^d . The symbol at position $v \in \mathbb{Z}^d$ is denoted c_v .

For a non-empty finite domain $D \subset \mathbb{Z}^d$, the elements of \mathcal{A}^D are *D-patterns*. We can observe patterns in a given configuration, the *D*-pattern occurring in c at position $\mathbf{v} \in \mathbb{Z}^d$ is the map

$$p: D \to \mathcal{A}$$

 $\boldsymbol{u} \mapsto c_{\boldsymbol{v}+\boldsymbol{u}}.$

The number of distinct *D*-patterns occurring in c, denoted $P_c(D)$, is the *D*-pattern complexity of c. We say that c has low complexity if $P_c(D) \leq |D|$ holds for some D.

We study what conditions on complexity imply that a configuration is periodic, that is, when there exists a non-zero vector \boldsymbol{u} such that $c_{\boldsymbol{v}} = c_{\boldsymbol{v}+\boldsymbol{u}}$ for all $\boldsymbol{v} \in \mathbb{Z}^d$. The situation in one dimension was described by Morse and Hedlund [MH38], let us denote $[n] = \{0, \ldots, n-1\}$:

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Theorem (Morse–Hedlund). Let c be a one-dimensional symbolic configuration. Then c is periodic if and only if there exists $n \in \mathbb{N}$ such that $P_c(\llbracket n \rrbracket) \leq n$.

As a corollary, non-periodic one-dimensional configurations satisfy $P_c(\llbracket n \rrbracket) \ge n+1$. Those for which equality holds for every n are Sturmian words, they are a central topic of combinatorics on words and have connections to discrete geometry, finite automata and mathematical physics [Lot02, AS03, DL99]. Note that Sander and Tijdeman [ST00] extended the Morse–Hedlund theorem for patterns of other shapes than $\llbracket n \rrbracket$, they showed that in fact any low complexity one-dimensional symbolic configuration is periodic.

Nivat's conjecture [Niv97] is a natural extension of the theorem to two-dimensions. To simplify notation we write $P_c(m,n) = P_c(\llbracket m \rrbracket \times \llbracket n \rrbracket)$.

Conjecture (Nivat). If a two-dimensional symbolic configuration c satisfies $P_c(m, n) \le mn$ for some $m, n \in \mathbb{N}$, then it is periodic.

Nivat's conjecture is tight in the sense that there exist non-periodic configurations satisfying $P_c(m,n) = mn+1$ for all $m,n \in \mathbb{N}$, all such configurations were classified by Cassaigne [Cas99]. Note that the conjecture is not an equivalence, the opposite implication is easily seen to be false.

There have been a number of partial results towards the conjecture. Cyr and Kra [CK16] proved that having $P_c(3,n) \leq 3n$ for some $n \in \mathbb{N}$ implies periodicity, which was an improvement on a previous result with constant 2 [ST02]. In another direction, there are results showing that having $P_c(m,n) \leq \alpha mn$ for some $m, n \in \mathbb{N}$ implies periodicity for a suitable real α . The best result to date is also by Cyr and Kra [CK15] with $\alpha = 1/2$, which improved on previous constants $\alpha = 1/16$ [QZ04] and $\alpha = 1/144$ [EKM03]. Recently, Kari and the author [KS15] proved an asymptotic version of the conjecture: If $P_c(m,n) \leq mn$ for infinitely many pairs $(m,n) \in \mathbb{N}^2$, the configuration is periodic.

The Morse–Hedlund theorem does not analogously generalize to higher dimensions. There exists a three-dimensional configuration with low block complexity which is not periodic [ST00].

Our contributions

In [KS15], Kari and the author introduced an algebraic view on symbolic configurations. Following their definition, let a *configuration* be any formal power series in d variables x_1, \ldots, x_d with complex coefficients, that is, an element of

$$\mathbb{C}[[X^{\pm 1}]] = \big\{ \sum_{\boldsymbol{v} \in \mathbb{Z}^d} c_{\boldsymbol{v}} X^{\boldsymbol{v}} \bigm| c_{\boldsymbol{v}} \in \mathbb{C} \big\}$$

where X^{v} is a shorthand for $x_1^{v_1} \cdots x_d^{v_d}$. If the configuration has only integer coefficients it is called *integral*, if they come from a finite set the configuration

¹ For the most of this paper, however, it is enough to consider configurations to be elements of $\mathbb{C}^{\mathbb{Z}^d}$.

is finitary. A symbolic configuration can be identified with a finitary integral configuration if the symbols from \mathcal{A} are chosen to be integers. Kari and the author in [KS15] proved:

Theorem (Decomposition theorem). Let c be a low complexity d-dimensional finitary integral configuration. Then there exists $k \in \mathbb{N}$ and periodic d-dimensional configurations c_1, \ldots, c_k such that $c = c_1 + \cdots + c_k$.

