On the Parallel Parameterized Complexity of the Graph Isomorphism Problem

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Abstract. In this paper, we study the parallel and the space complexity of the graph isomorphism problem (GI) for several parameterizations. Let $\mathcal{H} = \{H_1, H_2, \cdots, H_l\}$ be a finite set of graphs where $|V(H_i)| \leq d$ for all i and for some constant d. Let \mathcal{G} be an \mathcal{H} -free graph class i.e., none of the graphs $G \in \mathcal{G}$ contain any $H \in \mathcal{H}$ as an induced subgraph. We show that GI parameterized by vertex deletion distance to \mathcal{G} is in a parameterized version of AC^1 , denoted $\mathsf{Para-AC}^1$, provided the colored graph isomorphism problem for graphs in \mathcal{G} is in AC^1 . From this, we deduce that GI parameterized by the vertex deletion distance to cographs is in $\mathsf{Para-AC}^1$.

The parallel parameterized complexity of GI parameterized by the size of a feedback vertex set remains an open problem. Towards this direction we show that the graph isomorphism problem is in Para-TC⁰ when parameterized by vertex cover or by twin-cover.

Let \mathcal{G}' be a graph class such that recognizing graphs from \mathcal{G}' and the colored version of GI for \mathcal{G}' is in logspace (L). We show that GI for bounded vertex deletion distance to \mathcal{G}' is in L. From this, we obtain logspace algorithms for GI for graphs with bounded vertex deletion distance to interval graphs and graphs with bounded vertex deletion distance to cographs.

1 Introduction

Two graphs $G = (V_g, E_g)$ and $H = (V_h, E_h)$ are said to be isomorphic if there is a bijection $f: V_g \to V_h$ such that for all pairs $\{u, v\} \in {V_g \choose 2}$, $\{u, v\} \in E_g$ if and only if $\{f(u), f(v)\} \in E_h$. Given a pair of graphs as input the problem of deciding if the two graphs are isomorphic is known as the graph isomorphism problem (GI). Whether this problem has a polynomial-time algorithm is one of the outstanding open problem in the field of algorithms and complexity theory. It is in NP but very unlikely to be NP-complete as it is in NP \cap coAM [7]. Recently Babai [4] designed a quasi-polynomial time algorithm for GI improving the best previously known runtime $2^{O(\sqrt{n\log n})}$ [2,36]. However, efficient algorithms for GI have been discovered for various restricted classes of graphs e.g., planar graphs [25], bounded degree graphs [30], bounded genus graphs [32], bounded tree-width graphs [6] etc.

^{*} Supported by Tata Consultancy Services (TCS) research fellowship

For restricted classes of graphs the complexity of GI has been studied more carefully and finer complexity classifications within P have been done. Lindell [28] gave a deterministic logspace algorithm for isomorphism of trees. In the recent past, there have been many logspace algorithms for GI for restricted classes of graphs e.g., $K_{3,3}$ or K_5 minor free graphs [18], planar graphs [17], bounded tree-depth graphs [16], bounded tree-width graphs [20] etc. On the other hand parallel isomorphism algorithms have been designed for graphs with bounded eigenvalue multiplicity [3], bounded color class graphs [31] etc.

The graph isomorphism problem has been studied in the parameterized framework for several graph classes with parameters such as the tree-depth [8], the tree-distance width [35], the connected path distance width [33] and recently the tree-width which corresponds to a much larger class [29]. A more detailed list of FPT algorithms for GI in parameterized setting can be found in [9].

While there are many results on the parallel or the logspace complexity of problems in the parameterized framework [19], very little is known in this direction for GI. The parameterized analogues of classical complexity classes have also been studied in [12,22,21]. The class $\mathsf{Para-}\mathcal{C}$ is the family of parameterized problems that are in \mathcal{C} after a pre-computation on the parameter, where \mathcal{C} is a complexity class. In this paper we study the graph isomorphism problem from a parameterized space and parallel complexity perspective. Recently Chandoo [14] showed that GI for circular-arc graphs is in $\mathsf{Para-L}$ when parameterized by the cardinality of an obstacle set.

