# **Revisiting MITL to Fix Decision Procedures**

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Abstract. Metric Interval Temporal Logic (MITL) is a well studied realtime, temporal logic that has decidable satisfiability and model checking problems. The decision procedures for MITL rely on the automata theoretic approach, where logic formulas are translated into equivalent timed automata. Since timed automata are not closed under complementation, decision procedures for MITL first convert a formula into negated normal form before translating to a timed automaton. We show that, unfortunately, these 20-year-old procedures are incorrect, because they rely on an incorrect semantics of the  $\mathcal{R}$  operator. We present the right semantics of  $\mathcal{R}$  and give new, correct decision procedures for MITL. We show that both satisfiability and model checking for MITL are EXPSPACE-complete, as was previously claimed. We also identify a fragment of MITL that we call MITL<sub>WI</sub> that is richer than MITL<sub>0, $\infty$ </sub>, for which we show that both satisfiability and model checking are PSPACE-complete. Many of our results have been formally proved in PVS.

## 1 Introduction

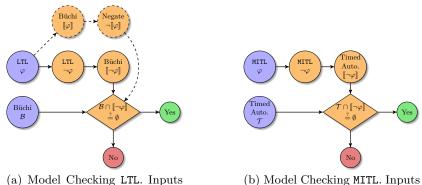
Specifications for real time systems often impose quantitative timing constraints between events that are temporally ordered. Classical temporal logics such as Linear Temporal Logic (LTL) [12] are therefore not adequate. Among the many real-time extensions of LTL, the most well studied is *Metric Temporal Logic* (*MTL*) [7]. The temporal modalities in this logic, like  $\mathcal{U}_{I}$ , are constrained by a time interval I which requires that the second argument of the  $\mathcal{U}$  operator be satisfied in the interval I. For example, the MTL formula in Equation 1 specifies that, at all times, every request should be followed by a response within 5 units of time, or in case there is no response during that time, an error should be raised within the next 0.1 units of time.

$$\Box \operatorname{req} \to \left(\Diamond_{[0,5)} \operatorname{resp} \lor \left(\Box_{[0,5)} \neg \operatorname{resp} \land \Diamond_{(5,5,1]} \operatorname{error}\right)\right) \tag{1}$$

Classical decision problems for any logic are satisfiability and model checking. For MTL, these problems are highly undecidable; both problems are  $\Sigma_1^1$ complete [2]. Undecidability in these cases arises because of specifications that

<sup>\*\*</sup> Part of this work was carried out while the first author was at the University of Illinois, Urbana-Champaign.

require events to happen exactly at certain time points, which can be described in the logic by having temporal operators decorated by singleton intervals (i.e., intervals containing exactly one point). If one considers the sublogic of MTL, called *Metric Interval Temporal Logic (MITL)*, which prohibits the use of singleton intervals, then both these problems are claimed to be EXPSPACE-complete [2].



(a) Model Checking LTL. Inputs are LTL formula  $\varphi$  and Büchi automaton  $\mathcal{B}$ .

(b) Model Checking MITL. Inputs are MITL formula  $\varphi$  and timed automaton  $\mathcal{T}$ .

Fig. 1. Model Checking Steps for LTL and MITL Formulas.

The decision procedures for satisfiability and model checking of MITL, follow the automata theoretic approach. In the automata theoretic approach to satisfiability or model checking, logical specifications are translated into automata such that the language recognized by the automaton is exactly the set of models of the specification. In case of LTL, this involves translating formulas to Büchi automata, and the model checking procedure is shown in Figure 1a. For MITL, formulas are translated into timed automata. Model checking timed automata against MITL specifications is schematically shown in Figure 1b. The specification  $\varphi$  is negated, a timed automaton  $[\neg \varphi]$  for  $\neg \varphi$  is constructed, and then one checks that the system represented as a timed automaton  $\mathcal{T}$  has an empty intersection with  $[\neg \varphi]$ . Since timed automata are not closed under complementation [1,9], MITL decision procedures crucially rely on transforming specifications  $\varphi$  (for the satisfaction problem) and  $\neg \varphi$  (for the model checking problem) into negated normal form, *i.e.*, one where all the negations have been pushed all the way inside to only apply to propositions. Using negated normal forms requires considering formulas with the full set of boolean operators (both  $\land$  and  $\lor$ ) and temporal operators (both  $\mathcal{U}$  and its dual  $\mathcal{R}$ ).

Unfortunately, the well known decision procedures for MITL [2] are *incorrect*. This is because we show that the semantics used for the  $\mathcal{R}$  operator, which is lifted from the semantics of  $\mathcal{R}$  in LTL, is not the dual of  $\mathcal{U}$  (see Example 5). Therefore, the timed automata constructed for the negated normal form of a formula in MITL, is not logically equivalent to the original formula. We give a new, correct semantics for  $\mathcal{R}$  (Definition 6 in Section 3). Defining the semantics

for  $\mathcal{R}$  in MTL is non-trivial because of the subtle interplay of open and closed intervals. Our definition uses 3 quantified variables (unlike 2, which is used in the semantics of  $\mathcal{U}$  in both LTL and MTL, and  $\mathcal{R}$  in LTL and the incorrect definition for MTL). We show that under fairly general syntactic conditions, one cannot use 2 quantified variables to correctly define the semantics of  $\mathcal{R}$  in MTL.

We present a new translation of MITL formulas into timed automata that uses our new semantics (Section 4). We show, using our new construction, that the complexity of the satisfiability and model checking problems remains EXPSPACEcomplete as was previous claimed [2]. MITL<sub>0, $\infty$ </sub> is a fragment of MITL that has PSPACE decision procedures for satisfiability and model checking. Our last result (Section 5) shows that MITL<sub>0, $\infty$ </sub> can be generalized. We introduce a new, richer fragment of MITL that we call MITL<sub>WI</sub>, for which we prove that satisfiability and model checking are both PSPACE-complete.

Proofs for results about MITL are subtle due to the presence of a continuous time domain and topological aspects like open and closed sets. This is evidenced by the fact that the errors we have exposed in this paper, have remained undiscovered for over 20 years, despite many researchers working on problems related to MITL. Therefore, to gain greater confidence in the correctness of our claims, we have formally proved many of our results in PVS [11]. The PVS proof objects can be downloaded from http://uofi.box.com/v/PVSProofsOfMITL.

**Related Work.** The complexity of satisfiability and model checking of MITL formulas was first considered in [2]. We show that the decision procedures are unfortunately flawed because of the use of an incorrect semantics for  $\mathcal{R}$ . A different translation of MITL to timed automata is presented in [8]. However, their setting is restricted in that all intervals are closed, and all signals are continuous from the right. Note that Example 5 and Theorem 12 in our paper, both use signals that are not continuous from right. Therefore, their algorithm does not fix the problem in [2]. Papers [4,5] present decision procedures for an event-based semantics for MITL which associates a time with every event. State-based semantics, considered here and in [2,8], is very different. For example, in a signal where p is only true in the interval [0, c], there is no time that can be ascribed to the event when p first becomes false. A recent survey of results concerning MTL and its fragments can be found in [10]. Finally, robust model checking of coFlat-MTL formulas with respect to sensor and environmental noise, is considered in [3].

### 2 Preliminaries

Sets and Functions. The set of natural, positive natural, real, positive real, and non-negative real numbers are respectively denoted by  $\mathbb{N}$ ,  $\mathbb{N}_+$ ,  $\mathbb{R}$ ,  $\mathbb{R}_+$ , and  $\mathbb{R}_{\geq 0}$ . For a set A, power set of A is denoted by  $2^A$ , Cartesian product of sets A and B is denoted by  $A \times B$ . Cardinality of A is denoted by |A|. The set of functions from A to B is denoted by  $A \to B$ . For a set A, we denote the fact that a is an element of A by the notation a : A. If A is a subset of  $\mathbb{R}$  then for any  $\epsilon : \mathbb{R}_{>0}$ , we define  $\mathsf{B}^{\epsilon}_{\infty}(A) := \{x : \mathbb{R} \mid \exists a : A \cdot |x - a| \leq \epsilon\}$  to be the  $\epsilon$ -ball around A. Finally, for any two numbers  $a, b : \mathbb{R}$ , we define  $a \doteq b$  to be  $\max\{a - b, 0\}$ .

Intervals. Every interval of real numbers is specified by a constraint of the form  $a \triangleleft_1 x \triangleleft_2 b$ , where  $a : \mathbb{R} \cup \{-\infty\}, b : \mathbb{R} \cup \{\infty\}$ , and  $\triangleleft_1, \triangleleft_2 : \{<, \leq\}$ . Also, if  $a \notin \mathbb{R}$  (or  $b \notin \mathbb{R}$ ) then we require  $\triangleleft_1 = <$  (or  $\triangleleft_2 = <$ ). We use the usual notation [a, b], (a, b), (a, b], and [a, b) to denote closed, open, left-open, or right-open intervals. The set of intervals and non-negative intervals over  $\mathbb{R}$ , are denoted by  $\mathcal{I}$  and  $\mathcal{I}_{\geq 0}$ , respectively. For any interval I, we use  $\underline{I}$  and  $\overline{I} = -\infty$ . Width of an interval, denoted by  $\|\mathbf{I}\|$ , is defined to be  $\overline{\mathbf{I}} - \underline{\mathbf{I}}$ . Thus the width of the empty interval is  $-\infty$ . Finally, an interval with only one element is called a singleton; the width of such an interval (by the above definition) is 0.

For any interval  $\mathbf{I} : \mathcal{I}$ , we use  $\langle \mathbf{I} \rangle := \overline{\mathbf{I}} \notin \mathbb{R} \vee \overline{\mathbf{I}} \in \mathbf{I}$  to check if  $\mathbf{I}$  is closed from right. Similarly, we use  ${}_{\underline{\mathbf{I}}} | \mathbf{I} \rangle$  and  ${}_{\underline{\mathbf{I}}} | \mathbf{I} \rangle$  to check if  $\mathbf{I}$  is closed/open from left. We use  $(|\mathbf{I}|) := \mathbf{I} \setminus \{\underline{\mathbf{I}}, \overline{\mathbf{I}}\}$  to denote the interval which is achieved after removing infimum and supremum of  $\mathbf{I}$  from it. We also use the following intervals:  $|\mathbf{I}\rangle := \mathbf{I} \cup \{\underline{\mathbf{I}}\};$   $|\mathbf{I}\rangle := (\mathbf{I} \cup \{\underline{\mathbf{I}}\}) \setminus \{\overline{\mathbf{I}}\};$  and  $(|\mathbf{I}| := (\mathbf{I} \cup \{\overline{\mathbf{I}}\}) \setminus \{\underline{\mathbf{I}}\}.$ 

Signal. Throughout this paper, AP is a non-empty set of atomic propositions <sup>3</sup>. Signal is any function of type  $\mathbb{R}_{\geq 0} \to 2^{AP}$ . Therefore, each signal is function that defines the set of atomic propositions that are true at each instant of time. For a signal f and time point  $r : \mathbb{R}_{\geq 0}$ , we define  $f^r : \mathbb{R}_{\geq 0} \to 2^{AP} : t \mapsto f(r+t)$  to be another signal that shifts f by r.

#### 2.1 Metric Temporal Logic

In this section, we first define the syntax of metric temporal logic (MTL) and its subclasses metric interval temporal logic (MITL), and metric temporal logic with restricted intervals (MITL<sub>0, $\infty$ </sub>). We then define the current semantics of these logics from the literature and call this the *old* semantics. Finally, we define the transformation to a negated normal form (**nnf**) and the finite variability condition (**fvar**) that are used in decision procedures for these logics.

**Definition 1** (Syntax of MTL, MITL, and  $\text{MITL}_{0,\infty}$ ). Syntax of a MTL formula is defined using the following BNF grammar, where by p and I, we mean an element of AP and  $\mathcal{I}_{\geq 0}$ .

 $\varphi ::= \top \mid \perp \mid p \mid \neg \varphi \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid \varphi \mathcal{U}_{\mathrm{I}} \varphi \mid \varphi \mathcal{R}_{\mathrm{I}} \varphi$ 

We assume  $\neg$  has the highest precedence. Syntax of a MITL formula is the same, except that singleton intervals cannot be used. Finally,  $\text{MITL}_{0,\infty}$  is the sublogic of MITL where every interval I appearing in a formula either has  $\underline{I} = 0$  or  $\overline{I} = \infty$ .

Definition 2, gives the semantics of MTL that was introduced in [2] and is commonly used in the literature. Since MITL and  $\text{MITL}_{0,\infty}$  are sublogics of MTL,

<sup>&</sup>lt;sup>3</sup> In Section 3.1 and Example 11, we require |AP| > 1.

their semantics follow from the semantics of MTL. Later in Example 5, we show that this is not the right semantics because  $\mathcal{U}$  and  $\mathcal{R}$  are *not* duals of each other. In Definition 6, we introduce a new semantics that fixes this problem. We distinguish the two semantics by putting words OLD and NEW, in gray, below the satisfaction relation  $\models$  (Definition 2 uses  $\models_{m}$  and Definition 6 uses  $\models_{m}$ ).

