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Relational Topology

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Preface

Over the years, the authors have encountered a multitude of topics that are ultimately related to general topology and the logics of spatial reasoning. On the other hand, they have long been working on and with relational methods in fields around computer science. Finally, programming was their daily lecturing task. They became increasingly unsatisfied with the many—but slightly diverging—approaches to the topics mentioned and decided to work on a unifying presentation.

Yet another stimulus was the idea to lift concepts to a relational level making them point-free as well as quantifier-free, thus liberating them from the style of first-order predicate logic and approaching the clarity of algebraic reasoning. For this, a calculus had already been invented, since the 1970s, introducing heterogeneous relations (i.e., relations between possibly different sets). Also the important domain construction steps of forming the direct product, direct sum, or direct power had in the meantime been given birth to, characterizing them uniquely *up to isomorphism*.

Treating a topic algebraically means to work with algebraic rules that are lastly based on axioms. As we know from Euclid's axioms of geometry, an axiomatic theory may admit not just one model. As early as in the 1980s, the problem of *sharp factorization* or *unsharpness* has been raised. One may best characterize it with the statement that the concept of predicate logic is insufficient in treating relations satisfactorily since it restricts us to just one model, namely the Boolean matrix model. There exist others that seem more appropriate when—more generally—considering processes.

In recent years, this relational approach has been extended introducing the constructs of a Kronecker operator, together with a strict fork and strict join operator. Axiomatic characterizations have been developed; the tool kit of rules and formulae is beginning to stabilize, and the effectivity of computing with them increases steadily.

Given this context, it was highly welcome that several concepts of topology, such as neighborhoods, transition to the open kernel, contact relations, proximity, etc., qualify for being typical application fields to be integrated under one common relational roof. All the transitions between such concepts may be formulated by concise relation-algebraic terms or rules. Any proof necessary lends itself to being executed algebraically, and in the near future possibly with machine assistance as earlier with RALF, if not via proof systems such as Isabelle/HOL and Coq.

First steps in this direction have already been made with the relational language TITUREL. When one is about to solve topological problems computationally, one often has to be able to convert the given topology to a suitable or favorable form which means to apply some step of transition that needs to be justified. Such justifications are here given for nearly all conceivable version switches.

Quotient topologies, product topologies, as well as relative topologies on a subset are handled in this way. It turned out that in all three cases, one comes close to the sharpness effect, which makes the intended point- and quantifier-free proofs unexpectedly complicated. Only when looking at these situations in full detail, one will recognize why. The typical situation is that an algebraic reasoning is allowed only via some additionally available relation seemingly peripheral to the problem proper and not even mentioned in its statement. We consider this as a deeper insight obtained during our work on the topic.

Furthermore, a study of several approaches to spatial reasoning on discreteness, proximity, nearness, apartness, betweenness, and Aumann contacts is presented, which are frequently performed by logicians. These concepts are heavily interrelated which we exhibit expressing one by means of the respective other concept. This would have hardly been possible when not with the relational shorthand expressions. We prove that these transitions are correct. In case of apartness, we had the opportunity to identify properties which to demand seems counterproductive.

Another point to be explained is that we do not make an overly detailed use of categories. Category theory has proved to be extremely versatile in studying concepts. Here, however, we also aim at computation and/or computational proofs. It is absolutely clear that in this context category theory is hardly used in its deeper sense. We go ahead and strip off overly detailed category theory, mentioning it just to the extent that typing is clarified.

Finally, some ideas about how to work relationally on simplicial complexes are demonstrated at least in examples. This differs from the approach taken for the algebraic transitions between related topics. Here, it seems possible to work practically using the computer. We could, of course, only give a very slight idea of how this might work.

This booklet rests on decades of work with colleagues and students, to whom we owe our sincere thanks. Without all the discussions, it would not have emerged. Special thanks are due to all those working and contributing to the by now well-established "intercontinental" RAMiCS conference series (Relational and Algebraic Methods in Computer Science). Also the European COST action 274 TARSKI (Theory and Applications of Relational Structures as Knowledge Instruments) from 2001 to 2005 with its meetings all over Europe and sometimes also in Canada gave many background ideas.

Direct input and repeated discussions and contributions have always been provided by Rudolf Berghammer and Wolfram Kahl—after earlier common work on such topics.

The authors are grateful to the publisher for having included this booklet in his program. They thank in particular for the agreeable cooperation with Ute McCrory.

Their deeply felt thanks go in particular to the anonymous reviewers. The sheer number of their suggestions made us feel that they were really interested in getting the authors to improve the text.

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Neubiberg, Germany — Gunther Schmidt
St. Catharines, ON, Canada — Michael Winter
April 27, 2018

Symbols

Sets

Union and intersection are denoted as $M \cup N$ and $M \cap N$—in the same way as later for relations. The complement is $\overline{M}$, provided the ground set is tacitly given. For a one-element set, we provide $\mathbb{1}$ as standard notation. The Cartesian product of sets is $M \times N$.

Logic

For metalanguage consequence, equivalence, and definition, "$\Longrightarrow$", "$\Longleftrightarrow$", and "$:\Longleftrightarrow$" are used. Definitional equality is denoted as "$:=$". The set of Boolean truth values is $\mathbb{B} = \{\, \mathbf{0}\,,\ \mathbf{1}\,\}$. In the context of propositional logic, "$\wedge$" , "$\vee$" are used for "and" and "or," together with "$\rightarrow$" for "if… then" and "$\leftrightarrow$" for "precisely when." In the context of predicate logic, "$\exists$" and "$\forall$" denote the existential quantifier and the universal quantifier.

Relations

$R : X \longrightarrow Y$	Relation with source and target	7
$\mathbb{1}$	One-element set	10
$\mathcal{P}(X)$	Powerset of X	14
$\mathbf{2}^X$	Powerset of X, variant	14
$R \cup S$	Union	7
$R \cap S$	Intersection	7

$\varepsilon_{\mathcal{O}}$	Membership-in-open-sets topology	81
$\mathcal{H}$	Closed hull-mapping topology	89
$\mathcal{C}_D$	Closed sets diagonal topology	89
$\varepsilon_{\mathcal{C}}$	Membership-in-closed-sets topology	89
$B^{\Uparrow}$	Positively oriented boundary operator	164
$B^{\downarrow\uparrow}$	Negatively oriented boundary operator	164
$\mathfrak{B}$	Joint boundary relation	166
$S_{\downarrow\uparrow}$	Orientation flip relation	165
$\mathit{\Gamma}$	Orientation adjacency relation	164
$\mathfrak{P}$	Commutativity flip	37
$\mathfrak{T}$	Associativity shuffling	38
$\mathfrak{D}$	Distributivity shuffling	41
$\mathfrak{K}$	Kronecker-fork shuffle	43

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