Since the graph isomorphism problem parameterized by tree-width has a logspace [20] as well as a separate FPT algorithm [29] it is natural to ask if we can design a parameterized parallel algorithm for this problem. In fact, the parallel complexity of GI parameterized by the well known but weaker parameter feedback vertex set number (FVS) is also unknown. We make some progress in this direction by showing that GI parameterized by the size of a vertex cover, which is a weaker parameter than the FVS, is parallelizable in the parameterized setting.

Let $\mathcal G$ be a graph class characterized by a finite set of forbidden induced subgraphs (see Section 3 for the formal definition). Kratsch et al. [27] gave an FPT algorithm for GI parameterized by the distance to $\mathcal G$ by taking a polynomial time colored graph isomorphism algorithm for graphs in $\mathcal G$ as a subroutine. In Section 3, we show that the result of [27] is parallelizable in the parameterized framework. More precisely, we give a Para-AC¹ algorithm for this problem. As a consequence, observe that GI parameterized by the distance to cographs is in Para-AC¹.

Using bounded search tree method we also design a parallel recognition algorithm for graphs parameterized by the distance to \mathcal{G} . One would ask if the problem is in Para-L using the same method as in [12] and [19]. However, the recent corrigendum Cai et al. [13] suggests that this may need completely new ideas.

In the above mentioned parallel analogue of the result by Kratsch et al. [27], \mathcal{G} is a class of graphs characterized by a finite set of forbidden induced subgraphs.

Instead of that if we take $\mathcal G$ to be the set of bounded tree-width graphs then the parallel parameterized complexity is again open. Note that the analogous preconditions of the theorem by Kratsch et al. [27] in this scenario is met by the logspace GI algorithm for bounded tree-width graphs by Elberfeld et al. [20]. In fact, the problem is open even when $\mathcal G$ is just the set of forests because this is the same problem: GI parameterized by feedback vertex set number. We study the graph isomorphism problem for bounded distance to any graph class $\mathcal G$ under reasonable assumptions: the colored version of GI for the class $\mathcal G$ and the recognition problem for $\mathcal G$ are in L. We give a logspace isomorphism algorithm for such classes of graphs.

In Section 4, we show that GI is in Para-TC⁰ when parameterized by the vertex cover number. By using the recognition algorithm for graphs parameterized by the vertex cover number due to [5], we first design a recognition algorithm for graphs parameterized by twin-cover number. We then prove that the graph isomorphism problem parameterized by twin-cover is in Para-TC⁰.

Parameter/Problem	Recognition	Graph Isomorphism
Vertex Cover	Para-AC ⁰ [5] Para-AC ⁰	Para-TC ⁰ [*]
Twin Cover		Para-TC ⁰ [*]
Distance to \mathcal{H} -free graphs	Para-AC ^{0↑} [*]	Para-AC ¹ [*]
Feedback Vertex set number	Open for Para-L	Open for Para-L

Table 1. Parallel/Space complexity results/status on the graph isomorphism problem parameterized by various parameters. [*] indicates results presented in this paper.

2 Preliminaries

The basic definitions and notations of standard complexity classes are from [1] and the definitions of parameterized versions of complexity classes are from [21,12,34]. A parameterized problem is pair (\mathcal{Q},k) of a language $\mathcal{Q}\subseteq \mathcal{L}^*$ and a parameterization $k:\mathcal{L}^*\to\mathbb{N}$ that maps input instances to natural numbers, their parameter values¹. The class Para- \mathcal{C} is defined to be the family of problems that are in \mathcal{C} after a precomputation on the parameter where \mathcal{C} is a complexity class.

Definition 1. [21] For a complexity class C, a parameterized problem (Q, k) belongs to the para class Para-C if there is an alphabet Π , a computable function $\pi: \mathbb{N} \to \Pi^*$ and a language $A \subseteq \Sigma^* \times \Pi^*$ with $A \in C$ such that for all $x \in \Sigma^*$ we have $x \in Q \Leftrightarrow (x, \pi(k(x))) \in A$.

