3D Visibility Representations of 1-planar Graphs

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Abstract. We prove that every 1-planar graph G has a z-parallel visibility representation, i.e., a 3D visibility representation in which the vertices are isothetic disjoint rectangles parallel to the xy-plane, and the edges are unobstructed z-parallel visibilities between pairs of rectangles. In addition, the constructed representation is such that there is a plane that intersects all the rectangles, and this intersection defines a bar 1visibility representation of G.

1 Introduction

Visibility representations are a classic research topic in Graph Drawing and Computational Geometry. Motivated by VLSI applications, seminal papers studied *bar visibility representations* of planar graphs (see, e.g., [22,25,26,27]), in which vertices are represented as non-overlapping horizontal segments, called *bars*, and edges correspond to vertical *visibilities* connecting pairs of bars, i.e., vertical segments that do not intersect any bar other than at their endpoints.

In order to represent non-planar graphs, more recent papers investigated models in which either two visibilities are allowed to cross, or a visibility can "go through" a vertex. Two notable examples are rectangle visibility representations and bar k-visibility representations. In a rectangle visibility representation of a graph, every vertex is represented as an axis-aligned rectangle and two vertices are connected by an edge using either a horizontal or a vertical visibility (see, e.g., [8,17,23]). A bar k-visibility representation is a bar visibility representation in which each visibility intersects at most k bars (see, e.g., [6,7,12]).

Extensions of visibility representations to 3D have also been studied. Of particular interest for us are z-parallel visibility representations (ZPRs), in which the vertices of the graph are isothetic disjoint rectangles parallel to the xy-plane, and the edges are visibilities parallel to the z-axis. Bose et al. [5] proved that K_{22} admits a ZPR, while K_{56} does not. Štola [24] reduced this gap by showing that K_{51} does not admit any ZPR. If the rectangles are restricted to unit squares, then K_7 is the largest representable complete graph [13]. Other 3D visibility models are box visibility representations [14], and 2.5D box visibility representations [1].



Fig. 1: (a) A 1-planar graph G. (b) The intersection of a 1-visible ZPR γ of G with the plane Y = 0; the red (bold) visibilities traverse a bar. (c) The projection to the yz-plane of γ (only the red visibilities are shown).

In this paper we study 3D visibility representations of 1-planar graphs. We recall that a graph is 1-planar if it can be drawn with at most one crossing per edge (see, e.g., [4,18,21]). The 1-planar graphs are among the most investigated families of "beyond planar graphs", i.e., graphs that extend planarity by forbidding specific edge crossings configurations (see, e.g., [16,20]). Brandeburg [6] and Evans et al. [12] proved that every 1-planar graph admits a bar 1-visibility representation. Later, Biedl et al. [3] proved that a 1-plane graph (i.e., an embedded 1-planar graph) admits a rectangle visibility representation if and only if it does not contain any of a set of obstructions, and that not all 1-planar graphs can be realized, regardless of their 1-planar embedding. On the other hand, every 1-planar graph can be represented with vertices that are orthogonal polygons with several reflex corners [11]. Our goal is to represent 1-planar graphs with vertices drawn as rectangles (rather than more complex polygons) by exploiting the third dimension. We prove that every 1-planar graph G has a ZPR γ . In addition, γ is 1-visible, i.e., there is a plane that is orthogonal to the rectangles of γ and such that its intersection with γ defines a bar 1-visibility representation of G (see Section 2 for formal definitions).

Our main contribution is summarized by the following theorem.

Theorem 1. Every 1-planar graph G with n vertices admits a 1-visible ZPR γ in $O(n^3)$ volume. Also, if a 1-planar embedding of G is given as part of the input, then γ can be computed in O(n) time.

An embedding is needed, as recognizing 1-planar graphs is NP-complete [15,19]. An example of a 1-visible ZPR is shown in Fig. 1. We also remark that, as pointed out by Kobourov et al. in a recent survey [18], very little is known on 3D representations of 1-planar graphs, and our result sheds some light on this problem.

From a high-level perspective, to prove Theorem 1 (see Section 3) we start by constructing a bar 1-visibility representation γ_1 of G, which is then used as the intersection of the ZPR γ with the plane Y = 0 (see, e.g., Fig. 1b). In particular, we transform each bar b of γ_1 into a rectangle R_b by computing the y-coordinates of its top and bottom sides, so that each visibility in γ_1 that traverses a bar b can be represented as a visibility in γ that passes above or below R_b (see, e.g., Fig. 1c). This is done by using two suitable acyclic orientations of the edges of G.