**Definition 2 (Semantics of MTL).** Let  $f : \mathbb{R}_{\geq 0} \to 2^{AP}$  be an arbitrary signal. For a MTL formula  $\varphi$ , satisfaction relation  $f \models_{\Box D} \varphi$  is defined using the following inductive rules:

Finally,  $f \not\models_{\Box D} \varphi$  is defined to be  $\neg(f \not\models_{\Box D} \varphi)$ .

The decision procedures for satisfiability and model checking of MITL introduced in [2], rely on translating the formulas into timed automata. Since timed languages are not closed under complementation [1], complementation cannot be handled as a first-class operation. Instead, one constructs an equivalent formula, where the negations are pushed all the way inside to only apply to propositions. We present this definition of the negation normal form (Definition 3) of a MTL formula next. The implicit assumption is that a formula is semantically equivalent to its negation normal form for certain special signals that are said to be *finitely variable*. We will define finite variability after presenting the definition of negation normal form.

**Definition 3 (Negated Normal Form).** For any MTL formula  $\varphi$ , its negated normal form, denoted by  $nnf(\varphi)$ , is a formula that is obtained by pushing all the negations inside operators. It is formally defined using the following inductive rules (p: AP is an atomic formula, and  $\varphi_1$  and  $\varphi_2$  are arbitrary MTL formulas):

 $nnf(\top) \coloneqq \top$  $nnf(\neg\top) \coloneqq \bot$  $nnf(p) \coloneqq p$  $nnf(\bot) := \bot$  $nnf(\neg \bot) := \top$  $nnf(\neg p) \coloneqq \neg p$  $nnf(\neg \neg \varphi) := nnf(\varphi)$  $\mathtt{nnf}(\varphi_1 \mathcal{U}_{\mathtt{I}} \varphi_2)$  $nnf(\varphi_1 \lor \varphi_2)$  $\coloneqq \mathtt{nnf}(\varphi_1) \lor \mathtt{nnf}(\varphi_2)$  $\coloneqq \mathtt{nnf}(\varphi_1) \mathcal{U}_{\mathtt{I}} \mathtt{nnf}(\varphi_2)$  $\mathtt{nnf}(\varphi_1 \wedge \varphi_2)$  $:= \operatorname{nnf}(\varphi_1) \wedge \operatorname{nnf}(\varphi_2)$  $nnf(\varphi_1 \mathcal{R}_I \varphi_2)$  $:= \mathtt{nnf}(\varphi_1) \mathcal{R}_{\mathtt{I}} \mathtt{nnf}(\varphi_2)$  $\mathtt{nnf}(\neg(\varphi_1 \lor \varphi_2)) \coloneqq \mathtt{nnf}(\neg\varphi_1) \land \mathtt{nnf}(\neg\varphi_2)$  $nnf(\neg(\varphi_1\mathcal{U}_{\mathrm{I}}\varphi_2)) \coloneqq nnf(\neg\varphi_1)\mathcal{R}_{\mathrm{I}}nnf(\neg\varphi_2)$  $\mathtt{nnf}(\neg(\varphi_1 \land \varphi_2)) \coloneqq \mathtt{nnf}(\neg\varphi_1) \lor \mathtt{nnf}(\neg\varphi_2)$  $nnf(\neg(\varphi_1 \mathcal{R}_{I}\varphi_2)) \coloneqq nnf(\neg\varphi_1)\mathcal{U}_{I}nnf(\neg\varphi_2)$ 

The semantics of the modal operators  $\mathcal{U}$  and  $\mathcal{R}$  are defined using quantifiers, and both of them are  $\exists \forall$  formulas. However,  $\mathcal{U}$  and  $\mathcal{R}$  are supposed to be duals of each other (see Definition 3) eventhough they are defined using formulas with the same quantifier alternation. Thus,  $\mathcal{U}$  and  $\mathcal{R}$  work as duals only for special signals that are *finitely variable* [2, 6, 10]. Intuitively, it means during any finite amount of time, number of times a signal changes its value is finite. Definition 4 formalizes this condition.

**Definition 4 (Finite Variability).** For an implicitly known satisfaction relation  $\models$ , a signal  $f : \mathbb{R}_{\geq 0} \to 2^{\mathsf{AP}}$  is said to be finitely variable from right with respect to a MTL formula  $\varphi$ , denoted by  $\mathsf{fvar}_{\mathsf{R}}(f, \varphi)$ , iff

$$\forall r : \mathbb{R}_{\geq 0} \bullet \left( \forall \epsilon : \mathbb{R}_{+} \bullet \exists t : (r, r + \epsilon) \bullet f^{t} \models \varphi \right) \Rightarrow \\ \left( \exists \epsilon : \mathbb{R}_{+} \bullet \forall t : (r, r + \epsilon) \bullet f^{t} \models \varphi \right)$$

f is said to be finitely variable from left with respect to a MTL formula  $\varphi$ , denoted by  $\texttt{fvar}_{L}(f, \varphi)$ , iff

$$\forall r : \mathbb{R}_+ \bullet \left( \forall \epsilon : (0, r) \bullet \exists t : (r - \epsilon, r) \bullet f^t \models \varphi \right) \Rightarrow \\ \left( \exists \epsilon : (0, r) \bullet \forall t : (r - \epsilon, r) \bullet f^t \models \varphi \right)$$

f is said to be finitely variable with respect to a MTL formula  $\varphi$ , denoted by  $\mathbf{fvar}(f,\varphi)$ , iff  $\mathbf{fvar}_{\mathbf{L}}(f,\varphi) \wedge \mathbf{fvar}_{\mathbf{R}}(f,\varphi)$ . f is said to be finitely variable (from left/right) iff for any MTL formula  $\varphi$ , f is finitely variable (from left/right) with respect to  $\varphi$ . Whenever we use finite variability, precise definition of  $\models$  will be clear from the context.

Finite variability as defined here (Definition 4), is formulated differently than the definition given in [6, 10]. However, the two definitions are equivalent, and we prefer the presentation given here because it makes the quantifier alternation in the definition explicit.

Definition 4 suggests that to establish finite variability of a signal, we need to consider all possible MTL formulas. However, it is known that a signal is finitely variable iff it is finitely variable over all atomic formulas; we will prove that this observation also holds for the new semantics for  $\mathcal{R}$  that we define in the next section (Lemma 8).

Every finitely variable signal can be specified using (finite or countably infinite) sequence of intervals paired with subsets of atomic propositions that are true during that interval. For example,  $([0,1], \{p\}), ((1,4), \{q\}), ([4,\infty), \{p,q\})$  specifies a signal that is  $\{p\}$  during  $[0,1], \{q\}$  during (1,4), and  $\{p,q\}$  at all other times. All our examples use this representation for (finitely variable) signals.

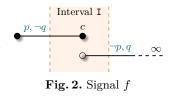
Equivalence for formulas in MTL will only be considered with respect to finitely variable signals. That is, two MTL formulas  $\varphi_1$  and  $\varphi_2$  are said to be *equivalent*, iff for any finitely variable signal f we have  $(f \models \varphi_1) \Leftrightarrow (f \models \varphi_2)$ ; here  $\models$  can either be taken to be the relation defined in Definition 2 or the one we will define in the next section (Definition 6).

## **3** Defining the Semantics of Release

The semantics of release as defined in Definition 2 does not ensure that  $\mathcal{R}$  and  $\mathcal{U}$  are duals. Example 5 describes a finite variable signal f such that  $f \not\models \mathcal{P}\mathcal{U}_{I}q$  (for

propositions p, q and interval I) and  $f \not\models \neg p \mathcal{R}_{I} \neg q$ . Thus, the transformation to negation normal form described in Definition 3 does not preserve the semantics. making decision procedures for satisfiability and model checking outlined in [2] incorrect. In this section, we identify the correct semantics of the release operator so that the transformation to negation normal form described in Definition 3 is semantically correct. Our semantics for  $\mathcal{R}$  is more complicated than the one in Definition 2, in that uses 3 quantified variables. We conclude this section by establishing that this increase in expression complexity is necessary — it is impossible to define the semantics of  $\mathcal{R}$  using a  $\exists \forall$  formula that uses only two quantified variables.

*Example 5.* Let c : ([I]) be an arbitrary point, and define f to be the signal  $([0,c], \{p\}), ((c,\infty), \{q\}).$ Clearly,  $f \not\models p\mathcal{U}_{\mathbf{I}}q$  and hence  $f \not\models \neg(p\mathcal{U}_{\mathbf{I}}q)$ . On the other hand,  $\neg q$  is not true throughout I and whenever  $\neg p$  is true,  $\neg q$  is false. Therefore,  $f \not\models \neg p \mathcal{R}_{\mathrm{I}} \neg q$ . Thus, Definition 3 does not preserve the semantics, making decision procedures for satisfiability and model check-



ing [2] of MITL that first convert a formula into negation normal form, incorrect. Since the semantics of release is incorrect (from the perspective of ensuring that  $\mathcal{U}$  and  $\mathcal{R}$  are duals), we define a new semantics for the release operator. Denseness of the time domain, along with subtleties introduced to due to open and closed endpoints of intervals, make proofs about MTL challenging to get right. Therefore, to have greater confidence in our results, we have proved most of our results in Prototype Verification Systems (PVS) [11]. We explicitly mark all lemmas and theorems that were proved in PVS<sup>4</sup>. Space limitations prevent these formal proofs to be part of this paper. However they can be downloaded from http://uofi.box.com/v/PVSProofsOfMITL.

**Definition 6** (New Semantics for MTL). Let  $f : \mathbb{R}_{>0} \to 2^{AP}$  be an arbitrary signal and  $r: \mathbb{R}_{\geq 0}$  be an arbitrary point in time. For a MTL formula  $\varphi$ , we define the satisfaction relation  $f \models \varphi$  as follows.

$$\begin{array}{ll} f & & | \\ \hline f & |$$

 $f \not\models \varphi$  is defined to be  $\neg(f \models \varphi)$ .

 $<sup>^4</sup>$  Each such result is annotated by  $\langle {\sf lemma-name} \rangle @\langle {\sf theory-name} \rangle.$  Theory name thry can be found in a file named thry.pvs.

Example 7. Consider the signal f from Example 5 that does not satisfy  $p\mathcal{U}_{\mathbf{I}}q$ . Observe that  $f \models_{\mathsf{HEV}} \neg p\mathcal{R}_{\mathbf{I}} \neg q$  by meeting the third condition for satisfying release operators under the new semantics as follows. Take  $t_1 = c$ , and  $t_2 = c + \epsilon$  such that  $[c, c + \epsilon] \subseteq \mathbf{I}$ . Now, for any  $t_3 \leq t_1$ ,  $f^{t_3} \models_{\mathsf{HEV}} \neg q$ , and for any  $t_1 < t_3 \leq t_2$ ,  $f^{t_3} \models_{\mathsf{HEV}} \neg p$ .

We will show that the new semantics (Definition 6) ensures that the transformation to negation normal form (Definition 3) preserves the semantics when considering finite variable signals. Before presenting this result (Theorem 9), we recall that a signal is finitely variable iff the truth of *every* formula in the logic changes only finitely many times within any bounded time. This is difficult to establish. Instead, in [2], it was shown that proving the finite variablity of a signal with respect to atomic propositions, guarantees its finite variability with respect to all formulas. We show that such an observation is also true for the new semantics we have defined.

**Lemma 8**  $_{\text{pvs}^5}$  (Finite Variability). Using the semantics in Definition 6, for any signal f, the following conditions hold:

$$\begin{split} & \left( \forall p : \mathtt{AP} \bullet \mathtt{fvar}_\mathtt{L}(f, p) \right) \Leftrightarrow \left( \forall \varphi : \mathtt{MTL} \bullet \mathtt{fvar}_\mathtt{L}(f, \varphi) \right) \\ & \left( \forall p : \mathtt{AP} \bullet \mathtt{fvar}_\mathtt{R}(f, p) \right) \Leftrightarrow \left( \forall \varphi : \mathtt{MTL} \bullet \mathtt{fvar}_\mathtt{R}(f, \varphi) \right) \end{split}$$

We now present the main result about the correctness of the new semantics.

**Theorem 9**  $_{\text{PVS}^6}$  (**Duality**). If a signal f is finitely variable from right then for any MTL formula  $\varphi$ ,  $f \models \varphi$  iff  $f \models \text{nnf}(\varphi)$ .