If the complexity class $\mathcal C$ is L then we get the complexity class Para-L. The following equivalent definition of Para-L is convenient when designing Para-L algorithms.

¹ Often we write k in stead of k(x).

Definition 2. [21] A parameterized problem (\mathcal{Q}, k) over Σ is in Para-L if there is function $f : \mathbb{N} \to \mathbb{N}$ such that the question $x \in \mathcal{Q}$ can be decided within space $f(k) + O(\log |x|)$.

The parameterized parallel complexity classes are defined by using the basic complexity classes in place of L in above and basic gates (AND and OR gates) as follows [34]:

Para-ACⁱ (Para-TCⁱ): The class of languages that are decidable via family of circuits over basic gates (resp. together with threshold gates) with unbounded fan-in, size $O(f(k)n^{O(1)})$, and depth $O(f(k) + \log^i n)$ if i > 0 and depth O(1) if i = 0. From the definition of Para- \mathcal{C} , we know that for two complexity classes \mathcal{C} and \mathcal{C}' , $\mathcal{C} \subseteq \mathcal{C}'$ if and only if Para- $\mathcal{C} \subseteq \operatorname{Para-C'}[5]$. Hence we have the following relation between complexity classes Para-AC⁰ $\subseteq \operatorname{Para-L} \subseteq \operatorname{Para-AC^1}$. There exists a circuit class Para-AC⁰ in between Para-AC⁰ and Para-AC¹ which is strictly more powerful than Para-AC⁰. The definition of Para-AC⁰ is as follows.

Definition 3. [5] Para-AC^{0↑} is a class of languages that are decidable via family of circuits over basic gates with unbounded fan-in, size $O(f(k)n^{O(1)})$, and depth g(k) where f and g are computable functions.

The depth of the circuits in this class is bounded by a function that depends only on the parameter. We have $Para-AC^0 \subseteq Para-AC^{0\uparrow} \subseteq Para-AC^1$ and $Para-AC^0 \subseteq Para-AC^1 = Para-AC^0$ and $Para-AC^0 = Para-AC^0$ and $Para-AC^0$ and $Para-AC^0$ and $Para-AC^0$ and $Para-AC^0$ a

In this paper, the graphs we consider are undirected and simple. For a graph G = (V, E), let V(G) and E(G) denote the vertex set and edge set of G respectively. An edge $\{u, v\} \in E(G)$ is denoted as uv for simplicity. For a subset $S \subseteq V(G)$, the graph G[S] denotes the subgraph of G induced by the vertices of G. We use notation $G \setminus S$ to refer the graph obtained from G after removing the vertex set G. For a vertex G0, G1, G2, G3, G3, G4, G4, G5, G5, G5, G6, G7, G8, G9, G9

In this paper we study problems similar to the graph modification problems where given a graph G, and a graph class \mathcal{G} the task is to apply some graph operations (such as vertex or edge deletions) on G to get a graph in \mathcal{G} . For example, if \mathcal{G} is the class of edgeless graphs then the number of vertices to be deleted from graph G to make it edgeless is the vertex cover problem. For a graph class \mathcal{G} , the distance to \mathcal{G} of a graph G is the minimum number of vertices to be deleted from G to get a graph in \mathcal{G} . For a positive integer k, we use $\mathcal{G} + kv$ to denote the family of graphs such that each graph in this family can be made into a graph in \mathcal{G} by removing at most k vertices.

Cographs are P_4 -free graphs i.e., they do not contain any induced paths on four vertices. Interval graphs are the intersection graphs of a family of intervals on the real line. A graph is a threshold graph if it can be constructed recursively by adding an isolated vertex or a universal vertex.

The parameterized Vertex Cover problem has input a graph G and a positive integer k. The problem is to decide the existence of a vertex set $X \subseteq$

V(G) of size at most k such that for every edge $uv \in E(G)$, either $u \in X$ or $v \in X$. A minimal vertex cover of a graph is a vertex cover that does not contain another vertex cover.