Note that the summands do not have to be finitary configurations. The minimal possible number of components k in the decomposition plays an important role. In this paper we prove:

Theorem 1. Let c be a two-dimensional configuration satisfying $P_c(m, n) \leq mn$ for some $m, n \in \mathbb{N}$. If c is a sum of two periodic configurations then it is periodic.

In the proof of the asymptotic version of Nivat's conjecture given in [KS16], configurations which are a sum of horizontally and vertically periodic configuration had to be handled separately using a rather technical combinatorial approach. Theorem 1 is of particular interest since it covers this case.

In this paper we revisit the method of Van Cyr and Bryna Kra [CK15,CK16]. They approach Nivat's conjecture from the point of view of symbolic dynamics. They use a refined version of the classical notion of expansiveness of a subshift, a so called *one-sided non-expansiveness*. A key definition of theirs is that of a balanced set – it is a shape $D \subset \mathbb{Z}^2$ which satisfies a particular condition on the complexity $P_c(D)$. (Note that this notion is different from balancedness usual in combinatorics on words.) The crucial tool they developed is a combinatorial lemma which links one-sided non-expansiveness and balanced sets to periodicity of a configuration. However, in order to obtain the main result of the paper from the lemma it still takes a rather lengthy technical analysis.

We combine the algebraic method with ideas of Cyr and Kra. We start the exposition with a very basic introduction to the topic of symbolic dynamics. In section 2 we define a subshift, in section 3 we fix some geometric terminology, and in section 4 we give definitions of non-expansiveness and one-sided non-expansiveness of a subshift.

In section 5 we introduce a simplified version of a balanced set and prove Lemma 4 which connects balanced sets with periodicity using the ideas of Cyr and Kra. We use the lemma together with decomposition theorem to prove Theorem 1 in section 6. As a corollary, we obtain an alternative proof of Theorem 1.2 of [CK15], the main result of their paper:

Theorem (Cyr, Kra). Let c be a configuration satisfying $P_c(m,n) \leq mn/2$ for some $m, n \in \mathbb{N}$. Then c is periodic.

2 Symbolic dynamics and subshifts

Let us recall basic facts from symbolic dynamics, for a comprehensive reference and proofs see [Kůr03].

Symbolic dynamics studies $\mathcal{A}^{\mathbb{Z}^d}$ as a topological space. Let us first make \mathcal{A} a topological space by endowing it with the discrete topology. Then $\mathcal{A}^{\mathbb{Z}^d}$ is considered to be a topological space with the product topology.

Open sets in this topology are for example sets of the following form. Let $D \subset \mathbb{Z}^d$ be finite and $p \colon D \to \mathcal{A}$ arbitrary. Then

$$Cyl(p) := \{ c \in \mathcal{A}^{\mathbb{Z}^d} \mid \forall v \in D \colon c_v = p_v \}$$

is an open set, also called a *cylinder*. In fact, the collection of cylinders Cyl(p) for all possible p forms a subbase of the topology on $\mathcal{A}^{\mathbb{Z}^d}$.

For a vector $\boldsymbol{u} \in \mathbb{Z}^d$, the *shift* operator $\tau_{\boldsymbol{u}} \colon \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ is defined by $(\tau_{\boldsymbol{u}}(c))_{\boldsymbol{v}} = c_{\boldsymbol{v}-\boldsymbol{u}}$. Informally, $\tau_{\boldsymbol{u}}$ shifts a configuration in the direction of vector \boldsymbol{u} .

The set $\mathcal{A}^{\mathbb{Z}^d}$ is called the *full shift*. A subset $X \subset \mathcal{A}^{Z^d}$ is called a *subshift* if it is a topologically closed set which is invariant under all shifts τ_u :

$$\forall \boldsymbol{u} \in \mathbb{Z}^d : c \in X \Rightarrow \tau_{\boldsymbol{u}}(c) \in X.$$

Subshifts are the central objects of study in symbolic dynamics.