We conclude this section by introducing a new (defined) temporal operator that we will use. For any MTL formula  $\varphi$ , let  $\bigcirc \varphi$  be defined as  $\varphi \mathcal{R}_{(0,\infty)} \varphi$ . Intuitively,  $f \models \bigcirc \varphi$  iff  $\varphi$  becomes true and stays true for some positive amount of time. Proposition 10 formalizes this observation. Note that instead if  $\infty$  in definition of  $\bigcirc \varphi$ , one can use any other positive number and obtain an equivalent formula <sup>7</sup>. In writing formulas, we assume  $\bigcirc$  has higher precedence than  $\mathcal{U}$  and  $\mathcal{R}$  operators but lower precedence than  $\lor$  and  $\land$  operators.

**Proposition 10**  $_{\text{PVS}^{7}}$  (**Operator**  $\bigcirc$ ). Let  $\models$  be the satisfaction relation given in Definition 2 or Definition 6. For any signal f we have  $f \models \bigcirc \varphi$  iff  $\exists \epsilon : \mathbb{R}_{+} \bullet \forall t :$  $(0, \epsilon) \bullet f^{t} \models \varphi$ .

The correctness of our semantics (Theorem 9) was only established for signals that were finitely variable from the right. Unfortunately, our next example shows that this assumption cannot be relaxed.

*Example 11.* Let  $\varphi$  be the following formula.

 $(\bigcirc q) \land \neg (p\mathcal{U}_{(0,1)}q) \land \neg (\neg p\mathcal{U}_{(0,1)}q)$ 

<sup>&</sup>lt;sup>5</sup> atom\_finitevar\_eqv\_fml\_finitevar\_left@mtl and atom\_finitevar\_eqv\_fml\_finitevar\_right@mtl.

<sup>6</sup> sat\_eqv\_nnfsat@mtl.

<sup>&</sup>lt;sup>7</sup> next\_def\_1@mtl, next\_def\_3@mtl.

 $\varphi$  is satisfied by a signal that is finitely variable from the left as follows. Consider f to be such that q is true at all times, and p is true at times  $t = \frac{1}{2^n}$  for  $n \in \mathbb{N}$  and false at all other times. First observe that f is finitely variable from the left and  $f \models \varphi$ , no matter whether  $\models$  is given by Definition 2 or by Definition 6.

Putting  $\varphi$  into its NNF we obtain the following formula which is not satisfiable (using either Definition 2 or Definition 6).

$$(\bigcirc q) \land (\neg p\mathcal{R}_{(0,1)} \neg q) \land (p\mathcal{R}_{(0,1)} \neg q)$$

#### 3.1 Necessity of Using Three Variables

The new semantics of the release operator, given in Definition 6, is defined by quantifying over 3 time points. A natural question to ask is whether this is necessary. Is there a "simpler" definition of the semantics of the release operation? In this section, we show that this is in some sense impossible. We show that no first order definition of the semantics of release that quantifies over only two time points can be correct.

Let us fix the formula  $\varphi = \neg(p\mathcal{U}_{\mathbf{I}}q)$ , where p and q are proposition. The goal is to show that  $\neg \varphi$  cannot be expressed by a "simple"  $\exists \forall$ -formula. Let us first define what we mean by "simple" formulas. Let  $\mathcal{L}_{p,q,\mathbf{I}}$  be the collection of first order formulas of the form

$$\bigwedge_{i:\{1,\dots,n\}} \bigvee_{j:\{1,\dots,i_n\}} \exists t_1 : \mathbb{R} \cdot \forall t_2 : \mathbb{R} \cdot \phi_{i,j}(f,t_1,t_2)$$
(2)

Here  $n : \mathbb{N}$  and  $i_n : \mathbb{N}$ , and formula  $\phi_{i,j}$  is given by the BNF grammar

$$\phi ::= \neg \phi \mid \phi \lor \phi \mid \alpha_1 t_1 + \alpha_2 t_2 \bowtie \beta \mid f^{t_u} \models \psi$$

where  $\alpha_1, \alpha_2, \beta : \mathbb{R}$  are arbitrary constants,  $\bowtie: \{<, \leq, >, \geq\}$  is an arbitrary relation symbol,  $u : \{1, 2\}$ , and  $\psi : \{p, q, \neg p, \neg q\}$ . We assume  $\models$  here is either  $\models$  or  $\models$ ; it doesn't make a difference because  $\psi$  is propositional. We call constraints of the form  $\alpha_1 t_1 + \alpha_2 t_2 \bowtie \beta$  domain constraints and constraints of the form  $f^{t_u} \models \psi$  signal constraints.

Before presenting the main theorem of this section, we examine the restrictions imposed on formulas in  $\mathcal{L}_{p,q,\mathbf{I}}$ . The requirements that  $\phi \in \mathcal{L}_{p,q,\mathbf{I}}$  be in conjuctive normal form, or that there be no  $\vee$  or  $\wedge$  operations between quantifiers, or that  $\psi$  in the BNF only be  $\{p, q, \neg p, \neg q\}$  do not restrict the expressive power. Any formula not satifying these conditions can be transformed into one that does. The main restrictions are that all domain constraints are linear and that f in the signal constraints only be shifted by  $t_1$  or  $t_2$  and not by an arithmetic combination of them.

**Theorem 12.** There is no formula in  $\mathcal{L}_{p,q,\mathbf{I}}$  that is logically equivalent to  $\neg(p\mathcal{U}_{\mathbf{I}}q)$ over finite variable signals. In fact, for any  $\phi \in \mathcal{L}_{p,q,\mathbf{I}}$ , there are signals  $f_1$  and  $f_2$  in which the truth of any atomic proposition changes at most 2 times such that  $f_1 \models \neg(p\mathcal{U}_{\mathbf{I}}q), f_2 \models (p\mathcal{U}_{\mathbf{I}}q)$  but either both  $f_1, f_2$  satisfy  $\phi$  or neither does. The rest of the section is devoted to proving Theorem 12. Suppose (for contradiction)

$$\phi = \bigwedge_{i:\{1,\dots,n\}} \bigvee_{j:\{1,\dots,i_n\}} \exists t_1 : \mathbb{R} \cdot \forall t_2 : \mathbb{R} \cdot \phi_{i,j}(f,t_1,t_2)$$

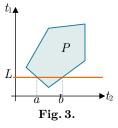
is logically equivalent to  $\neg(p\mathcal{U}_{\mathrm{I}}q)$ . We begin by observing that  $\phi$  can be assumed to be in a special canonical form. We then identify two parameters r and  $\delta$  that are used in the construction of signals that demonstrate the inequivalence of  $\phi$ and  $\neg(p\mathcal{U}_{\mathrm{I}}q)$ . Finally, we use these parameters to construct the signals and prove the inequivalence.

**Cannonical Form of**  $\phi$ **.** We can assume without loss of generality, that  $\phi$  has the following special form.

- 1. Negations are pushed all the way inside, and are only applied to p or q. This is always possible since  $\{<, \leq, >, \geq\}$  is closed under negation and  $\neg(f^t \models \psi)$  is, by definition, equivalent to  $f^t \models \neg \psi$ . Note that after this step,  $\phi_{i,j}$  may contain  $\land$  operator.
- 2. Each  $\phi_{i,j}$  is a conjunction of clauses that we denote as  $\phi_{i,j,k}$ .
- 3. Every clause in  $\phi_{i,j}$ , has at most one signal constraint of the form  $f^{t_1} \models \psi_1$ and one signal constraint of the form  $f^{t_2} \models \psi_2$  where  $\psi_1$  and  $\psi_2$  are boolean combinations of p and q.
- 4. For an arbitrary clause  $\phi_{i,j,k}$  in  $\phi_{i,j}$ , let S and P be, respectively, the set of signal and *negated* domain constraints in  $\phi_{i,j,k}$ .  $\phi_{i,j,k}$  is equivalent to  $(\bigwedge_{\theta:P} \theta) \rightarrow (\bigvee_{s:S} s)$ . The left hand side of this implication defines a 2dimensional convex polyhedron using variables  $t_1$  and  $t_2$ .

For the rest of the proof, wlog., we assume every clause in every  $\phi_{i,j}$  is of the form  $P \to S$ , where P is a polyhedron over  $t_1$  and  $t_2$ , and S is a disjunction of 0, 1, or 2 signal constraints. For any polyhedron P, we define  $\llbracket P \rrbracket \coloneqq \{(t_1, t_2) \mid P(t_1, t_2)\}$  to be the set of points in P. Also,  $cl(\llbracket P \rrbracket)$  is defined to be the closure of  $\llbracket P \rrbracket$ . Finally, let  $\mathcal{P}$  be the set of all polyhedra used in  $\phi$ .

Figure 3 shows a geometrical interpretation of the polyhedral representation of clauses in  $\phi_{i,j}$ . Let  $\phi_{i,j,k}$  be a clause that is specified by  $P \to S$ . An arbitrary horizontal line L, may or may not have intersection with P. Either way, L witnesses  $\exists t_1 \cdot \forall t_2 \cdot \phi_{i,j,k}$  iff for all points in this possibly empty intersection, S is satisfied. Every  $\phi_{i,j}$  is a set of constraints of the form  $P \to S$ . Therefore, L witnesses  $\exists t_1 \cdot \forall t_2 \cdot \phi_{i,j}$  iff L witnesses all clauses in  $\phi_{i,j}$ . Further-



more,  $\exists t_1 \bullet \forall t_2 \bullet \phi_{i,j}$  is true iff there is a horizontal line L that witnesses it.

**Identifying Parameters**  $\delta$  and r. For any  $P : \mathcal{P}$ , define  $V_P$  to be the set of vertices of P, and  $L_P$  to be the set of points on *vertical* edges of P (that is, segments of a line of the form  $t_2 = c$  for some  $c : \mathbb{R}$ ). Let  $C_1 := \bigcup_{P:\mathcal{P}} (V_P \cup L_P)$ . Define  $C_2 := \{t_2 : \mathbb{R} \mid \exists t_1 : \mathbb{R} \cdot (t_1, t_2) \in C_1\}$  be the projection of points in  $C_1$  on  $t_2$ . Take  $C_3 := C_2$  if  $\overline{\mathbf{I}} = \infty$ , and  $C_3 := C_2 \cup \{\overline{\mathbf{I}}\}$ , otherwise. Observe that  $C_3$  is always a finite set. Therefore, for some  $\epsilon : \mathbb{R}_+$ ,  $\mathbf{I} \setminus \mathsf{B}^{\epsilon}_{\infty}(C_3) \neq \emptyset$ . Fix  $r : \mathbf{I} \setminus C_3$  such that for some  $\epsilon, \mathsf{B}^{\epsilon}_{\infty}(r) \subseteq \mathbf{I} \setminus C_3$ .

For any  $P : \mathcal{P}$  and  $c : \mathbb{R}$ ,  $cl(\llbracket P \rrbracket) \cap \llbracket t_1 = c \rrbracket$  is equal to  $\{c\} \times J$ , for some (possibly empty) interval J. Define  $\|cl(\llbracket P \rrbracket) \cap \llbracket t_1 = c \rrbracket\|$  to be  $\|J\|$ . The main property we exploit about our choice of r, is that if  $(r, c) \in J$  then  $\|J\|$  is either  $\leq 0$  or "large". This is the content of our next lemma.

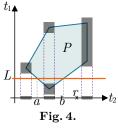
**Lemma 13.** There is a  $\delta : \mathbb{R}_+$  such that for any  $P : \mathcal{P}$  and  $c : \mathbb{R}$ , we have that if  $(c, r) \in cl(\llbracket P \rrbracket)$  then  $\|cl(\llbracket P \rrbracket) \cap \llbracket t_1 = c \rrbracket \| \notin (0, \delta]$ .

*Proof.* For the purpose of contradiction, let us assume that the lemma does not hold. Since  $\mathcal{P}$  is a finite set, we therefore have,

$$\exists P: \mathcal{P} \cdot \forall n: \mathbb{N}_+ \cdot \exists c_n: \mathbb{R} \cdot (c_n, r) \in \mathsf{cl}(\llbracket P \rrbracket) \land \|\mathsf{cl}(\llbracket P \rrbracket) \cap \llbracket t_1 = c_n \rrbracket\| \in \left(0, \frac{1}{n}\right)$$

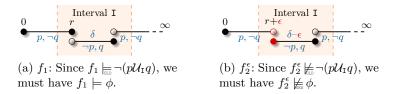
Let P be the polyhedron witnessing the violation of the lemma as in the above equation. If  $\llbracket P \rrbracket$  is an empty set, point or a line/line segment/half line that is not horzontal then its intersection with  $\llbracket t_1 = c_n \rrbracket$  is either empty or has width 0, which contradicts the fact that P violates the lemma. Otherwise, if  $\llbracket P \rrbracket$  is a horizontal line, a horizontal half line, or a horizontal line segment, its intersection with  $\llbracket t_1 = c_n \rrbracket$  is either empty or has a fixed width, which again contradicts P violating the lemma. Therefore, consider the case when P has a non-empty interior. Since P has a finite number of vertices, an infinite subsequence of  $(c_n, r)$  converges to a vertex of P. However, this is also a contradiction since our choice of r ensures that  $(c_n, r)$  is always  $\epsilon$  away from any point in  $C_1$ .