Definition 4. Let G be a graph. The set $X \subseteq V(G)$ is said to be twin-cover of G if for every edge $uv \in E(G)$ either (a) $u \in X$ or $v \in X$, or (b) u and v are twins².

An edge between a pair of twins is called a *twin edge*. The graph G' obtained by removing a twin-cover from G is a disjoint collection of cliques [23].

A kernel for a parameterized problem Q is an algorithm which transforms an instance (I, k) of Q to an equivalent instance (I', k') in polynomial time such that $k' \leq k$ and $|I'| \leq f(k)$ for some computable function f. For more details on parameterized complexity see [19].

In this paper, a coloring of a graph is just a mapping of the vertices of a graph to a set of colors, and it need not be proper.

Definition 5. The colored graph isomorphism problem is to decide the existence of a color preserving isomorphism between a pair of colored graphs G = (V, E) and G' = (V', E'), i.e., there exists a bijection mapping $\varphi : V \to V'$, satisfying the following conditions: 1) $(u, v) \in E \Leftrightarrow (\varphi(u), \varphi(v)) \in E'$ for all $u, v \in V$ 2) $color(v) = color(\varphi(v))$ for all $v \in V$.

3 GI for Distance to a Graph Class is in Para-AC¹

In this Section, first we give a generic method to solve GI for graphs from $\mathcal{G} + kv$ in Para-L provided there is a logspace colored GI algorithm for graphs in \mathcal{G} and a Para-L algorithm for enumerating vertex deletion sets.

Theorem 1. Let \mathcal{G} be a any graph class. Suppose enumerating all the vertex deletion sets of $\mathcal{G} + kv$ is in Para-L and the colored graph isomorphism problem for graphs from \mathcal{G} is in L. Then the graph isomorphism problem for graphs from $\mathcal{G} + kv$ is in Para-L.

Proof. Let $\mathcal{A}_{\mathcal{I}}$ be a logspace algorithm to check whether two given input colored graphs G_1 and G_2 from \mathcal{G} are isomorphic. We assume that graphs G_1 and G_2 are at a distance at most k from \mathcal{G} . If G_1 and G_2 belong to \mathcal{G} then use the algorithm $\mathcal{A}_{\mathcal{I}}$ to check the isomorphism between G_1 and G_2 . Otherwise we consider a vertex deletion set $S \subseteq V(G_1)$ of minimum size (say s) such that $G_1 \setminus S \in \mathcal{G}$ and all possible vertex deletion sets S_1, S_2, \dots, S_m of size s for G_2 such that $G_2 \setminus S_i \in \mathcal{G}$ for all $i \in [m]$ given as input. Notice that $m \leq f(k)$.

For each $i \in [m]$, the algorithm iterates over all possible isomorphisms between G[S] to $G[S_i]$, and tries to extend them to isomorphisms from G_1 to G_2 with the help of the colored graph isomorphism algorithm $\mathcal{A}_{\mathcal{I}}$ applied on some colored versions of $G_1 \setminus S$ and $G_2 \setminus S_i$, where the colors of the vertices are determined by their neighbors in the corresponding deletion set. A crucial observation

Two vertices u and v are twins if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$.

is that any bijective mapping from S to S_i can be viewed as a string in $[s]^s$ and can be encoded as a string of length $O(k \log k)$. The string is $x_1 \cdots x_s$ encodes the map that sends the *i*th vertex in S to the x_i th vertex in S_i .