Let c be a symbolic configuration. We denote by X_c the *orbit closure* of c, that is, the smallest subshift which contains c. It can be shown that c contains exactly those configurations c' whose finite patterns are among the finite patterns of c. In particular, for any $c' \in X_c$ and a finite domain D we have $P_{c'}(D) \leq P_c(D)$.

Example 1. Let us give an example of taking orbit closure. Let $c \in \{0,1\}^{\mathbb{Z}^2}$ be such that $c_{ij} = 1$ if i = 0 or j = 0, and $c_{ij} = 0$ otherwise. When pictured, the configuration c consists of a large cross with its center at (0,0). The orbit closure X_c then consist of four types of configurations: a cross, a horizontal line, a vertical line and all zero configurations, with all possible translations, see Figure 1. It is easy to see that any pattern which occurs in them also occurs in c, and not difficult to prove that those are all such configurations.



Fig. 1: Four types of configurations in the orbit closure X_c from Example 1. The gray color corresponds to value 1, white is 0.

3 Geometric notation and terminology

In the sequel we will be concerned with the geometry of \mathbb{Z}^2 . Let us establish some notation and terminology.

We view \mathbb{Z}^2 as a subset of the vector space \mathbb{Q}^2 . A direction is an equivalence class of $\mathbb{Q}^2 \setminus \{(0,0)\}$ modulo the equivalence relation $u \sim v$ iff $u = \lambda v$ for some $\lambda > 0$. By a slight abuse of notation, we identify a non-zero vector $\mathbf{u} \in \mathbb{Z}^2$ with the direction $\mathbf{u}\mathbb{Q}^+$.

Let $u \in \mathbb{Z}^2$ be non-zero. An (undirected) line in \mathbb{Z}^2 is a set of the form

$$\{ \boldsymbol{v} + q\boldsymbol{u} \mid q \in \mathbb{Q} \} \cap \mathbb{Z}^2$$

for some $v \in \mathbb{Z}^2$. We call both u and -u a direction of the line. We define a directed line to be a line augmented with one of the two possible directions.

Let ℓ be a directed line in direction u going through $v \in \mathbb{Z}^2$. The half-plane determined by ℓ is defined by

$$H_{\ell} = \{ v + w \mid w \in \mathbb{Z}^2, w_1 u_2 - u_1 w_2 \ge 0 \}.$$

With the usual choice of coordinates it is the half-plane "on the right" from the line. Let H_u denote the half-plane determined by the directed line in direction u going through the origin.

We say that a non-empty $D \subset \mathbb{Z}^2$ is *convex* if D can be written as an intersection of half-planes. Convex hull of D, denoted Conv(D), is the smallest convex set containing D. Assume ℓ is a directed line in direction \boldsymbol{u} such that $D \subset H_\ell$ and $\ell \cap D$ is non-empty. If $|\ell \cap D| > 1$ we call it the edge of D in direction \boldsymbol{u} , otherwise we call it the vertex of D in direction \boldsymbol{u} . Note that a vertex is a vertex for many directions, but an edge has a unique direction (as long as D is not contained in a line). See Figure 2 for an example.

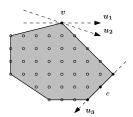


Fig. 2: A convex set. The point v is a vertex of the set for both directions u_1 and u_2 . The set of three marked points e is the edge in direction u_3 .

Let u be a direction and ℓ, ℓ' two directed lines in direction u. If

$$S = H_{\ell} \setminus H_{\ell'}$$

is non-empty, then S is called a *stripe* in direction \boldsymbol{u} . We call ℓ, ℓ' the *inner* and outer boundary of S respectively. Let $S^{\circ} = S \setminus \ell$ be the *interior* of S.

For $A, B \subset \mathbb{Z}^2$, we say that A fits in B if there exists a translation of A which is a subset of B.

4 Non-expansiveness and one-sided non-expansiveness

It can be verified that the topology on $\mathcal{A}^{\mathbb{Z}^d}$ is compact and also metrizable. Note that shift operators τ_u are continuous maps on $\mathcal{A}^{\mathbb{Z}^d}$. Expansiveness can be defined in general for a continuous action on a compact metric space, the definition is however too general for our purposes. We give a definition specific to the case of $\mathcal{A}^{\mathbb{Z}^2}$.

Let $X \subset \mathcal{A}^{\mathbb{Z}^2}$ be a subshift and \boldsymbol{u} a direction. Then \boldsymbol{u} is an expansive direction for X if there exists a stripe S in direction \boldsymbol{u} such that

$$\forall c, e \in X: c \upharpoonright_S = e \upharpoonright_S \Rightarrow c = e.$$

Informally speaking, u is an expansive direction for X if a configuration in X is uniquely determined by its coefficients in a wide enough stripe in direction u.