For the rest of this section, let us fix r as above, and take  $\delta$  to be such that in addition to Lemma 13, it satisfies  $\overline{1}-r > \delta$ . Figure 4 shows a geometric interpretation for the parameters  $r, \delta$  we have identified. For any clause  $P \to S$ in  $\phi$  and for any horizontal line L defined by  $t_1 = c$  (for any  $c : \mathbb{R}$ ), if  $(c, r) \in \llbracket P \rrbracket$  then we have 1. either a = b = r, or 2. if S contains a constraint of the form  $f^{t_2} \models \psi$  then then S is checked for all values of  $t_2$  in an inversal of size  $> \delta$  around r.



**Constructing Signal**  $f_1$ . Figure 5a shows the signal  $f_1$ .  $f_1$  is the signal  $([0,r], \{p, \neg q\}), ((r, r + \delta], \{\neg p, q\}), ((r + \delta, \infty), \{p, \neg q\})$ . It is easy to see that  $f_1 \models \neg (p\mathcal{U}_1q)$  (where  $\models$  is either  $\models$  or  $\models$ ); the reason is similar to Example 5. Therefore, if  $\phi$  is equivalent to  $\neg (p\mathcal{U}_1q)$ , then  $f_1$  also satisfies  $\phi$ .

**Constructing signal**  $f_{\underline{\epsilon}}^{\epsilon}$ . Since  $f_1$  satisfies  $\phi$ , there are  $c_1, c_2, \ldots c_n : \mathbb{R}$  and  $j_1, \ldots, j_n : \mathbb{N}$  such that for any  $i : \{1, \ldots, n\}$ , line  $L_i := (t_1 = c_i)$  witnesses the satisfaction of  $\exists t_1 \cdot \forall t_2 \cdot \phi_{i,j_i}$ . Consider a clause  $\phi_{i,j_i,k}$  of  $\phi_{i,j_i}$  of the form  $P_{i,j_i,k} \to S_{i,j_i,k}$ . We know that line  $L_i$  witnesses the satisfaction of this clause. Define interval  $J_{i,j_i,k}$  to be the interval given by  $[\![P_{i,j_i,k}]\!] \cap [\![L_i]\!] = \{c_i\} \times J_{i,j_i,k}$ .



**Fig. 5.** Signals  $f_1$  and  $f_2^{\epsilon}$  (interval widths are not proportional).

Choose  $\epsilon_{i,j_i,k} : \mathbb{R}_+$  to be such that either  $(r, r + \epsilon_{i,j_i,k}) \subseteq J_{i,j_i,k}$  or  $(r, r + \epsilon_{i,j_i,k}) \cap J_{i,j_i,k} = \emptyset$ . Such a choice of  $\epsilon_{i,j_i,k}$  always exists no matter what  $J_{i,j_i,k}$  is. Fix the parameter  $\epsilon$  to be

$$\epsilon \coloneqq \frac{1}{2} \min(\{\epsilon_{i,j_i,k} \mid \text{any } i, j_i, k\} \cup \{c_i - r \mid c_i > r\}).$$

Notice that our choice of  $\epsilon$  ensures that for  $i, j_i, k$ ,  $(r, r + \epsilon)$  is either contained in  $J_{i,j_i,k}$  or is disjoint from it.

Having defined  $\epsilon$ , we are ready to describe the signal  $f_2^{\epsilon}$  which is shown in Figure 5b.  $f_2^{\epsilon}$  is given as  $([0, r + \epsilon), \{p, \neg q\}), ([r + \epsilon, r + \delta], \{\neg p, q\}), ((r + \delta, \infty), \{p, \neg q\})$ . Notice  $f_1$  and  $f_2^{\epsilon}$  only differ in the interval  $(r, r + \epsilon)$ . Further,  $f_2^{\epsilon}$ satisfies  $p\mathcal{U}_1q$ .

**Deriving a Contradiction.** Let  $c_1, c_2, \ldots c_n, L_1, L_2, \ldots L_n$ , and  $j_1, \ldots j_n$ , as defined above, be the witness that demonstrates that  $f_1$  satisfies  $\phi$ . We will show that these also witness the fact that  $f_2^{\epsilon}$  satisfies  $\phi$ , giving us the desired contradiction. That is, we will show that the lines  $L_i := (t_1 = c_i)$  witness the fact that  $f_2^{\epsilon}$  satisfies  $\exists t_1 \cdot \forall t_2 \cdot \phi_{i,j_i}$ . Consider any clause  $P_{i,j_i,k} \to S_{i,j_i,k}$  of  $\phi_{i,j_i}$ .

- Suppose  $S_{i,j_i,k}$  is of the form  $f^{t_1} \models \psi$ , where  $\psi$  is a boolean combination of propositions p, q. Observe that by construction  $t_1 \notin (r, r + \epsilon)$ , and so  $f_1(t_1) = f_2^{\epsilon}(t_1)$ . Therefore, since  $f_1$  satisfies  $S_{i,j_i,k}$ , so does  $f_2^{\epsilon}$ .
- Suppose  $S_{i,j_i,k}$  is of the form  $f^{t_2} \models \psi$ . Let  $\mathbf{J}_{i,j_i,k}$  be as defined above. By our choice of  $\epsilon$ , we know that either  $(r, r+\epsilon) \cap \mathbf{J}_{i,j_i,k} = \emptyset$  or  $(r, r+\epsilon) \subseteq \mathbf{J}_{i,j_i,k}$ . In the first case, we have  $f_1(t) = f_2^{\epsilon}(t)$  for all  $t \in \mathbf{J}_{i,j_i,k}$ . Therefore,  $f_1$  satisfies  $\forall t_2 \in \mathbf{J}_{i,j_i,k} \bullet S_{i,j_i,k}$  iff  $f_2^{\epsilon}$  satisfies the same condition. Now, let us consider the more interesting case when  $(r, r+\epsilon) \subseteq \mathbf{J}_{i,j_i,k}$ . Observe that in this case  $r \in \mathsf{cl}(\llbracket \mathbf{J}_{i,j_i,k} \rrbracket)$ , and so Lemma 13 applies, and we can conclude that  $\lVert \mathsf{cl}(\llbracket \mathbf{J}_{i,j_i,k} \rrbracket) \rVert > \delta$ . This means that either there is a t < r such that  $t \in \mathbf{J}_{i,j_i,k}$  or there is a  $t > r+\delta$  such that  $t \in \mathbf{J}_{i,j_i,k}$ . Thus, for any  $t_2 \in (r, r+\epsilon)$ , there is a  $t \in \mathbf{J}_{i,j_i,k}$  such that  $f_2^{\epsilon}(t_2) = f_1(t)$ . Hence, once again we can conclude that  $L_i$  witnesses the satisfaction of  $\forall t_2 \bullet \mathbf{S}_{i,j_i,k}$  by  $f_1^{\epsilon}$  since  $f_1$  does.
- The last case to consider is when  $S_{i,j_i,k}$  is of the form  $f^{t_1} \models \psi_1 \lor f^{t_2} \models \psi_2$ . In this case also we can conclude that  $f_2^{\epsilon}$  satisfies this clause using the reasoning in the previous two cases.

# 4 Satisfiability and Model Checking MITL Formulas

The satisfiability and model checking problems for MITL are as follows. In satisfiability, given an MITL formula  $\varphi$ , one needs to determine if there is a finite variable signal f that satisfies  $\varphi$ . The model checking problem is, given a timed automaton  $\mathcal{T}$  and a MITL formula  $\varphi$ , determine if every finite variable signal produced by  $\mathcal{T}$  satisfies  $\varphi$ . Algorithms for both these problems rely on translating the MITL formula  $\varphi$  (or its negation, in the case of model checking) to a timed automaton  $\llbracket \varphi \rrbracket$  and then either checking emptiness of  $\llbracket \varphi \rrbracket$  (for satisfiability) or checking the emptiness of the intersection of two timed automata (for model checking). Since timed automata are not closed under complementation, decision procedures rely on translating a formula in NNF. As observed in Example 5, since the semantics of  $\mathcal{R}$  is incorrect, the decision procedures for satisfiability and model checking given in [2, 6, 10] are incorrect. In this section, we describe a translation of MITL to timed automata with respect to the correct semantics given in Definition 6.

The translation given in [2] from MITL in NNF to timed automata, is correct when the semantics of  $\mathcal{R}$  is taken to be as given in Definition 2. We will exploit this construction to give a translation with respect to the semantics in Definition 6. More precisely, in Definition 14, we transform an MITL formula  $\varphi$  into  $\mathsf{old}(\varphi)$  such that for any signal f, we have  $(f \models \varphi) \Leftrightarrow (f \models \mathsf{old}(\varphi))$ .

**Definition 14.** The transformation old is inductively defined as follows. In this definition,  $\varphi'_1$  and  $\varphi'_2$  are old( $\varphi_1$ ) and old( $\varphi_2$ ), respectively.

$$\begin{array}{lll} \mathsf{old}(\top) \coloneqq \top & \mathsf{old}(p) \coloneqq p & \mathsf{old}(\varphi_1 \lor \varphi_2) \coloneqq \varphi_1' \lor \varphi_2' \\ \mathsf{old}(\bot) \coloneqq \bot & \mathsf{old}(\neg \varphi) \coloneqq \neg \mathsf{old}(\varphi) & \mathsf{old}(\varphi_2 \land \varphi_2) \coloneqq \varphi_1' \land \varphi_2' \\ \mathsf{old}(\varphi_1 \mathcal{U}_{\mathrm{I}} \varphi_2) \coloneqq \varphi_1' \mathcal{U}_{\mathrm{I}} \varphi_2' \\ \mathsf{old}(\varphi_1 \mathcal{R}_{\mathrm{I}} \varphi_2) \coloneqq (\varphi_1' \mathcal{R}_{\mathrm{I}} \varphi_2') \lor (\bigcirc \varphi_1' \mathcal{R}_{\mathrm{I}} \varphi_2') & \text{if } \underline{\mathrm{I}} > 0 \\ & (\varphi_1' \mathcal{R}_{\mathrm{I}} \varphi_2') \lor (\bigcirc \varphi_1' \mathcal{R}_{\mathrm{I}} \varphi_2') \lor \bigcirc \varphi_1' & \text{if } \underline{\mathrm{I}} = 0 \land_{\S}(\mathrm{I}) \\ & (\varphi_1' \mathcal{R}_{\mathrm{I}} \varphi_2') \lor (\bigcirc \varphi_1' \mathcal{R}_{\mathrm{I}} \varphi_2') \lor (\varphi_2' \land \bigcirc \varphi_1') & \text{otherwise} \end{array}$$

The transformation old ensures that the semantics of the transformed formula  $\mathsf{old}(\varphi)$  with respect to  $\models$ , is the same as the semantics of  $\varphi$  with respect to  $\models$ .

**Lemma 15** PVS<sup>8</sup>. For any signal f and MTL formula  $\varphi$ , we have  $(f \models \varphi) \Leftrightarrow (f \models \mathfrak{old}(\varphi))$ .

*Proof (idea).* Use induction on the structure of  $\varphi$ . Special treatment for  $\underline{I}$  is needed because both Definition 2 and Definition 6 define what is called *strict* semantics, in which value of signal at time 0 is *not* important when  $\underline{I} > 0$ .  $\Box$ 

It is worth emphasizing that Definition 14 and Lemma 15 apply to any MTL formula (not just MITL), and the soundness of the transformation holds for *any* signal (and not just finite variable signals).

<sup>&</sup>lt;sup>8</sup> sat\_and\_isat@mtl.

Lemma 15 immediately gives us a procedure for transforming a negated normal formula into a timed automaton according to Definition 6. For any MITL formula  $\varphi$ , we transform  $old(nnf(\varphi))$  into a timed automaton according to [2]. Note that output of old is in NNF iff its input is <sup>9</sup>. Using Theorem 9 and Lemma 15 we know that the transformation is correct.

The main problem with this approach and Definition 14, is that  $\operatorname{old}(\varphi)$  could be exponentially larger than  $\varphi$ . So we need to address the concern that this might change the complexity of satisfiability and model checking. The complexity of the transformation in [2] for MITL and  $\operatorname{MITL}_{0,\infty}$  depends only on the number of *distinct* subformulas in  $\varphi$ , and *not* on the formula size of  $\varphi$  itself <sup>10</sup>. In Proposition 16, we show that the number of subformulas of  $\operatorname{old}(\varphi)$  is *linearly* related to the number of subformulas of  $\varphi$ . Thus using the construction in [2] for  $\operatorname{old}(\operatorname{nnf}(\varphi))$  does not change the complexity results for satisfiability and model checking <sup>11</sup>.