Some proofs and technicalities have been moved to the appendix.

2 Preliminaries and definitions

We assume familiarity with the concepts of planar drawings and planar embeddings, see, e.g., [9]. The *planarization* of a non-planar drawing is a planar drawing obtained by replacing every crossing with a *dummy vertex*. An *embedding* of a graph is an equivalence class of drawings whose planarized versions have the same planar embedding. A 1-*plane* graph is a 1-planar graph with a 1-*planar embedding*, i.e., an embedding where each edge is incident to at most one dummy vertex. A *kite* is a 1-plane graph isomorphic to K_4 in which the outer face is composed of four vertices and four crossing-free edges, while the remaining two edges cross each other. Given a 1-plane graph G and a kite $K = \{a, b, c, d\}$, with $K \subseteq G$, kite K is *empty* if it contains no vertex of G inside the 4-cycle $\langle a, b, c, d \rangle$.

A (partial) orientation \mathcal{O} of a graph G is an assignment of directions to (a subset of) the edges of G. The graph obtained by orienting the edges of Gaccording to \mathcal{O} is the directed (or mixed) graph $G_{\mathcal{O}}$. A planar st-(multi)graph Gis a plane acyclic directed (multi)graph with a single source s and a single sink t, with both s and t on its outer face [10]. The sets of incoming and outgoing edges incident to each vertex v of G are bimodal, i.e., they are contiguous in the cyclic ordering of the edges at v. Each face f of G is bounded by two directed paths with a common origin and destination, called the left path and right path of f. Face f is the left (resp., right) face for all vertices on its right (resp., left) path except for the origin and for the destination. A topological ordering of a directed acyclic (multi)graph is a linear ordering of its vertices such that for every directed edge from vertex u to vertex v, u precedes v in the ordering.

A set \mathcal{R} of disjoint rectangles in \mathbb{R}^3 is *z*-parallel, if each rectangle has its sides parallel to the *x*- and *y*-axis. Two rectangles of \mathcal{R} are *visible* if and only if they contain the ends of a closed cylinder *C* of radius $\varepsilon > 0$ parallel to the *z*-axis and orthogonal to the *xy*-plane, and that does not intersect any other rectangle.

Definition 1. A z-parallel visibility representation (ZPR) γ of a graph G maps the set of vertices of G to a z-parallel set of disjoint rectangles, such that for each edge of G the two corresponding rectangles are visible³. If there is a plane that is orthogonal to the rectangles of γ and such that its intersection with γ defines a bar k-visibility representation of G, then γ is a k-visible ZPR.

³ Our visibility model is often called *weak*, to be distinguished with the *strong* model in which visibilities and edges are in bijection. While this distinction is irrelevant when studying complete graphs (e.g., in [5,24]), the weak model is commonly adopted to represent sparse non-planar graphs in both 2D and 3D (see, e.g., [1,3,6,11,12]).

3 Proof of Theorem 1

Let G = (V, E) be a 1-plane graph with *n* vertices. To prove Theorem 1, we present a linear-time algorithm that takes *G* as input and computes a 1-visible ZPR of *G* in cubic volume. The algorithm works in three steps, described in the following.

Step 1. We compute a bar 1-visibility representation γ_1 of G by applying Brandenburg's linear-time algorithm [6], which produces a representation with integer coordinates on a grid of size $O(n^2)$. This algorithm consists of the following steps. a) A 1-plane multigraph $G' = (V, E' \supseteq E)$ is computed from G such that: The four end-vertices of each pair of crossing edges of G' induce an empty kite; no edge can be added to G' without introducing crossings; if two vertices are connected by a set of k > 1 parallel edges, then all of them are uncrossed and non-homotopic. We remark that the embedding of G' may differ from the one of G due to the rerouting of some edges. b) Let P be the plane multigraph obtained from G' by removing all pairs of crossing edges. Let \mathcal{O} be an orientation of P such that $P_{\mathcal{O}}$ is a planar st-multigraph. Then the algorithm by Tamassia and Tollis [25] is applied to compute a bar visibility representation of $P_{\mathcal{O}}$. c) Finally, all pairs of crossing edges are reinserted through a postprocessing step that extends the length of some bars so to introduce new visibilities. The newly introduced visibilities traverse at most one bar each. In addition, each bar is traversed by at most one visibility.