A two-dimensional configuration is *doubly periodic* if it has two linearly independent period vectors. The following classical theorem links double periodicity of a configuration with expansiveness. It is a corollary of a theorem by Boyle and Lind [BL97].

Theorem 2. Let c be a symbolic configuration. Then c is doubly periodic iff all directions are expansive for X_c .

Let $X \subset \mathcal{A}^{\mathbb{Z}^2}$ be a subshift and \boldsymbol{u} a direction. Then \boldsymbol{u} is a *one-sided expansive direction* for X if

$$\forall c, e \in X : c \upharpoonright_{H_n} = e \upharpoonright_{H_n} \Rightarrow c = e.$$

Equivalently, \boldsymbol{u} is a one-sided expansive direction for X if there exists a wide enough stripe S in direction \boldsymbol{u} such that $\forall c, e \in X : c \upharpoonright_S = e \upharpoonright_S \Rightarrow c \upharpoonright_{H_{-\boldsymbol{u}}} = e \upharpoonright_{H_{-\boldsymbol{u}}}$. See Figure 3 for a comparison of the notion of expansiveness and one-sided expansiveness.

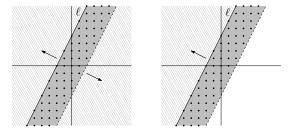


Fig. 3: The figure on the left illustrates expansiveness – values of the configuration inside the stripe determine the whole configuration. On the right we see one-sided expansiveness in direction (1,2) – values in the half-plane H_{ℓ} , or equivalently in a wide enough stripe, determine the values in the half-plane $\mathbb{Z}^2 \setminus H_{\ell}$.

Example 2 (Ledrappier's subshift). It is possible for a subshift to be one-sided expansive but non-expansive in the same direction. Consider a subshift $X \subset \{0,1\}^{\mathbb{Z}^2}$ consisting of configurations c which satisfy $c_{ij} \equiv c_{i,j+1} + c_{i+1,j+1} \pmod{2}$. Upper half-plane of a configuration determines the whole, since any single row determines the one below it. Therefore (-1,0) is a one-sided expansive direction for X. However, no stripe in direction (-1,0) determines a configuration from the subshift; for any row, there are always two possibilities for the row above it (they are complements of each other). Any horizontal stripe can be extended to the upper half-plane in infinitely many ways.

We are primarily interested in non-expansive directions. In our setup, it is known that there are only finitely many of them, we omit the proof for space reasons. (See Appendix.)

Lemma 1. Let c be a low complexity two-dimensional configuration. Then there are at most finitely many one-sided non-expansive directions for X_c .

For later use it will be practical to define non-expansiveness explicitly. Let $X \subset \mathcal{A}^{\mathbb{Z}^2}$ be a subshift and S a stripe in direction u. We say that S is an ambiguous stripe in direction u if there exist $c, e \in X$ such that

$$c \upharpoonright_{S^{\circ}} = e \upharpoonright_{S^{\circ}}, \text{ but } c \upharpoonright_{S} \neq e \upharpoonright_{S}.$$
 (1)

We say that $c \in X$ contains an ambiguous stripe S if there exists $e \in X$ satisfying (1). Informally, a stripe is ambiguous if its interior does not determine the inner boundary.

Definition. Let u be a direction and $X \subset A^{\mathbb{Z}^2}$ a subshift. Then u is one-sided non-expansive direction if there exists an ambiguous stripe in direction u of arbitrary width.

We leave the proof that this is the converse of the earlier definition of onesided expansiveness to the reader.

5 Balanced sets

Let c be a fixed symbolic configuration.

Definition 1. Let $B \subset \mathbb{Z}^2$ be a finite and convex set, \mathbf{u} a direction and E an edge or a vertex of B in direction \mathbf{u} . Then B is \mathbf{u} -balanced if:

- (i) $P_c(B) \leq |B|$
- (ii) $P_c(B) < P_c(B \setminus E) + |E|$
- (iii) Intersection of B with all lines in direction \mathbf{u} is either empty or of size at least |E|-1.