**Proposition 16.** For any MTL formula  $\varphi$ , we have  $|S_{\mathsf{old}}(\varphi)| \leq 6|S_{\varphi}|$ , where for any MTL formula  $\psi$ ,  $S_{\psi}$  is the set of subformulas of  $\psi$  (including  $\psi$ , itself).

The proof of Proposition 16 is deferred to the appendix in the interests of space. It is worth noting that this proposition also applies to any MTL formula and not just MITL. Using Proposition 16, we can conclude that the complexity of satisfiability and model checking remain unchanged in the new semantics.

**Corollary 17.** With respect to the semantics in Definition 6, the satisfiability and model checking problems for  $MITL_{0,\infty}$  and MITL are PSPACE-complete and EXPSPACE-complete, respectively.

# 5 MITL with Wide Intervals (MITL<sub>WI</sub>)

One important result in [2] is the identification of sublogic  $MITL_{0,\infty}$  of MITL, for which the satisfiability and model checking problems are in PSPACE, as opposed to EXPSPACE for MITL. In this section we prove that this result can be generalized. We identify a more expressive sublogic of MITL for which satisfiability and model checking are in PSPACE.

For a formula  $\varphi$  of MITL, the size of  $\varphi$  is the size of the formula, where the constants appearing in the intervals are represented in binary. Here we do not restrict constants in  $\varphi$  to be natural numbers (as in [2]), but instead allow them to be rational numbers; as is standard, we represent a rational number as a pair of binary strings encoding the numerator and denominator of the fractional representation. Define MITL<sub>WI</sub> to be the collection of MITL formulas  $\varphi$  such that

<sup>&</sup>lt;sup>9</sup> toISatNNF@mtl.

<sup>&</sup>lt;sup>10</sup> The complexity depends on the size of the DAG representation of the formula, and not its syntactic representation.

<sup>&</sup>lt;sup>11</sup> There are multiple initial transformations in [2], and each one of them can make the size of formula exponentially bigger. However, the number of distinct subformulas remains linear to the size of original formula.

every interval I appearing in  $\varphi$ , either has 1.  $\underline{I} = 0$  or 2.  $\overline{I} = \infty$  or 3.  $\frac{\overline{I}}{\overline{I} - \underline{I}} \leq n$ when  $0 < I < \overline{I} < \infty$ , where *n* is the size of  $\varphi$ .

Notice that every  $\text{MITL}_{0,\infty}$  formula is also a  $\text{MITL}_{WI}$  formula, and there are many  $\text{MITL}_{WI}$  formulas that are not  $\text{MITL}_{0,\infty}$  formulas. Thus,  $\text{MITL}_{WI}$  is a richer fragment of MITL. Condition 3 above in the definition of  $\text{MITL}_{WI}$  says that when there is an interval not conforming to the restrictions of  $\text{MITL}_{0,\infty}$ , and it has a large supremum, then the size of the interval must also be large. Thus, intervals in  $\text{MITL}_{WI}$  can be thought of as being "wide" (and hence the name). The main result of this section is the following.

**Theorem 18.** For any  $MITL_{WI}$  formula  $\varphi$  of size n, there is a timed automaton  $[\![\varphi]\!]$  satisfying the following properties.

1. For any finite variable signal f, f is in the language of  $\llbracket \varphi \rrbracket$  Iff  $f \models_{\blacksquare} \varphi$ .

2.  $\llbracket \varphi \rrbracket$  has at most  $2^{\mathcal{O}(n^2)}$  many locations and edges.

3.  $\llbracket \varphi \rrbracket$  has at most  $\mathcal{O}(n^2)$  clocks.

4.  $\llbracket \varphi \rrbracket$  has at most  $\mathcal{O}(n)$  distinct integer constants, each bounded by  $2^{\mathcal{O}(n)}$ . Furthermore,  $\llbracket \varphi \rrbracket$  can be constructed in polynomial space from  $\varphi$ .

The proof of this result will be presented over the course of this section, but it is worth noting that Theorem 18 immediately gives a PSPACE algorithm for satisfiability and model checking of  $MITL_{WI}$ .

**Corollary 19.** Model checking and satisfiability problems for  $MITL_{WI}$  is PSPACEcomplete.

*Proof.* Being in PSPACE is an immediate consequence of Theorem 18, and PSPACE-hardness follows from the PSPACE-hardness of  $MITL_{0,\infty}$ .

The rest of this section is devoted to proving Theorem 18. We begin by highlighting the crucial features of  $MITL_{0,\infty}$  that make it easier to decide than MITL (Section 5.1). In Section 5.2, we sketch the proof of Theorem 18, by drawing on the observations in Section 5.1.

#### 5.1 MITL vs. $MITL_{0,\infty}$

The algorithm (from [2]) for constructing a timed automaton for a MITL formula  $\varphi$  applies a series of syntactic transformations to  $\varphi$  such that the resulting formula 1. is in negated normal form, 2. has at most linearly more distinct subformulas, 3. has the same maximum constant as the original formula, and most importantly, 4. is in the normal form given in Definition 20. These transformations can be carried out in polynomial time, and the construction of the timed automaton assumes that the MITL formula is in the normal form given by Definition 20 below.

**Definition 20 (Normal Form** [2, **Definition 4.1]).** *MITL formula*  $\varphi$  *is said to be in* normal form *iff it is built from propositions and negated propositions using conjunction, disjunction, and temporal subformulas of the following six types:* 

1. $\langle I_{\mathbf{I}} \varphi' with_{\mathfrak{g}}(\mathbf{I}), \mathbf{I} = 0, and \mathbf{\overline{I}} \in \mathbb{R},$	4. $\varphi_1 \mathcal{U}_{\mathbb{I}} \varphi_2 \text{ with } \underline{\mathbb{I}} > 0, \text{ and } \overline{\mathbb{I}} \in \mathbb{R},$
2. $\Box_{I}\varphi'$ with $\mathfrak{g}(I)$ , $\underline{I} = 0$ , and $\overline{I} \in \mathbb{R}$ ,	5. $\varphi_1 \mathcal{R}_{\mathbf{I}} \varphi_2$ with $\underline{\mathbf{I}} > 0$ , and $\overline{\mathbf{I}} \in \mathbb{R}$ ,
$3. \ \Box \varphi'.$	6. $\varphi_1 \mathcal{U} \varphi_2$ ,

The main challenge (in terms of complexity) is in handling formulas of Type 4 and Type 5. If the formula you start with is in  $\text{MITL}_{0,\infty}$  then it can be seen that the normal form does not have any subformulas of Type 4 and Type 5. Hence, the timed automaton constructed for  $\text{MITL}_{0,\infty}$  formulas is "small", which results in PSPACE decision procedures.

To see the difficulty of transforming Type 4 and Type 5 formulas into timed automata, consider  $\Box_{(3,4)}(p \to \Diamond_{(1,2)}q)$ . Intuitively, the formula says, during the 4<sup>th</sup> unit of time, every p should be followed by a q within 1 to 2 units of time. A naïve approach, starts and dedicates a clock after seeing every p during 4<sup>th</sup> unit of time, and uses that clock to ensure that there will be at least one q, 1 to 2 units of time after the corresponding p was seen. However, this approach does not work, since there is no bound on number of p that one can expect to see during any period of time, which makes number of required clocks infinite.

Instead, the construction in [2] divides  $\mathbb{R}_{\geq 0}$  into  $[0, 1), [1, 2), \ldots$  intervals. Two important facts are central to the construction. 1. For any interval [n, n+1) and any Type 4  $\mathcal{U}$  or Type 5  $\mathcal{R}$  formula  $\varphi$ , the subset of times in [n, n+1) for which  $\varphi$  is true is exactly union of two possibly empty intervals. Using this property, for each interval [n, n + 1), we first guess those two intervals and then use at most 4 clocks to verify our guess. 2. We can start reusing a clock at most  $\overline{\mathbf{I}}$  units of time after we started using it. Therefore, total number of clocks required for checking each Type 4 and Type 5 formula is bounded by  $4\overline{\mathbf{I}}$ . Since  $\overline{\mathbf{I}}$  could be exponentially big, the resulting timed automaton could have exponentially many clocks <sup>12</sup>.

#### 5.2 Witness Points and Intervals

Let us define *step size* of an interval I as follows.

$$\mathbf{sz}(\mathbf{I}) \coloneqq \begin{cases} \overline{\mathbf{I}} - \underline{\mathbf{I}} & \text{if } \overline{\mathbf{I}} - \underline{\mathbf{I}} < \underline{\mathbf{I}} \\ \underline{\mathbf{I}} & \text{otherwise} \end{cases}$$
(3)

The crucial observation needed to prove Theorem 18 is that the truth of Type 4 and Type 5 does not change very frequently. We show that for a bounded non-empty interval I with  $\underline{I} > 0$ , using constantly many clocks, the timed automaton can monitor the truth of a formula of Type 4 or Type 5 for intervals  $[0, c), [c, 2c), \ldots$ , where c := sz(I), instead of intervals  $[0, 1), [1, 2), \ldots$  as in the construction given in [2]. This has two important consequences.

<sup>&</sup>lt;sup>12</sup> The construction in [2], keeps track of the subset of clocks that are free (*i.e.* can be reused) in the discrete modes of the timed automaton. This makes the number of locations doubly exponential. However, it is possible to reuse clocks in a queue like fashion and instead of encoding a subset of free clocks in discrete modes, one can just encode the index of the next free clock. This approach exponentially decreases number of required discrete modes. This optimization however does not change the asymptotic complexity.

- 1. If a formula is in MITL<sub>WI</sub>, then number of required clocks will be at most linear in the size of formula. For example, verifying  $\Box_{[n,2n]}\varphi$  requires constant number of clocks, as opposed to exponentially many clocks in [2].
- 2. Consider satisfiability of  $\varphi \coloneqq \Box_{[1,2]} \Diamond_{[0.01,0.02]} \varphi'$  formula. The algorithm in [2], first changes  $\varphi$  to an "equivalent" formula  $\varphi \coloneqq \Box_{[100,200]} \Diamond_{[1,2]} \varphi'$ , because if observation intervals are  $[0, 1), [1, 2), \ldots$  then all constants in the input formula must be natural numbers. Therefore, timed automaton will have hundreds of clocks. However, we show there is no need for observation intervals to have natural numbers as endpoints. This means that the timed automaton for  $\varphi$  requires at most 8 clocks for each of  $\Box_{[1,2]}$  and  $\Diamond_{[0.01,0.02]}$ sub-formulas. Note that the algorithm to check emptiness of timed automata will replace all rational numbers by natural numbers by scaling, when constructing the region graph [1]. However, in spite of this, it is worth observing that the complexity of emptiness checking of timed automata has an exponentially worse dependence on the number of clocks, than on constants [1, Lemma 4.5]. Thus, our observations may lead to better running times in practice even for  $MITL_{0,\infty}$ .

Witness Points for  $\mathcal{U}$  Operators. For the rest of this section, let us fix an arbitrary signal f. We begin by presenting some technical definitions of "witnesses" that demonstrate when an  $\mathcal{U}$ -formula is satisfied.

**Definition 21 (Witness Sets for**  $\mathcal{U}$ ). For every MTL formulas  $\varphi_1$  and  $\varphi_2$ , and  $i : \{1, 2, 3\}$ , we define witness<sup>i</sup><sub> $\mathcal{U}</sub>(\varphi_1, \varphi_2)$  to be a subset of  $\mathbb{R}^2_{\geq 0}$  defined by the</sub> following predicates over (r, w):

$$1. \ r \leq w \land \left( \forall t : (r, w) \bullet f^t \models_{\mathsf{NEW}} \varphi_1 \right) \land f^w \models_{\mathsf{NEW}} \varphi_2$$
$$2. \ r < w \land \exists \epsilon : \mathbb{R}_+ \bullet \left( \forall t : (r, w) \bullet f^t \models_{\mathsf{NEW}} \varphi_1 \right) \land \left( \forall t : (w \doteq \epsilon, w) \bullet f^t \models_{\mathsf{NEW}} \varphi_2 \right)$$
$$3. \ r < w \land \exists \epsilon : \mathbb{R}_+ \bullet \left( \forall t : (r, w + \epsilon) \bullet f^t \models_{\mathsf{NEW}} \varphi_1 \right) \land \left( \forall t : (w, w + \epsilon) \bullet f^t \models_{\mathsf{NEW}} \varphi_2 \right)$$

Notice, that if (r, w) is in any of the witness sets given in Definition 21, then it provides proof that certain until formulas are true. This is captured by the definition of proof sets, given next.