For all i algorithm iterates over all s! bijective mappings from S to S_i using string of length $O(s \log s)$. Next it checks whether the bijective mapping is actually an isomorphism from $G_1[S]$ to $G_2[S_i]$. For each isomorphism φ from S to S_i , we need to check whether this isomorphism can be extended to an isomorphism from $G_1 \setminus S$ to $G_2 \setminus S_i$ by using algorithm $\mathcal{A}_{\mathcal{I}}$. We color the vertices of $G_1 \setminus S$ according to their neighbourhood in S. Two vertices of $G_1 \setminus S$ get same color if they have the same neighbourhood in S. A vertex u in $G_1 \setminus S$ and a vertex v in $G_2 \setminus S_i$ will get same color if $\varphi(N(u) \cap S) = N(v) \cap S_i$. We query algorithm $\mathcal{A}_{\mathcal{I}}$ with input the graphs $G_1 \setminus S$ and $G_2 \setminus S_i$ colored as above. If the algorithm $\mathcal{A}_{\mathcal{I}}$ says 'yes' then $G_1 \cong G_2$ and the algorithm accepts the input. Otherwise it tries the next isomorphism from S to S_i . If for all i and all isomorphisms from S to S_i , the algorithm $\mathcal{A}_{\mathcal{I}}$ rejects then the we conclude that $G_1 \ncong G_2$ and the algorithm rejects the input.

We note few more details of the algorithm to demonstrate that it uses small space. The enumeration over the S_i 's can be done using a $\log m$ bit counter. To check if two vertices u in $G_1 \setminus S$ and v in $G_2 \setminus S_2$ have same color in logspace we can inspect each vertex in G_1 , find out if it in S, find out if it is a neighbour of u, and check if its image under φ is a neighbour of v. This needs constantly many counters.

Next we give a Para-AC^{0↑} recognition algorithm for graphs parameterized by the distance to a graph class \mathcal{G} by using the bounded search tree technique, where \mathcal{G} is characterized by finitely many forbidden induced subgraphs.

Definition 6. [27] A class \mathcal{G} of graphs is characterized by finitely many forbidden induced subgraphs if there is a finite set of graphs $\mathcal{H} = \{H_1, H_2, \dots, H_l\}$ such that a graph G is in \mathcal{G} if and only if G does not contain H_i as an induced subgraph for any $i \in \{1, 2, \dots, l\}$.

Let \mathcal{G} and \mathcal{H} be classes as defined above. We use the bounded search tree technique [19,11] to find a set S of size at most k such that $G \setminus S \in \mathcal{G}$. In this method we can compute all deletion sets of size at most k. Let d be the size of the largest forbidden induced subgraph in \mathcal{H} . The algorithm constructs a tree T as follows. The root of the tree is labelled with the empty set. It finds a forbidden induced subgraph $H_i \in \mathcal{H}$ of size at most d in G. Any vertex deletion set S must contain a vertex of H_i . We add $|V(H_i)|$ many children to the root labelled with vertices of H_i . In general if a node is labelled with a set P, then we find a forbidden induced subgraph H_j in $G \setminus P$ and create $|V(H_j)|$ many children for the node labeled P and label each child with $P \cup \{v_i\}$, where $v_i \in H_j$. If there exists a node labeled with a set S in T of size at most k such that $G \setminus S \in \mathcal{G}$, then S is a required vertex deletion set. From this, we also know that there are at most d^k minimal vertex deletion sets of size at most k. Using the same process we can also find all the minimal vertex deletion sets of size at most S is a required vertex deletion sets of size at most S is a same process we can also find all the minimal vertex deletion sets of size at most S is a find all the minimal vertex deletion sets of size at most S is a most S.

Cai et al. [12] implemented bounded search tree method and kernelization to find the vertex cover in Para-L in 1997. However, the implementation of bounded

search tree method in Para-L was reported to have some errors [13]. Thus, this paper seems to give the first implementation of bounded search tree method in Para-AC $^{0\uparrow}$. Let us recall form Section 2, that there is no known relation between Para-AC $^{0\uparrow}$ and Para-L.

Lemma 1. Let \mathcal{G} be a class of graphs characterized by finitely many forbidden induced subgraphs $\mathcal{H} = \{H_1, H_2, \cdots, H_l\}$ with $|V(H_i)| \leq d$ for all $1 \leq i \leq l$ where d is a constant. On input a graph G, the problem of computing all vertex deletion sets of size at most k is in Para- $AC^{0\uparrow}$ where k is the parameter.