The three conditions of the definition can be interpreted as follows. The first one simply states that B is a low complexity shape. The second condition limits the number of $(B \setminus E)$ -patterns which do not extend uniquely to a B-pattern, there is strictly less than |E| of them. The third condition is implied if the length of the edge in direction \boldsymbol{u} is smaller or equal to the length of the edge in the opposite direction, as can be seen in the next proof.

Lemma 2. Let c be such that $P_c(m,n) \leq mn$ holds for some $m,n \in \mathbb{N}$ and \boldsymbol{u} be a direction. Then there exists a \boldsymbol{u} -balanced or $(-\boldsymbol{u})$ -balanced set. Moreover, if \boldsymbol{u} is horizontal or vertical, then there exists a \boldsymbol{u} -balanced set.

Proof. Let D be an $m \times n$ rectangle, we have $P_c(D) \leq |D|$. Let us define a sequence of convex shapes $D = D_0 \supset D_1 \supset \cdots \supset D_k = \emptyset$ such that $D_i \setminus D_{i+1}$ is the edge of D_i in direction $(-1)^i \boldsymbol{u}$. Informally, the sequence represents shaving off an edge (or a vertex) of the shape alternately in directions \boldsymbol{u} and $-\boldsymbol{u}$. See Figure 4 for an illustration.

Consider the expression $P_c(D_i) - |D_i|$ as a function of i. For i = 0 its value is non-positive and for i = k its value is 1. Let $i \in [0, k - 1]$ be smallest such that $0 < P_c(D_{i+1}) - |D_{i+1}|$, then we have

$$P_c(D_i) - |D_i| \le 0 < P_c(D_{i+1}) - |D_{i+1}|.$$

Denote $E = D_i \setminus D_{i+1}$, it is an edge or a vertex of D_i in direction \boldsymbol{u} or $-\boldsymbol{u}$. Adding $|D_i|$ to the inequality and rewriting gives $P(D_i) \leq |D_i| < P(D_i \setminus E) + |E|$.

We show that $B = D_i$ is a balanced set by showing that *(iii)* of Definition 1 holds. Without loss of generality let the direction of E be \boldsymbol{u} . Then, by construction, the length of E is smaller or equal to the edge in direction $-\boldsymbol{u}$. In fact, if we consider the convex hull of B in \mathbb{Q}^2 , any line in direction \boldsymbol{u} intersects it in a line segment longer or equal to d, the length of the edge. Any line segment of length at least d in direction \boldsymbol{u} intersects either none or at least |E|-1 integer points, and we are done.

If u is either horizontal or vertical, instead of alternating the direction of shaved off edges, we can always shave off the edge in direction u. It will be always the shortest edge in direction u, therefore verification of part (iii) goes through.

Next we present Lemma 4 which connects non-expansiveness and balanced sets with periodicity, based on the method of Cyr and Kra. Periodicity in the proof first arises in a stripe from the use of Morse–Hedlund theorem. This part of the proof follows Lemma 2.24 from [CK15]. The periodicity is then extended to the whole configuration by the following lemma, which is a corollary of Lemma 39 from [KS16]. We omit the proof for space reasons. (See Appendix.)

Lemma 3. Let c be a two-dimensional configuration and D a non-empty finite subset of \mathbb{Z}^2 such that $P_c(D) \leq |D|$. Let S be a stripe in direction \boldsymbol{u} such that D fits in S. If S° is periodic with a period in direction \boldsymbol{u} then also c is periodic with a period in direction \boldsymbol{u} .

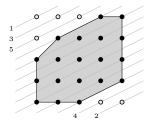


Fig. 4: Shaving off edges or vertices of a 5×5 rectangle alternately in directions (2,1) and (-2,-1). Small numbers indicate the order in which the edges or vertices were removed.

Lemma 4. Let c be a configuration and B a u-balanced set. Assume that c contains an ambiguous stripe for X_c in direction u such that B fits in the stripe. Then c is periodic in direction u.

Proof. Let E be the edge or vertex of B in direction u, denote S the stripe and let ℓ be the inner boundary of S in direction u. Without loss of generality assume $B \subset S$, $E \subset \ell$, and that u is not an integer multiple of a smaller vector. Let $e \in X_c$ be such that Equation 1 holds.