**Definition 22** (Proof Sets for  $\mathcal{U}$ ). For every MTL formulas  $\varphi_1$  and  $\varphi_2$ , times  $r, w : \mathbb{R}_{\geq 0}$ , interval  $I : \mathcal{I}_{\geq 0}$ , and  $i : \{1, 2, 3\}$ , we define  $\operatorname{proofset}^{i}_{\mathcal{U}}(\varphi_{1}, \varphi_{2}, r, w, I)$ to be a subset of  $\mathbb{R}_{\geq 0}$  defined by the following predicates over t:

- 1. witness<sup>1</sup><sub> $\mathcal{U}$ </sub> $(r, w) \land r \leq t \land w t \in \mathbf{I}$ 2. witness<sup>2</sup><sub> $\mathcal{U}$ </sub> $(r, w) \land r \leq t \land w t \in (|\mathbf{I}|$ 3. witness<sup>3</sup><sub> $\mathcal{U}$ </sub> $(r, w) \land r \leq t \land w t \in [\mathbf{I}]$

A proof set  $\operatorname{proofset}^{i}_{\mathcal{U}}(\varphi_1, \varphi_2, r, w, \mathbf{I})$  establishes the fact that  $\varphi_1 \mathcal{U}_{\mathbf{I}} \varphi_2$  is true at time r in signal f. This is proved next.

**Proposition 23**  $_{\text{PVS}^{13}}$  (**Proof Sets for**  $\mathcal{U}$ ). For any MTL formulas  $\varphi_1$  and  $\varphi_2$ , times  $r, w : \mathbb{R}_{\geq 0}$ , interval  $I : \mathcal{I}_{\geq 0}$ ,  $i : \{1, 2, 3\}$ , and  $t : \texttt{proofset}^i_{\mathcal{U}}(\varphi_1, \varphi_2, r, w, I)$  we have  $f^t \models \varphi_1 \mathcal{U}_{\mathbf{I}} \varphi_2$ .

<sup>&</sup>lt;sup>13</sup> def\_until\_witness\_{1,2,3}\_proofset@mtl\_witness.

In Proposition 23, the signal f need not be finitely variable. Also, the formulas  $\varphi_1, \varphi_2$  could be any MTL formulas. The truth of  $\varphi_1 \mathcal{U}_{\mathbf{I}} \varphi_2$  within  $[0, \mathbf{sz}(\mathbf{I}))$  changes only finitely many times. This crucial observation helps limit the number of clocks needed to monitor the truth of  $\mathcal{U}$ -subformulas.

**Theorem 24**  $_{\text{PVS}^{14}}$  (Finite Variability of  $\mathcal{U}$ ). For any MTL formulas  $\varphi_1$  and  $\varphi_2$ , and interval  $I : \{I : \mathcal{I} \mid \underline{I}, \overline{I} \in \mathbb{R}_+\}$ , there are two intervals  $T_1 : \mathcal{I}_{\geq 0}$  and  $T_2: \mathcal{I}_{\geq 0}$  with the following properties:

- $\forall t_1 : T_1, t_2 : T_2 \bullet t_1 < t_2$ , and
- $\forall t : \mathbb{R}_{>0} \bullet (t < \mathbf{sz}(\mathbf{I}) \land f^t \models \varphi_1 \mathcal{U}_{\mathbf{I}} \varphi_2) \Leftrightarrow (t \in T_1 \cup T_2)$

It is worth noting that Theorem 24 is not restricted to MITL or to finite variable signals. Since within  $[0, \mathbf{sz}(\mathbf{I}))$ , the times when  $\varphi_1 \mathcal{U}_{\mathbf{I}} \varphi_2$  is true can be partitioned into two intervals, suggests that a timed automaton checking this property can just guess these intervals. But how can such intervals be guessed? Definition 21 provides an answer. These observations are combined in the next theorem, to identify what the timed automaton needs to guess and check for  $\mathcal{U}$ -formulas.

**Theorem 25**  $_{\text{PVS}^{15}}$  (Witness Point for  $\mathcal{U}$ ). In Theorem 24, if fvar(f) then  $T_1$ and  $T_2$  have the following properties:

- If  $T_1 \neq \emptyset$  then there are  $w_1 : \mathbb{R}_+$  and  $i : \{1, 2\}$  such that:
  - 1. if i = 1 then  $w_1 T_1 \in I$ , otherwise,  $w_1 T_1 \in \langle I ]$
  - 2.  $(T_1, w_1) \in witness^i_{\mathcal{U}}(\varphi_1, \varphi_2)$
- 3.  $T_1 \subseteq \text{proofset}_{\mathcal{U}}^i(\varphi_1, \varphi_2, \underline{T_1}, w_1)$  If  $T_2 \neq \emptyset$  then there are  $w_2 : \mathbb{R}_+$  and  $i : \{1, 3\}$  such that:
  - 1.  $w_2 \underline{T_2} \in \langle \mathtt{I} ]$

  - 2.  $(\underline{T_2}, w_2) \in \texttt{witness}^i_{\mathcal{U}}(\varphi_1, \varphi_2)$ 3.  $T_2 \subseteq \texttt{proofset}^i_{\mathcal{U}}(\varphi_1, \varphi_2, \underline{T_2}, w_2)$

In Theorem 25, the 1<sup>st</sup> property bounds possible values of  $w_i$ , and hence bounds possible values that should be guessed by timed automaton. The  $2^{nd}$ property specifies what  $w_i$  should satisfy (*i.e.* what timed automaton should verify about the guess), and the  $3^{\rm rd}$  property states that  $w_i$  is enough for proving that  $\varphi_1 \mathcal{U}_{\mathbf{I}} \varphi_2$  is satisfied by f at all times in  $T_i$ .

Witness Intervals for  $\mathcal{R}$  Operators. We now identify how a timed automaton can check  $\mathcal{R}$ -formulas. We will repeat the steps from the previous section. We will identify witness intervals, and proof sets for  $\mathcal{R}$ -formulas. As in the case of  $\mathcal{U}$ , these provide proofs of when a  $\mathcal{R}$  formula is true.

**Definition 26** (Witness Interval for  $\mathcal{R}$ ). For every MTL formulas  $\varphi_1$  and  $\varphi_2$ , and  $i : \{1, \ldots, 4\}$ , we define witness<sup>*i*</sup><sub>*R*</sub>( $\varphi_1, \varphi_2$ ) to be a subset of *I* defined by the

<sup>&</sup>lt;sup>14</sup> until\_witness\_interval\_2@mtl\_witness.

<sup>&</sup>lt;sup>15</sup> until\_witness\_interval\_3@mtl\_witness

following predicates over I:

1.  $\forall t : \mathbf{I} \cdot \mathbf{f}^t \models_{\overline{\mathbf{I}} \otimes \varphi_2}$ 2.  $\mathbf{I} \neq \emptyset \land_{\mathbb{Q}}[\mathbf{I} \land \forall t : \mathbf{I} \cdot \mathbf{f}^t \models_{\overline{\mathbf{I}} \otimes \varphi_1}$ 3.  $\mathbf{I} \neq \emptyset \land \langle \mathbf{I} ]_{\ell} \land \forall t : \mathbf{I} \cdot \mathbf{f}^t \models_{\overline{\mathbf{I}} \otimes \varphi_2} \land \mathbf{f}^{\overline{\mathbf{I}}} \models_{\overline{\mathbf{I}} \otimes \varphi_1}$ 4.  $[\mathbf{I} \rangle \neq \emptyset \land \langle \mathbf{I} ]_{\ell} \land \forall t : \mathbf{I} \cdot \mathbf{f}^t \models_{\overline{\mathbf{I}} \otimes \varphi_2} \land \exists \epsilon : \mathbb{R}_+ \cdot \forall t : (\overline{\mathbf{I}}, \overline{\mathbf{I}} + \epsilon) \cdot \mathbf{f}^t \models_{\overline{\mathbf{I}} \otimes \varphi_1}$ 

**Definition 27 (Proof Sets for**  $\mathcal{R}$ ). For every MTL formulas  $\varphi_1$  and  $\varphi_2$ , intervals  $I, J : \mathcal{I}_{\geq 0}$ , and  $i : \{1, 2, 3, 4\}$ , we define  $\operatorname{proofset}^i_{\mathcal{U}}(\varphi_1, \varphi_2, I, J)$  to be a subset of  $\mathbb{R}_{>0}$  defined by the following predicates over  $t : \mathbb{R}_{>0}$ :

 $\begin{array}{l} 1. \hspace{0.2cm} \textbf{I} \in \mathtt{witness}_{\mathcal{R}}^{1}(\varphi_{1},\varphi_{2}) \wedge t + \textbf{J} \subseteq \textbf{I} \\ 2. \hspace{0.2cm} \textbf{I} \in \mathtt{witness}_{\mathcal{R}}^{2}(\varphi_{1},\varphi_{2}) \wedge t + \textbf{J} \subseteq (\underline{\textbf{I}},\infty) \hspace{0.2cm} \wedge t < \overline{\textbf{I}} \wedge \underline{\textbf{J}} > 0 \\ 3. \hspace{0.2cm} \textbf{I} \in \mathtt{witness}_{\mathcal{R}}^{3}(\varphi_{1},\varphi_{2}) \wedge t + \textbf{J} \subseteq \textbf{I} + \mathbb{R}_{\geq 0} \wedge t < \overline{\textbf{I}} \\ 4. \hspace{0.2cm} \textbf{I} \in \mathtt{witness}_{\mathcal{R}}^{4}(\varphi_{1},\varphi_{2}) \wedge t + \textbf{J} \subseteq \textbf{I} + \mathbb{R}_{\geq 0} \wedge t \leq \overline{\textbf{I}} \end{array}$ 

**Proposition 28**  $_{\text{PVS}^{16}}$  (Proof Sets for  $\mathcal{R}$ ). For any MTL formulas  $\varphi_1$  and  $\varphi_2$ , intervals  $I, J : \mathcal{I}_{\geq 0}, i : \{1, 2, 3, 4\}$ , and  $t : \text{proofset}^i_{\mathcal{R}}(\varphi_1, \varphi_2, I, J)$  we have  $f^t \models_{\mathcal{R}} \varphi_1 \mathcal{R}_J \varphi_2$ .

Like  $\mathcal{U}$ -formulas, a formula  $\varphi_1 \mathcal{R}_{I} \varphi_2$  changes its truth only finitely many times in the interval [0, sz(I)).

**Theorem 29**  $_{\text{PVS}^{17}}$  (Finite Variability of  $\mathcal{R}$ ). For any MTL formulas  $\varphi_1$  and  $\varphi_2$ , and non-empty positive bounded interval I, there are four intervals  $T_1, \ldots, T_4$  with the following properties:

 $-\forall i, j : \{1, \dots, 4\}, t_i : T_i, t_j : T_j \bullet i < j \Rightarrow t_i < t_j, \text{ and} \\ -\forall t : \mathbb{R}_{\geq 0} \bullet (t < \mathbf{sz}(\mathbb{I}) \land f^t \models_{\mathbb{H}\mathbb{H}} \varphi_1 \mathcal{R}_{\mathbb{I}} \varphi_2) \Leftrightarrow (t \in \bigcup_{i:1,\dots,4} T_i)$ 

Like in Theorem 25, we can combine Theorem 29 and Definition 26 to come up with how a timed automaton can check such  $\mathcal{R}$  formulas.

**Theorem 30**  $_{\text{PVS}^{17}}$  (Witness Interval for  $\mathcal{R}$ ). In Theorem 29, if fvar(f) then  $T_1, \ldots, T_4$  have the following properties:

- If $T_3 \neq \emptyset$ then $\exists I : \mathcal{I}$ such that:
1. $I \subseteq [sz(J), sz(J) + \overline{J})$
$2. \hspace{0.1 cm} \texttt{witness}^4_{\mathcal{R}}(arphi_1, arphi_2, \mathtt{I})$
3. $T_3 \subseteq \texttt{proofset}^4_\mathcal{R}(\varphi_1, \varphi_2, \mathtt{I}, \mathtt{J})$
- If $T_4 \neq \emptyset$ then $\exists I : \mathcal{I}, i : \{2, 3, 4\}$
such that:
1. $I \subseteq [sz(J), sz(J) + \overline{J})$
$2. \ \mathtt{witness}^i_\mathcal{R}(arphi_1,arphi_2,\mathtt{I})$
3. $T_4 \subseteq \texttt{proofset}^i_\mathcal{R}(\varphi_1, \varphi_2, \mathtt{I}, \mathtt{J})$

<sup>16</sup> def\_release\_witness\_{1,2,3,4}\_proofset@mtl\_witness.

<sup>&</sup>lt;sup>17</sup> release\_witness\_interval\_1@mtl\_witness.