Step 2. We transform each bar b_v of γ_1 to a preliminary rectangle R_v . We assume that γ_1 lies on the xz-plane and that the bars are parallel to the x-axis. Let z(v) be the z-coordinate of b_v and let $x_L(v)$ and $x_R(v)$ be the x-coordinates of the left and right endpoints of b_v , respectively. The rectangle R_v lies on the plane parallel to the xy-plane with equation Z = z(v). Also, its left and right sides have x-coordinates of the top and bottom sides of R_v . We preliminarily set the y-coordinates of the top and bottom sides of R_v . We preliminarily set the y-coordinates of the bottom sides and of the top sides of all the rectangles to -1 and +1, respectively. All the visibilities of γ_1 that do not traverse any bar can be replaced with cylinders of radius $\varepsilon < \frac{1}{2}$. Let P' be the subgraph of G' induced by all such visibilities, and let γ_2 be the resulting ZPR. The next lemma follows.

Lemma 1. γ_2 is a ZPR of P'.

Step 3. To realize the remaining visibilities of γ_1 , we modify the y-coordinates of the rectangles. The idea is to define two partial orientations of the edges of P, denoted by \mathcal{O}_1 and \mathcal{O}_2 , to assign the final y-coordinates of the top sides and of the bottom sides of the rectangles, respectively. In particular, an edge oriented from u to v in \mathcal{O}_1 (\mathcal{O}_2) encodes that the top side (bottom side) of R_u will have y-coordinate greater (smaller) than the one of R_v . The orientations are such that if two vertices u and v see each other through a third vertex w in γ_1 , then their top (bottom) sides both have larger (smaller) y-coordinate than the one of w. Hence, both \mathcal{O}_1 and \mathcal{O}_2 are defined based on γ_1 , using the following three rules.



Fig. 2: (a)-(b) A right wing. (c)-(d) A left wing. (e)-(f) A diamond.

Let $f = \{o, u, v, d\}$ be a face of $P_{\mathcal{O}}$ (and hence of P) such that $\{o, u, v, d\}$ are part of an empty kite of G'. In what follows we assume that o is the origin and d is the destination of the face. We borrow some terminology from [6], refer to Fig. 2 (the black thin edges only). If the left (resp., right) path of f is composed of the single edge (o, d), then f is called a *right wing* (resp., *left wing*). If both the left path and the right path of f consist of two edges, then f is a *diamond*.

(**R.1**) If f is a right wing, we may assume that b_v is above b_u . Consider the restriction of γ_1 with respect to $\{o, u, v, d\}$. Either the visibility between b_u and b_d traverses b_v (as in Fig. 2b), or the visibility between b_o and b_v traverses b_u . In both cases we only orient edges in \mathcal{O}_1 . In the first case we orient (u, v) from u to v and (v, d) from d to v (see the green bold edges in Fig. 2a). In the second case we orient (o, u) from o to u and (u, v) from v to u. (**R.2**) If f is a left wing, we may assume that b_v is above b_u . As for a right wing, either the visibility between b_u and b_d traverses b_v (as in Fig. 2d), or the visibility between b_o and b_v traverses b_u . We orient the edges as for a right wing, but we only consider \mathcal{O}_2 (see, e.g., the blue bold edges in Fig. 2c). (**R.3**) If f is a diamond, we may assume that b_u is to the left of b_v . Either the visibility between b_o and b_d traverses b_v , or the visibility between b_o and b_d traverses b_v . In the first case we orient (o, v) from o to v and (v, d) from d to v in \mathcal{O}_1 (see the green bold edges in Fig. 2e). In the second case we orient (o, u) from o to u and (u, d) from d to v in \mathcal{O}_2 .

By applying the above three rules for all left and right wings, and for all diamonds of $P_{\mathcal{O}}$, we obtain \mathcal{O}_1 and \mathcal{O}_2 . Note that the above procedure is correct, in the sense that no edge is assigned a direction twice. This is due to the fact that a direction in \mathcal{O}_1 (resp., \mathcal{O}_2) is assigned to an edge only if it belongs to the right (resp., left) path of a right (resp., left) wing or of a diamond. On the other hand, an edge belongs only to one right path and to one left path. In what follows, we prove that both $P_{\mathcal{O}_1}$ and $P_{\mathcal{O}_2}$ are acyclic, i.e., they have no oriented cycles.