Denote points in E consecutively by e_1, \ldots, e_n (see Figure 5). Define a sequence $B = D_n \supset \cdots \supset D_1 \supset D_0 = B \setminus E$ by setting $D_{i-1} = D_i \setminus \{e_i\}$. Consider the values $P(D_i) - |D_i|$. Since B is a balanced set, by (ii) we have $P_c(D_n) - |D_n| < P_c(D_0) - |D_0|$, let $k \in [0, n-1]$ be such that

$$P_c(D_{k+1}) - |D_{k+1}| < P_c(D_k) - |D_k|.$$

Adding $|D_{k+1}|$ to both sides yields $P_c(D_{k+1}) < P_c(D_k) + 1$. On the other hand, $P_c(D_k) \le P_c(D_{k+1})$ since $D_k \subset D_{k+1}$, and therefore we have $P_c(D_k) = P(D_{k+1})$. In other words, a D_k -pattern uniquely determines the value at position e_{k+1} .

We will show that $\forall i \colon c \upharpoonright_{D_k + i \boldsymbol{u}} \neq e \upharpoonright_{D_k + i \boldsymbol{u}}$. For the contrary, assume that there is j such that $c \upharpoonright_{D_k + j \boldsymbol{u}} = e \upharpoonright_{D_k + j \boldsymbol{u}}$. Using the property of D_k , we have $c \upharpoonright_{e_{k+1} + j \boldsymbol{u}} = e \upharpoonright_{e_{k+1} + j \boldsymbol{u}}$. Therefore $c \upharpoonright_{D_k + (j+1)\boldsymbol{u}} = e \upharpoonright_{D_k + (j+1)\boldsymbol{u}}$ and we can proceed by induction to show $c \upharpoonright_{D_k + j' \boldsymbol{u}} = e \upharpoonright_{D_k + j' \boldsymbol{u}}$ for all j' > j. Analogously, by constructing sets D_i by removing edge points from the other end, it can be shown that also $c \upharpoonright_{D_k + j' \boldsymbol{u}} = e \upharpoonright_{D_k + j' \boldsymbol{u}}$ for all j' < j. We proved $c \upharpoonright_S = e \upharpoonright_S$, which is a contradiction with ambiguity of S.

We have that all $(B \setminus E)$ -patterns $c \upharpoonright_{(B \setminus E) + i\boldsymbol{u}}$ have at least two possible extensions into a B-pattern. Part (ii) of Definition 1 implies that there are at most |E|-1 such patterns. Let T be a thinner stripe in direction \boldsymbol{u} defined by $T = \bigcup_{i \in \mathbb{Z}} (B \setminus E) + i\boldsymbol{u}$. Using part (iii) of Definition 1, values of c on every line $\lambda \subset T$ in direction \boldsymbol{u} contain at most |E|-1 distinct subsegments of length at least |E|-1. By Morse–Hedlund theorem, the values on the line repeat periodically. Therefore $c \upharpoonright_T$ is periodic in direction \boldsymbol{u} .

B fits in the stripe $T \cup \ell$ and its interior T is periodic in direction \boldsymbol{u} . By Lemma 3 also c is periodic in direction \boldsymbol{u} .

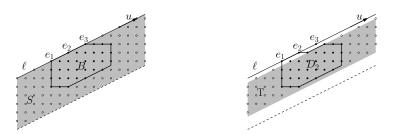


Fig. 5: Illustration of the proof of Lemma 4.

6 Main result

Theorem (Theorem 1). Let c be a two-dimensional configuration satisfying $P_c(m,n) \leq mn$ for some $m,n \in \mathbb{N}$. If c is a sum of two periodic configurations then it is periodic.

Proof. For contradiction assume c is non-periodic and denote c_1, c_2 periodic configurations such that $c = c_1 + c_2$. Let u_1, u_2 be their respective vectors of periodicity. If they are linearly dependent, c is periodic and we are done. Otherwise, define a parallelogram

$$D = \{ a\mathbf{u_1} + b\mathbf{u_2} \mid a, b \in [0, 1) \} \cap \mathbb{Z}^2.$$

We can choose u_1, u_2 large enough so that an $m \times n$ rectangle fits in. We can also assume that $u_2 \in H_{u_1}$. Denote $D_j = D + ju_2$ and define a sequence of stripes $S_j = \bigcup_{i \in \mathbb{Z}} D_j + iu_1$. The setup is illustrated in Figure 6.