Constructing a timed automaton for  $MITL_{WI}$ . One can use Theorem 25 and Theorem 30 and follow the same ideas outlined in [2] to construct a timed automaton for Type 4 and Type 5 formulas. Let us outline how this works for Type 4 formulas. If the automaton guesses that  $T_1$  is not empty then it must make this guess at time exactly  $T_1$ . At the same time, the automaton takes two more actions: First, it resets a *free* clock x and remembers that this clock is not free anymore. Second, it guesses whether i should be 1 or 2. Suppose, i is chosen to be 1. As long as x is not free, the automaton makes sure that the input signal satisfies  $\varphi_1$ . Note that this is a different proof obligation and will be considered by an induction on the structure of input formula. At the same time or at some time later, the automaton should guess whether the current time is  $\overline{T_1}$ . At any point in time, if the automaton does not make that call (*i.e.* decides the current time is not  $\overline{T_1}$ ), it means the automaton wants to prove that the input signal satisfies the  $\mathcal{U}$  formula at all points in time between  $T_1$  and sometime in the future. As soon as the automaton guesses that the current time is  $\overline{T_1}$ , it resets a new free clock y and marks it as non-free. The automaton then makes sure that when current values of x and y belong to I, the input signal satisfies  $\varphi_2$ at least once. As soon as  $\varphi_2$  becomes true during during this period, the proof obligation is over and x and y will both be marked as free clocks (note that  $\varphi_1$  does not need to be true when  $\varphi_2$  becomes true). Using Theorem 25, we know what the automaton checks, guarantees  $\forall t : T_1 \bullet f^t \models_{\text{\tiny HEV}} \varphi_1 \mathcal{U}_{\text{I}} \varphi_2$ . However, the automaton has only finitely many clocks and it cannot reuse a clock while it is not free. The significance of Theorem 24 is that it guarantees simultaneously guessing and proving at most  $\lceil \frac{\underline{I}}{sz(\underline{I})} \rceil + 1$  number of  $T_1, T_2$  intervals is enough. Since clocks x and y will be freed at most  $\overline{I}$  units of time after they became non-free, number of required clocks for each Type 4 formula will be only twice the number of simultaneous proof obligations. The same argument holds for  $\mathcal{R}$ operators, except that the automaton has to simultaneously guess and prove at most  $\lceil \frac{1}{\mathbf{sz}(\mathbf{I})} \rceil + 1$  number of  $T_1, T_2, T_3, T_4$  intervals.

#### 6 Conclusion

We proved that the classical decision procedures for satisfiability and model checking of MITL [2] are incorrect. This is because they rely on a semantics for the  $\mathcal{R}$  operator which is not the dual of  $\mathcal{U}$ . We give a new semantics of  $\mathcal{R}$  and prove that it behaves like the dual of  $\mathcal{U}$  over signals that are finitely variable. Identifying the right semantics for  $\mathcal{R}$  is subtle as we show that it is not possible to give a correct semantics using characterization that uses only two quantified variables. Using the new semantics, we give a translation from MITL to timed automata and thereby correcting the decision procedures for MITL. Finally, we also identify a fragment of MITL called MITL<sub>WI</sub>, that is more expressive than MITL<sub>0, $\infty$ </sub>, but nonetheless has decision procedures in PSPACE.

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## A Proofs

Most of our positive results have been already proven in PVS and references to those proofs have already been given in the main part of this paper. In this section, we put proof ideas for many of those results to give a flavor of what is done in those proofs. Proposition 16 is not proved in PVS, so we have its full proof here. Also, Theorem 9 is an immediate consequence of new Lemma 31 and new Lemma 32, which we sketch their proofs here and refer the reader to PVS for the full proofs.

**Lemma 8.** Using the semantics in Definition 6, for any signal f, the following conditions hold:

$$\begin{split} & \left(\forall p: \mathtt{AP} \bullet \mathtt{fvar}_\mathtt{L}(f, p)\right) \Leftrightarrow \left(\forall \varphi: \mathtt{MTL} \bullet \mathtt{fvar}_\mathtt{L}(f, \varphi)\right) \\ & \left(\forall p: \mathtt{AP} \bullet \mathtt{fvar}_\mathtt{R}(f, p)\right) \Leftrightarrow \left(\forall \varphi: \mathtt{MTL} \bullet \mathtt{fvar}_\mathtt{R}(f, \varphi)\right) \end{split}$$

*Proof (idea).* From right to left direction is trivially true. The other direction can be proved using induction on the structure of  $\varphi$ . The only interesting cases are when  $\varphi$  is in the form of  $\varphi_1 \mathcal{U}_{\mathrm{I}} \varphi_2$  or  $\varphi_1 \mathcal{R}_{\mathrm{I}} \varphi_2$ . Either way, use induction hypothesis and let  $\epsilon_1, \ldots, \epsilon_6 : \mathbb{R}_+$  be finite variability constants for  $\varphi_1$  and  $\varphi_2$ 

at times 0,  $\underline{I}$ , and  $\overline{I}$  (if supremum is finite). Let  $\epsilon : \mathbb{R}_+$  be any value that is  $< \min\{\epsilon_1, \ldots, \epsilon_6\}$  and prove the new satisfiability relation does not change its value during  $(0, \epsilon)$ .

**Lemma 31**  $_{\text{PVS}^{18}}$  (Duality-1). For any signal f that is finitely variable from right and MTL formula  $\varphi$ , we have  $f \models_{\varphi} \varphi$  implies  $f \models_{\varphi} \operatorname{nnf}(\varphi)$ .

*Proof (sketch).* Proof is by induction on the structure of  $\varphi$ . We prove the case  $\varphi \coloneqq \neg(\varphi_1 \mathcal{R}_{I} \varphi_2)$  here. We know all the following conditions are true:

 $\begin{aligned} \exists t_1 : \mathbf{I} \bullet f^{t_1} &\models \neg \varphi_2 \\ \forall t_1 : \mathbb{R}_+ \bullet (f^{t_1} \models \varphi_1) \Rightarrow \exists t_2 : [0, t_1] \cap \mathbf{I} \bullet f^{t_2} \models \neg \varphi_2 \\ \forall t_1 : [\mathbf{I}\rangle, t_2 : \mathbf{I} \cap (t_1, \infty) \bullet \exists t_3 : \mathbf{I} \bullet (t_3 \le t_1 \land f^{t_3} \models \neg \varphi_2) \lor \\ (t_1 < t_3 \le t_2 \land f^{t_3} \models \neg \varphi_1) \end{aligned}$ 

Let  $A := \{t : \mathbf{I} \mid f^t \models_{\mathbb{N}} \neg \varphi_2\}$  and  $t_1 := \inf A$ . Pick any  $t : \mathbb{R}_+$  and assume  $f^t \models_{\mathbb{N}} \varphi_1$  is true (if no such t exists then the first condition immediately give us  $f \models_{\mathbb{N}} \neg \varphi_1 \mathcal{U}_1 \neg \varphi_2$ ). Because of the second condition, there must be a  $t' : [0, t] \cap \mathbf{I}$  such that  $f^{t'} \models_{\mathbb{N}} \neg \varphi_2$ . Therefore,  $t_1 \leq t'$  and hence  $t_1 \leq t$ . This means  $\forall t_2 : (0, t_1) \cdot f^{t_2} \models_{\mathbb{N}} \neg \varphi_1$ .

- If  $(f^{t_1} \models \neg \varphi_2) \land t_1 \in I$  then  $f \models \neg \varphi_1 \mathcal{U}_{I} \neg \varphi_2$  is true.
- If  $(f^{t_1} \models \varphi_2) \wedge t_1 \in \mathbf{I}$  then fix  $t_1$  in the third condition and, knowing  $\mathbf{I} \cap (t_1, \infty) \neq \emptyset$ , let  $t_2 : \mathbf{I} \cap (t_1, \infty)$  be an arbitrary element. For some  $t_3 : \mathbf{I}$  we have  $(t_3 \leq t_1 \wedge f^{t_3} \models \neg \varphi_2) \vee (t_1 < t_3 \leq t_2 \wedge f^{t_3} \models \neg \varphi_1)$ . Since  $t_1 = \inf A$  the left disjunct is false. Therefore, we must have  $(t_1 < t_3 \leq t_2 \wedge f^{t_3} \models \neg \varphi_1)$ . Since we can make  $t_2$  arbitrary close to  $t_1$ , using  $\mathbf{fvar}_{\mathbf{R}}(f, \neg \varphi_1)$ , wlog., we assume  $t_2 : \mathbf{I} \cap (t_1, \infty)$  is such that  $\forall t_3 : (t_1, t_2] \cdot f^{t_3} \models \neg \varphi_1$ .
  - If  $f^{t_1} \models_{\overline{uv}} \neg \varphi_1$  then  $\forall t' : (0, t_2] \bullet f^{t'} \models_{\overline{uv}} \neg \varphi_1$ , and since  $t_1 = \inf A$ , we know  $\exists t : [0, t_2] \cap \mathbf{I} \bullet f^t \models_{\overline{uv}} \neg \varphi_2$ . Therefore,  $f \models_{\overline{uv}} \neg \varphi_1 \mathcal{U}_{\mathbf{I}} \neg \varphi_2$ .
  - If  $f^{t_1} \models \varphi_1$  then because of the second condition there is  $t'' : [0, t_1] \cap \mathbb{I}$  such that  $f^{t''} \models \neg \varphi_2$ . But this is contradictory to the facts  $t_1 = \inf A$  and  $f^{t_1} \models \varphi_2$ .
- If  $t_1 \notin I$  then we know  $t_1 \in [I]$ . Follow the steps of the previous case to obtain the same contradiction.

Looking at the proof of Lemma 31, we see that for temporal operators  $\varphi_1 \mathcal{U}_I \varphi_2$ and  $\varphi_1 \mathcal{R}_I \varphi_2$ , only right-side finite variability of  $\varphi_1$  is used. Therefore, as far as Lemma 31 and hence Theorem 9 are concerned, right-side finite variability of atomic propositions that are only appeared in the right hand side of temporal operators is not required.

**Lemma 32**  $_{\text{PVS}^{19}}$  (Duality-2). For any signal f and MTL formula  $\varphi$ , we have  $f \models_{\text{INF}} \text{nnf}(\varphi)$  implies  $f \models_{\text{FV}} \varphi$ .

<sup>&</sup>lt;sup>18</sup> sat\_implies\_nnfsat@mtl.

<sup>&</sup>lt;sup>19</sup> nnfsat\_implies\_sat@mtl.

*Proof (sketch).* Proof is by induction on the structure of  $\varphi$ . We prove the case  $\varphi \coloneqq \neg(\varphi_1 \mathcal{R}_{\mathrm{I}} \varphi_2)$  here. If  $\mathbf{I} = \emptyset$  then  $f \not\models_{\mathrm{Ev}} \mathrm{nnf}(\varphi)$ , therefore for the rest of this proof assume  $\mathbf{I} \neq \emptyset$ . Let  $t_1 : \mathbf{I}$  be any element that satisfies  $(f^{t_1} \models_{\mathrm{Ev}} \neg \varphi_2) \land \forall t_2 : (0, t_1) \bullet (f^{t_2} \models_{\mathrm{Ev}} \neg \varphi_1)$ . We consider every case in Definition 6 for  $f \models_{\mathrm{Ev}} \varphi_1 \mathcal{R}_{\mathrm{I}} \varphi_2$  and show they are all lead to contradiction.

- $\text{ If } \forall t: \mathtt{I} \bullet f^t \models_{\mathtt{MEV}} \varphi_2 \text{ then we have a contradiction since } t_1 \in \mathtt{I} \land (f^{t_1} \models_{\mathtt{MEV}} \neg \varphi_2).$
- If for some  $t : [\mathbf{I}\rangle$  and  $t' : \mathbf{I} \cap (t, \infty)$  we have  $\forall t'' : \mathbf{I} \cdot (t'' \leq t \to f^{t''} \models_{\mathtt{NEW}} \varphi_2) \wedge (t < t'' \leq t' \to f^{t''} \models_{\mathtt{NEW}} \varphi_1)$  then from  $t_1 \in \mathbf{I}$ ,  $f^{t_1} \models_{\mathtt{NEW}} \neg \varphi_2$ , and  $\forall t'' : \mathbf{I} \cap [0, t] \cdot f^{t''} \models_{\mathtt{NEW}} \varphi_2$  we conclude  $t < t_1$ . Knowing t < t' and by setting  $t'' \coloneqq \frac{1}{2}(t + \min\{t_1, t'\})$  we have  $t < t'' < \min\{t_1, t'\}$  and hence  $t'' \in \mathbf{I}$ . Therefore,  $f^{t''} \models_{\mathtt{NEW}} \varphi_1$  which is contrary to the assumption  $\forall t_2 : (0, t_1) \cdot (f^{t_2} \models_{\mathtt{NEW}} \neg \varphi_1)$ .

**Theorem 9.** If a signal f is finitely variable from right then for any MTL formula  $\varphi, f \models \varphi$  iff  $f \models nnf(\varphi)$ .

*Proof.* Immediate from Lemma 31 and Lemma 32.  $\hfill \Box$ 

**Proposition** 10. Let  $\models$  be the satisfaction relation given in Definition 2 or Definition 6. For any signal f we have  $f \models \bigcirc \varphi$  iff  $\exists \epsilon : \mathbb{R}_+ \bullet \forall t : (0, \epsilon) \bullet f^t \models \varphi$ .