Lemma 2. Both $P_{\mathcal{O}_1}$ and $P_{\mathcal{O}_2}$ are acyclic.

Sketch of proof. We prove that $P_{\mathcal{O}_1}$ is acyclic. The argument for $P_{\mathcal{O}_2}$ is symmetric. Suppose, for a contradiction, that $P_{\mathcal{O}_1}$ contains a directed cycle $C = \langle e_1, e_2, \ldots, e_c \rangle$, as shown in Fig. 3a. First, note that c > 2. If c = 2, there are two non-homotopic parallel edges that are both part of the right path of a right wing or of a diamond in $P_{\mathcal{O}}$. But this is impossible since each pair of crossing



Fig. 3: Illustration for the proof of Lemma 2. In black (thin) we show the orientation of the edges according to \mathcal{O} , while in green (bold) according to \mathcal{O}_1 .

edges in G' forms an empty kite. Some edges of C have opposite orientations in \mathcal{O} and \mathcal{O}_1 , since \mathcal{O} is acyclic. In particular, there is at least a non-empty maximal subsequence $S = \langle e_i, e_{i+1}, \ldots, e_j \rangle$ of C with this property. We distinguish two cases, whether C is oriented clockwise or counter-clockwise in a closed walk along its boundary. Let a and b be the origin of e_i and the destination of e_j , respectively. Note that there is a directed path from b to a in $P_{\mathcal{O}}$ (and from a to b in $P_{\mathcal{O}_1}$).

Case 1. Refer to Fig. 3b. Since e_j is oriented in $P_{\mathcal{O}_1}$, it belongs to the right path of a right wing or of a diamond f of $P_{\mathcal{O}}$ by **R**.1 and **R**.3. Also, b is the origin of f, as otherwise b would have an incoming edge between e_j and e_{j+1} in counterclockwise order from e_j , which violates the bimodality of the edges around b or the fact that the source s of $P_{\mathcal{O}}$ is on the outer face. But then the orientation of e_j in \mathcal{O}_1 contradicts **R**.1 or **R**.3.

Case 2. This case can be handled similarly by observing that a is the destination of a face f having e_{i-1} in its right path.

For each maximal subsequence of the edges of $P_{\mathcal{O}_1}$ such that each edge is oriented and the induced subgraph is connected, compute a topological ordering. Concatenate all such topological orderings, and append at the beginning or at the end of the sequence possible vertices that are not incident to any oriented edge. This gives a total ordering of the vertices of $P_{\mathcal{O}_1}$, denoted by σ_1 . Set the *y*-coordinate of the top side of the rectangle representing the *i*-th vertex in σ_1 equal to n - i + 1. Apply a symmetric procedure for $P_{\mathcal{O}_2}$, by computing a total ordering σ_2 , and by setting the *y*-coordinate of the bottom side of the rectangle representing the *i*-th vertex in σ_2 equal to i - n - 1. This concludes the construction of γ (possible dummy edges inserted by the augmentation procedure of *Step 1(a)* are simply ignored in γ). The correctness of γ easily follows.

Lemma 3. γ is a 1-visible ZPR of G.

Since γ_1 takes $O(n^2)$ area, and each rectangle of γ has height at most 2n, it follows that γ takes $O(n^3)$ volume. Also, each step of the algorithm can be performed in linear time. This concludes the proof of Theorem 1.

4 Open problems

Our research suggests interesting research directions, such as: (i) The algorithm in [6] can be adjusted to compute bar 1-visibility representations of optimal 2-planar graphs [2] (i.e., 2-planar graphs with maximum density), and our construction can be also modified to obtain 1-visible ZPRs for these graphs. Does every 2-planar graph admit a 1-visible ZPR? (ii) Can we generalize our result so to prove that every graph admitting a bar 1-visibility representation also admits a 1-visible ZPR? (iii) Our algorithm computes ZPRs in which all the rectangles are intersected by the plane Y = 0. Can this plane contain all bottom sides of the rectangles? If this is not possible, we wonder if every 1-planar graph admits a 2.5D-visibility representation (i.e., vertices are axis-aligned boxes whose bottom faces lie on a same plane, and visibilities are both vertical and horizontal).