Assume that there are $j \neq j'$ such that $c \upharpoonright_{D_j} = c \upharpoonright_{D_{j'}}$. We claim that then $c \upharpoonright_{S_j} = c \upharpoonright_{S_{j'}}$. Note that since $c = c_1 + c_2$, for $\boldsymbol{v} \in \mathbb{Z}^2$ we have

$$(c_{(v+u_1)+ju_2} - c_{(v+u_1)+j'u_2}) - (c_{v+ju_2} - c_{v+j'u_2}) = 0.$$

In particular, if $c_{\boldsymbol{v}+j\boldsymbol{u_2}}=c_{\boldsymbol{v}+j'\boldsymbol{u_2}}$, then also $c_{(\boldsymbol{v}+\boldsymbol{u_1})+j\boldsymbol{u_2}}=c_{(\boldsymbol{v}+\boldsymbol{u_1})+j'\boldsymbol{u_2}}$. Since $c_{\boldsymbol{v}+j\boldsymbol{u_2}}=c_{\boldsymbol{v}+j'\boldsymbol{u_2}}$ holds for $\boldsymbol{v}\in D$, it also holds for $\boldsymbol{v}\in D+\boldsymbol{u_1}$, and by induction $c|_{S_j}=c|_{S_{j'}}$.

Since c is finitary there are only finitely many possible D-patterns, let N be an upper bound on their number. There are also finitely many stripe patterns $c \upharpoonright_{S_j}$ since the pattern in S_j is determined by the pattern in D_j . Because c is not periodic, there exists $k \in \mathbb{Z}$ such that $c \upharpoonright_{S_k} \neq c \upharpoonright_{S_{k-N!}}$.

By Lemma 2, there is either a u_1 -balanced or $(-u_1)$ -balanced set B, without loss of generality assume the former. Since c is non-periodic, by Lemma 4 there

is no ambiguous stripe in c in direction u_1 in which B fits. B fits in any stripe S_j , therefore values in any stripe S_j determine the values in the whole half-plane on the side of the inner boundary of S_j .

By pigeonhole principle, there are $j < j' \in [0, N]$ such that $c \upharpoonright_{S_{k+j}} = c \upharpoonright_{S_{k+j'}}$. The two stripes extend uniquely to the half-planes on the side of their inner boundary. Therefore the half-plane $H = \bigcup_{i \leq j'} S_i$ has period $(j' - j) \mathbf{u_2}$. Since j' - j divides N! and $S_k, S_{k-N!} \subset H$, we have a contradiction with $c \upharpoonright_{S_k} \neq c \upharpoonright_{S_{k-N!}}$.

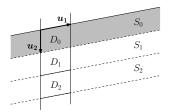


Fig. 6: Proof of Theorem 1.

Corollary 1. If a non-periodic configuration c is a sum of two periodic ones, then $P_c(m,n) \geq mn+1$ for all $m,n \in \mathbb{N}$.

We finish the exposition by reproving the result of Cyr and Kra from [CK15]. To do that, we need additional theory from [KS16]. Multiplication of a two-dimensional configuration c by a polynomial $f \in \mathbb{C}[x_1, x_2]$ is well defined. If fc = 0, we call f an annihilator of c. The following two lemmas we state without a proof, they are direct corollaries of Corollary 24 and Lemma 32 of [KS16], respectively.

Lemma 5. Let c be a low complexity two-dimensional integral configuration. Then there exists $k \in \mathbb{N}$ and polynomials $\phi_1, \ldots, \phi_k \in \mathbb{C}[x_1, x_2]$ with the following properties:

Every annihilator of c is divisible by $\phi_1 \cdots \phi_k$. Furthermore, c can be written as a sum of k, but no fewer periodic configurations. If g is a product of $0 \le \ell < k$ of the polynomials ϕ_i , then gc can be written as a sum of $k - \ell$, but no fewer periodic configurations.

Any polynomial in $\mathbb{C}[x_1, x_2]$ can be written as $f = \sum_{\boldsymbol{v} \in \mathbb{Z}^2} a_{\boldsymbol{v}} X^{\boldsymbol{v}}$. The support of f, denoted supp(f), is defined as the finite set of vectors $\boldsymbol{v} \in \mathbb{Z}^2$ such that $a_{\boldsymbol{v}} \neq 0$. We say that f fits in a subset $D \subset \mathbb{Z}^2$ if its support fits in D.

Lemma 6. Let c be a finitary configuration. Then the symbols of A can be changed to suitable integers such that if $P_c(D) \leq |D|$ for some $D \subset \mathbb{Z}^d$, then there exists an annihilator f which fits in -D.

Theorem 3. Let c be a configuration such that $P_c(m,n) \leq mn/2$ for some $m,n \in \mathbb{N}$. Then c is periodic.