*Proof (idea).* Apply two easy inductions on the structure of  $\varphi$  (one for Definition 2 and one for Definition 6).

**Proposition 16.** For any MTL formula  $\varphi$ , we have  $|\mathcal{S}_{\mathsf{old}}(\varphi)| \leq 6|\mathcal{S}_{\varphi}|$ , where for any MTL formula  $\psi$ ,  $\mathcal{S}_{\psi}$  is the set of subformulas of  $\psi$  (including  $\psi$ , itself).

*Proof.* During this proof,  $\varphi'$ ,  $\varphi'_1$  and  $\varphi'_2$  stand for  $old(\varphi)$ ,  $old(\varphi_1)$ , and  $old(\varphi_2)$ , respectively. Define ops as a function that maps any MTL formula  $\varphi$  to the set of MTL formulas that are operands of  $\varphi$ . Also, define height as a function that maps  $\top$ ,  $\bot$ , and every atomic proposition to 1, and maps every other MTL formula to 1 plus height of its highest operand. Let  $\mathcal{A}$  be an arbitrary finite non-empty set of MTL formulas, and define height( $\mathcal{A}$ ) := max $_{\varphi:\mathcal{A}}$  height( $\varphi$ ) to be the height of a highest formula in  $\mathcal{A}$ .

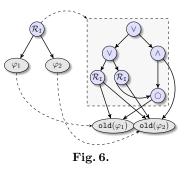
We use induction on  $\operatorname{height}(\mathcal{A})$  to prove  $|\bigcup_{\varphi:\mathcal{A}} S_{\varphi'}| \leq 6|\bigcup_{\varphi:\mathcal{A}} S_{\varphi}|$ . Base of induction is where  $\mathcal{A} \subseteq \{\top, \bot\} \cup AP$ , which is trivially true. For the inductive step, let  $\mathcal{B} \coloneqq \{\varphi: \mathcal{A} \mid \operatorname{height}(\mathcal{A}) = \operatorname{height}(\varphi)\}$  be the set of all formulas in  $\mathcal{A}$  that have the same height as  $\mathcal{A}$ . Also, let  $\mathcal{B}_1 \subseteq \mathcal{B}$  be the set of formulas in  $\mathcal{B}$  that are *not* of the form  $\varphi_1 \mathcal{R}_1 \varphi_2$ , and let  $\mathcal{B}_2 \coloneqq \mathcal{B} \setminus \mathcal{B}_1$  be everything else in  $\mathcal{B}$ . We know

$$\bigcup_{\varphi:\mathcal{A}} \mathcal{S}_{\varphi} = \bigcup_{\varphi:\mathcal{A} \setminus \mathcal{B}} \mathcal{S}_{\varphi} \cup \bigcup_{\varphi:\mathcal{B}} \mathcal{S}_{\varphi} = \overbrace{\bigcup_{\varphi:\mathcal{A} \setminus \mathcal{B}} \mathcal{S}_{\varphi} \cup \bigcup_{\psi:\mathcal{B}} \mathcal{S}_{\varphi}}^{\mathcal{C}} \cup \underset{\psi:\mathcal{B}}{\bigcup_{\varphi: \mathsf{ops}} (\psi)} \mathcal{S}_{\varphi} \cup \mathcal{B}_{1} \cup \mathcal{B}_{2}$$

$$\bigcup_{\varphi:\mathcal{A}} \mathcal{S}_{\varphi'} = \bigcup_{\varphi:\mathcal{A}\setminus\mathcal{B}} \mathcal{S}_{\varphi'} \cup \bigcup_{\varphi:\mathcal{B}} \mathcal{S}_{\varphi'} \subseteq \overbrace{\bigcup_{\varphi:\mathcal{A}\setminus\mathcal{B}} \mathcal{S}_{\varphi'} \cup \bigcup_{\psi:\mathcal{B}} \mathcal{S}_{\varphi'}}^{\mathcal{C}'} \cup \bigcup_{\psi:\mathcal{B}} \mathcal{S}_{\varphi'} \cup \bigcup_{\varphi:\mathcal{B}_{1}} \overset{\mathcal{B}'_{1}}{\bigcup_{\varphi:\mathcal{B}_{1}} \{\varphi'\}} \cup \underbrace{\mathcal{B}'_{2}}_{\varphi:\operatorname{ops}(\psi)} \underbrace{\mathcal{B}'_{2}}_{\varphi:\operatorname{ops}(\psi)} \mathcal{S}_{\varphi'} \cup \underbrace{\mathcal{B}'_{2}}_{\varphi:\mathcal{B}_{1}} \{\varphi'\}}_{\varphi:\mathcal{B}_{1}} \bigcup_{\varphi:\mathcal{B}_{1}} \{\varphi', \underbrace{\mathcal{B}'_{2}}_{\psi_{1}}, \underbrace{\mathcal{B}'_{2}}_{\psi_{2}}, \underbrace{\mathcal{B}'_{2}}, \underbrace{\mathcal{B}'_{2}}_{\psi_{2}}, \underbrace{\mathcal{B}'_{2}}, \underbrace{\mathcal{B}'_{2}}_{\psi_{2}}, \underbrace{\mathcal{B}'_{2}}, \underbrace{\mathcal{B}'_{2}}_{\psi_{2}}, \underbrace{\mathcal{B}'_{2}}_{\psi_{2}}, \underbrace{\mathcal{B}'_{2}}, \underbrace{\mathcal{B}'_{2}}_{\psi_{2}}, \underbrace{\mathcal{B}'_{2}}_{\psi_{2}}, \underbrace{\mathcal{B}'_{2}}, \underbrace{\mathcal{B}''_{2}}, \underbrace{\mathcal{B}''_{2}}, \underbrace{\mathcal{B}''_{2}}, \underbrace{\mathcal{B}''_{2}}, \underbrace{\mathcal$$

The inclusion holds, because subformulas of operands of formulas in  $\mathcal{B}$  are already considered in  $\mathcal{C}'$ . Using induction hypothesis and knowing  $\bigcup_{\varphi:\mathcal{C}} \mathcal{S}_{\varphi} = \mathcal{C}$  and  $\bigcup_{\varphi:\mathcal{C}'} \mathcal{S}_{\varphi} = \mathcal{C}', \text{ we conclude } |\mathcal{C}'| \leq 6|\mathcal{C}|. \text{ The proof is complete once we notice, } \mathcal{C},$  $\mathcal{B}_1$ , and  $\mathcal{B}_2$  are pairwise disjoint,  $|\mathcal{B}'_1| \leq |\mathcal{B}_1|$ , and  $|\mathcal{B}'_2| \leq 6|\mathcal{B}_2|$ . 

Figure 6 shows the proof idea. We look at  $\varphi$  as a directed acyclic graph, with nodes being subformulas and edges between every node and its operands. Function old changes this graph. The most interesting case is where  $\varphi$  is of the form  $\varphi_1 \mathcal{R}_I \varphi_2$ . As it is shown in this figure,  $\mathcal{R}$ nodes are replaced by at most 6 nodes. Note that  $\lor$  and  $\land$  operators are associative. Also, expanding  $\bigcirc$  to its definition, does not change number of nodes.



**Proposition 23.** For any MTL formulas  $\varphi_1$  and  $\varphi_2$ , times  $r, w : \mathbb{R}_{>0}$ , interval  $I: \mathcal{I}_{\geq 0}, i: \{1, 2, 3\}, \text{ and } t: \texttt{proofset}_{\mathcal{U}}^{i}(\varphi_{1}, \varphi_{2}, r, w, I) \text{ we have } f^{t} \models_{\mathbb{H}} \varphi_{1}\mathcal{U}_{I}\varphi_{2}.$ 

*Proof (idea).* This is a simple application of  $\mathcal{U}$  semantics. The case i = 1 holds trivially. For the other two cases, show the extra condition in Definition 22 is enough to find w' such that (r, w') is in the  $\mathcal{U}$  proof set of type 1. 

**Theorem 24.** For any MTL formulas  $\varphi_1$  and  $\varphi_2$ , and interval  $I : \{I : \mathcal{I} \mid I, \overline{I} \in I\}$  $\mathbb{R}_+$ }, there are two intervals  $T_1: \mathcal{I}_{\geq 0}$  and  $T_2: \mathcal{I}_{\geq 0}$  with the following properties:  $- \forall t_1 : T_1, t_2 : T_2 \bullet t_1 < t_2$ , and  $- \forall t : \mathbb{R}_{>_0} \bullet (t < \mathbf{sz}(\mathbf{I}) \land f^t \models \varphi_1 \mathcal{U}_{\mathbf{I}} \varphi_2) \Leftrightarrow (t \in T_1 \cup T_2)$ 

*Proof (idea).* Let  $T_1$  be the set of all points t : [0, sz(I)) for which not only  $f^t \models$  $\varphi_1 \mathcal{U}_{\mathbf{I}} \varphi_2$  is true, but also there is a  $w : \mathbf{I}$  such that  $t \in \mathtt{proofset}^1_{\mathcal{U}}(\varphi_1, \varphi_2, \underline{T_1}, w, \mathbf{I})$ . Let  $T_2$  be the set of all points  $t : [0, \mathbf{sz}(\mathbf{I}))$  for which  $f^t \models_{\mathsf{HW}} \varphi_1 \mathcal{U}_{\mathbf{I}} \varphi_2$  and  $t \notin T_1$ are both true. Use the fact that every point in  $T_1$  and  $T_2$  has a  $\mathcal{U}$  witness of type 1 and show  $T_1$  and  $T_2$  are both convex sets. Therefore, we can look at them as intervals. This proves the second condition of the theorem. To prove the first condition, first note that, by definition,  $T_1 \cap T_2 = \emptyset$ . Use the fact that witness w for any point in  $T_1$  belong to I and  $T_1 \cap T_2 = \emptyset$ , to show that every witness w for a point in  $T_2$  is not only outside of I but also non-strictly larger than supremum of I. Conclude every point in  $T_1$  is strictly smaller than all points in  $T_2$ .  **Theorem 25.** In Theorem 24, if fvar(f) then  $T_1$  and  $T_2$  have the following properties:

- If  $T_1 \neq \emptyset$  then there are  $w_1 : \mathbb{R}_+$  and  $i : \{1, 2\}$  such that:
  - 1. if i = 1 then  $w_1 \underline{T_1} \in I$ , otherwise,  $w_1 \underline{T_1} \in \langle I ]$
- 2.  $(\underline{T}_1, w_1) \in \mathtt{witness}_{\mathcal{U}}^{\overline{i}}(\varphi_1, \varphi_2)$ 3.  $T_1 \subseteq \mathtt{proofset}_{\mathcal{U}}^i(\varphi_1, \varphi_2, \underline{T}_1, w_1)$  If  $T_2 \neq \emptyset$  then there are  $w_2 : \mathbb{R}_+$  and  $i : \{1, 3\}$  such that:

  - 1.  $w_2 \underline{T_2} \in \langle I ]$ 2.  $(\underline{T_2}, w_2) \in \texttt{witness}^i_{\mathcal{U}}(\varphi_1, \varphi_2)$ 3.  $T_2 \subseteq \texttt{proofset}^i_{\mathcal{U}}(\varphi_1, \varphi_2, \underline{T_2}, w_2)$

*Proof (idea).* Continue the proof of Theorem 24 and consider the set of witnesses that can be used for points in  $T_1$  and the set of witnesses that can be used for points in  $T_2$ . Call these sets  $W_1$  and  $W_2$ . Define  $w_1 \coloneqq \overline{W_1}$  and  $w_2 \coloneqq W_2$ . If  $w_1 \in W_1$  then use  $\mathcal{U}$  witness of type 1 (*i.e.* i = 1 in the first part). Otherwise, knowing f is finitely variable from left, use  $\mathcal{U}$  witness of type 2 (*i.e.* i = 2 in the first part). Similarly, if  $w_2 \in W_2$  then use  $\mathcal{U}$  witness of type 1 (*i.e.* i = 1 in the second part). Otherwise, knowing f is finitely variable from right, use  $\mathcal{U}$  witness of type 3 (*i.e.* i = 3 in the second part). The rest is about proving the choices for i and  $w_1, w_2$  satisfy all the conditions in this theorem. 

Proofs of Proposition 28, Theorem 29, and Theorem 30 follow the exact same steps as in the corresponding results for  $\mathcal{U}$  operator. However, since semantics of  $\mathcal{R}$  operator, as defined in Definition 6, is more complex than the semantics of  $\mathcal U$  operator, proofs are more involved. For example, instead of only two intervals  $T_1$  and  $T_2$ , we needed four intervals  $T_1, \ldots, T_4$  for  $\mathcal{R}$  operator.