Proof. The idea to implement the bounded search tree method in Para-AC $^{0\uparrow}$ is as follows:

Consider the set of all subsets of size at most d that induce a forbidden subgraph in G. We order these subsets lexicographically to obtain a list $\mathcal{L} =$ A_1, \dots, A_m where for each i, $G[A_i]$ is isomorphic to some graph in \mathcal{H} . Notice that $m = O(n^d)$. The list \mathcal{L} can be computed in Para-AC^{0↑} by first producing all subsets of V(G) of size at most d and then keeping only those that induce a subgraphs isomorphic to some H in \mathcal{H} . Observe that any vertex deletion set must contain at least one vertex from each A_i for all i. The algorithm uses all strings $\Gamma = \gamma_1 \cdots \gamma_k \in [d]^k$ in parallel to pick the vertex deletion sets S of size at most k as follows: Let us concentrate on the part of the circuit that processes a particular string $\Gamma = \gamma_1 \cdots \gamma_k$. Initially the deletion set S is empty. The algorithm puts the γ_i th vertex (in lexicographic order) of A_1 in S if $|A_1| \geq \gamma_1$. If $|A_1| < \gamma_1$ the computation ends in this part of the circuit. Suppose the algorithm has already picked i vertices using $\gamma_1 \cdots \gamma_i$. It picks the (i+1)th vertex using γ_{i+1} . To do so it picks the first set A_j in the list \mathcal{L} such that $A_j \cap S = \phi$ (if $A_j \cap S \neq \phi$ we say that A_i is 'hit' by S). Then it puts the γ_{i+1} th vertex of A_i in S if $|A_i| \geq \gamma_{i+1}$. Otherwise the computation ends in the part processing Γ . If on or before reaching γ_k we have obtained a set S such that $A_i \cap S \neq \phi$ for all j, the algorithm has successfully found a vertex deletion set. We say that the algorithm is in phase i if it processing γ_i .

To see that the algorithm can be implemented in Para-AC^{0↑}, we just need to observe that in each phase the algorithm has to maintain the list of sets in \mathcal{L} that are not yet hit by S. The depth of the circuit is O(k) and the total size is $d^k poly(n)$.

We implemented the bounded search tree method in Para-AC^{0↑}. This implementation can be used not only to recognize the graph class defined in the Definition 6 but also, as we can show, for designing Para-AC^{0↑} algorithms for the problems RESTRICTED ALTERNATING HITTING SET and WEIGHT $\leq k$ q-CNF SATISFIA-BILITY. The problems are as follows:

Problem 1: [19,12] RESTRICTED ALTERNATING HITTING SET

Instance: A collection C of subsets of a set B with $|S| \leq k_1$ for all $S \in C$.

Parameter: Two positive integers (k_1, k_2) .

Question: Does Player I have a win in at most k_2 moves in the following game? Players play alternatively and choose unchosen elements, until, for each $S \in C$ some member of S has been chosen. The player whose choice this happens to be

wins.

Problem 2: [19,12] WEIGHT $\leq k$ q-CNF SATISFIABILITY

Instance: Boolean formula φ in conjunctive normal form with maximum clause size q where q is fixed.

Parameter: A positive integer k.

Question: Does φ have a satisfying assignment with at most k literals true?

Theorem 2. The following problems are in Para-AC $^{0\uparrow}$:

- (i) RESTRICTED ALTERNATING HITTING SET.
- (ii) Weight $\leq k$ q-Cnf Satisfiability.

Downey et al. [19] gave FPT algorithms for these two problems by using bounded search tree method. We implemented the bounded search tree method in Para- $AC^{0\uparrow}$. Thus, these two problems are also in Para- $AC^{0\uparrow}$.

The next theorem is obtained by replacing the complexity class Para-L by Para-AC¹ in Theorem 1. The proof of the theorem uses similar ideas and the implementation is easier. Moreover, because of Lemma 1 we do not have to assume the existence of an algorithm that outputs all the vertex deletion sets.