Proof. Assume that the symbols of \mathcal{A} have been renamed as in Lemma 6, then there exists f an annihilator of c which fits in an $m \times n$ rectangle. By Lemma 5, we can write $f = \phi_1 \cdots \phi_k h$. If $k \leq 2$ then c is periodic by Theorem 1. Assume $k \geq 3$, we will show that it leads to a contradiction.

Let $g = \phi_3 \cdots \phi_k$, c' = gc and let $m_g, n_g \in \mathbb{N}$ be smallest such that g fits in an $(m_g+1)\times (n_g+1)$ rectangle, see Figure 7. Note that an $(m-m_g)\times (n-n_g)$ block in c' is determined by multiplication by g from an $m \times n$ block in c. Therefore $P_c(m,n) \geq P_{c'}(m-m_g,n-n_g)$.

By Lemma 5, c' is a sum of two but no fewer periodic configurations. Thus it is not periodic, and by Theorem 1,

$$P_c(m,n) \ge P_{c'}(m-m_q,n-n_q) > (m-m_q)(n-n_q).$$

Let v be an arbitrary vertex of the convex hull of $-\sup(g)$. Consider all translations of $-\sup(g)$ which are a subset of the rectangle $[m] \times [n]$, denote R the locus of v under these translations. There are $(m-m_g)(n-n_g)$ such translations, therefore the size of R is the same number.

Now let us define a shape $U = \llbracket m \rrbracket \times \llbracket n \rrbracket \setminus R$. It is a shape such that no polynomial multiple of g fits in -U. In particular no annihilator of c fits in -U, and thus by Lemma 6,

$$P_c(m,n) \ge P_c(U) > |U|.$$

Since either $(m-m_g)(n-n_g) = |R| \ge mn/2$ or $|U| \ge mn/2$, we have $P_c(m,n) > mn/2$, a contradiction.

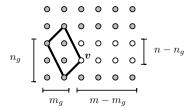


Fig. 7: Proof of Theorem 3. The quadrilateral depicts the convex hull of $-\sup(g)$ for a polynomial g, positioned in the bottom left corner of an $m \times n$ block. The white points form the set R and the shaded points form the set U. We have $|U| \ge mn/2$ or $|R| \ge mn/2$.

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A Appendix

Proofs in the appendix use definitions from section 6.

A.1 Proof of Lemma 1

The lemma also follows from existence of generating sets introduced by Cyr and Kra [CK15]. Here we show a proof using polynomials:

Proof. By Lemma 5 of [KS16], there exists a non-trivial annihilator of the configuration. Let F denote convex hull of it support. It has finitely many edges. We claim that only directions of the edges can be one-sided non-expansive for X_c .

Let u be a direction such that F has a vertex v in direction u. Let ℓ be the line in direction u which is the closest to H_u but lies outside of H_u . Then F can be translated such that $F \setminus \{v\}$ lies in H_u and $v \in \ell$. Linear combination given by the annihilator determines the value of c_v from values in H_u , and by translation in the whole line ℓ . Moving to the next and next line in direction u, all the values of c are determined. We proved that u is a one-sided expansive direction for X_c .

A.2 Proof of Lemma 3

The proof is by reduction to Lemma 39 of [KS16]:

Lemma (Lemma 39). Let c be a counterexample candidate and $v \in \mathbb{Z}^2$ a non-zero vector. Let S be an infinite stripe in the direction of v of maximal width such that ϕ does not fit in. Then c restricted to the stripe S is non-periodic in the direction of v.

We assume the reader is comfortable with notions used in its statement. Let us however briefly describe some of them. A two-dimensional configuration is a counterexample candidate if it is normalized non-periodic finitary integral configuration which has a non-trivial annihilator. Without going into further details, normalized configurations have the property from Lemma 6 and any configuration can be made normalized by changing the symbols in \mathcal{A} . The polynomial ϕ is the largest polynomial (w.r.t. polynomial division) which divides every annihilator, it is product of polynomials ϕ_i from the statement of Lemma 5.

Proof (of Lemma 3). Without loss of generality assume that c is normalized and for the contrary assume that it is non-periodic, then c is a counterexample candidate. By Lemma 6 there is an annihilator which fits in -D and therefore also in S. Then also ϕ fits in S. Let $T \subset S^{\circ}$ be a stripe in direction u of maximal width such that ϕ does not fit in. Since $c \upharpoonright_T$ is periodic in direction u, by Lemma 39 also c is.