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Appendix

Lemma 2. Both $P_{\mathcal{O}_1}$ and $P_{\mathcal{O}_2}$ are acyclic.

Proof. We prove that $P_{\mathcal{O}_1}$ is acyclic. The argument for $P_{\mathcal{O}_2}$ is symmetric. Suppose, for a contradiction, that $P_{\mathcal{O}_1}$ contains a directed cycle $C = \langle e_1, e_2, \ldots, e_c \rangle$, as shown in Fig. 3a. First, note that c > 2. If c = 2, there are two non-homotopic parallel edges that are both part of the right path of a right wing or of a diamond in $P_{\mathcal{O}}$. But this is impossible since each pair of crossing edges in G' forms an empty kite. Some edges of C have opposite orientations in \mathcal{O} and \mathcal{O}_1 , since \mathcal{O} is acyclic. In particular, there is at least a non-empty maximal subsequence $S = \langle e_i, e_{i+1}, \ldots, e_j \rangle$ of C with this property. We distinguish two cases, whether C is oriented clockwise or counter-clockwise in a closed walk along its boundary. Let a and b be the origin of e_i and the destination of e_j , respectively. Note that there is a directed path from b to a in $P_{\mathcal{O}}$ (and from a to b in $P_{\mathcal{O}_1}$).

Case 1. Refer to Fig. 3b. Since e_j is oriented in $P_{\mathcal{O}_1}$, it belongs to the right path of a right wing or of a diamond f of $P_{\mathcal{O}}$ by **R.1** and **R.3**. Also, b is the origin of f, as otherwise b would have an incoming edge between e_j and e_{j+1} in counterclockwise order from e_j , which violates the bimodality of the edges around b or the fact that the source s of $P_{\mathcal{O}}$ is on the outer face (recall that e_j is directed outgoing from b in $P_{\mathcal{O}}$). But then the orientation of e_j in \mathcal{O}_1 contradicts **R.1** or **R.3**.

Case 2. Refer to Fig. 3c. Since edge e_{i-1} is oriented in $P_{\mathcal{O}_1}$, it belongs to the right path of a right wing or of a diamond f of $P_{\mathcal{O}}$ by **R**.1 and **R**.3. Also, a is the destination of f, as otherwise a would have an outgoing edge between e_i and e_{i-1} in counterclockwise order from e_i , which violates the bimodality of the edges around a or the fact that the sink t of $P_{\mathcal{O}}$ is on the outer face (recall that e_{i-1} is directed towards b in $P_{\mathcal{O}}$). But then the orientation of e_{i-1} in \mathcal{O}_1 contradicts **R**.1 or **R**.3.

Lemma 3. γ is a 1-visible ZPR of G.

Proof. By Lemma 1 we know that γ_2 realizes all the edges of G whose visibilities do not cross any bar in γ_1 . Note that the top sides of the first and of the last vertex of σ_1 receive y-coordinates n and 1, respectively. Similarly, the bottom sides of the first and of the last vertex of σ_2 receive y-coordinates -n and -1, respectively. Hence all visibilities in γ_2 are preserved in γ .

Each visibility connecting a bar b_u to a bar b_v and traversing a bar b_w in γ_1 can now be replaced with a cylinder of radius $\varepsilon < \frac{1}{2}$ and y-coordinate equal either to the one of the top side of R_w plus $\frac{1}{2}$ or to the bottom side of R_w minus $\frac{1}{2}$. In fact, the above construction ensures the top sides of R_v and R_u have y-coordinates greater than the one of R_w (by at least one unit), or that the bottom sides of R_v and R_u have y-coordinates smaller than the one of R_w . Also, there is no rectangle R_q that obstructs the visibility (u, v), as otherwise b_q would be traversed by (u, v) in γ_1 , which is not possible.

The 1-visibility of γ is obtained by construction, being γ_1 the intersection of γ with the plane Y = 0.



Fig. 4: Running example for the algorithm. In (g), we only show (in red) the visibilities that cross a bar in (d). For the sake of presentation, we chose two total orderings for $P_{\mathcal{O}_1}$ and $P_{\mathcal{O}_2}$ such that no red visibility crosses the projection of a rectangle. The two partial orderings are $\sigma_1 = \{t, a, h, f, b, e, g, s\}$ and $\sigma_2 = \{h, t, g, e, b, f, a, s\}$.