Theorem 3. Let \mathcal{G} be a class of graphs characterized by finitely many forbidden induced subgraphs $\mathcal{H} = \{H_1, H_2, \cdots, H_l\}$ with $|V(H_i)| \leq d$ for all $1 \leq i \leq l$ where d is a constant. Suppose the colored graph isomorphism problem for graphs from \mathcal{G} is in AC^1 . Then the graph isomorphism problem for graphs from $\mathcal{G} + kv$ is in Para- AC^1 .

Corollary 1. The graph isomorphism problem parameterized by the distance to cographs is in $\mathsf{Para}\text{-}\mathsf{AC}^1$.

Proof. Recall that cographs are graphs without any induced P_4 . The colored graph isomorphism for cographs was shown to be in L using logspace algorithm to find the modular decomposition [24]. From this along with Theorem 3 and Lemma 1, we deduce that the graph isomorphism problem for distance to cographs is in Para-AC¹.

As a consequence of the above corollary, we can also solve graph isomorphism problem for some of the other graph classes e.g., distance to cluster (disjoint union of cliques), distance to threshold graphs in Para-AC¹ by using the generalized meta Theorem 3.

For larger parameters like vertex-cover, distance to clique and twin-cover, we can get better complexity theoretic results which we discuss in the following section.

4 GI Parameterized by Vertex Cover is in Para-TC⁰

In this section we give a parameterized parallel algorithm for GI parameterized by vertex cover. Sam Buss [10] showed that VERTEX COVER admits a polynomial kernel. Based on this kernelization result, Cai et al. [12], Elberfeld et al. [21] and

Bannach et al. [5] showed that VERTEX COVER is in Para-L, Para-TC⁰ and Para-AC⁰ respectively. These methods not only determines the existence of a vertex cover of size at most k but can also output all vertex covers of size at most k in Para-AC⁰. We give a brief overview of the procedure to enumerate all vertex covers of size at most k by using kernelization method given in [12,5,21].

Observe that any vertex of degree more than k must belong to any vertex cover of a given graph G. For the graph G=(V,E), consider the set $V_H=\{v\in V(G)\big|d(v)>k\}$. If $|V_H|$ is more than k then we declare that there is no k sized vertex cover. Let us assume $|V_H|=b$. Consider the set $V_L=\{v\in V(G)\big|d(v)\le k$ and $N(v)\setminus V_H\neq\emptyset\}$ of vertices that have at least one neighbour outside V_H . Notice that none of the edges in $G[V_L]$ are covered by V_H . Let S' be a vertex cover of $G[V_L]$. It is easy to see that $V_H\cup S'$ forms a vertex cover of G. On the other hand if S is a vertex cover of G then $V_L\cap S$ is a vertex cover of $G[V_L]$. If the cardinality of V_L is more than (k-b)(k+1) then reject (because the graph induced by vertices V_L with k-b vertex cover and all vertices degree bounded by k has no more than (k-b)(k+1) vertices). So the cardinality of V_L is not more than (k-b)(k+1). We can use the best known vertex cover algorithm [15] to find the (k-b) vertex cover on the sub graph induced by vertices V_L . Elberfeld et al. [21] pointed that the parallel steps of this process are the following:

- i. Checking whether the vertex belongs to V_H .
- ii. Checking whether $|V_H|$ at most k.
- iii. Checking whether $|V_L|$ at most k(k+1).
- iv. Computing the induced subgraph $G[V_L]$ from G.

The computation of above steps can be implemented by Para-AC⁰ circuits [5]. The above process finds all vertex covers of size at most k by enumerating all the $2^{|V_L|}$ possible binary strings on length $|V_L|$.

Theorem 4. The graph isomorphism problem parameterized by vertex cover is in Para-TC⁰.

Proof. Given two input graphs G and H with vertex cover of size at most k, we need to test if G to H are isomorphism in Para-TC⁰. Using the kernelization method of Bannach et al. [5] we can recognize whether these two graphs have same sized vertex covers or not. For the graph G we find a minimal vertex cover S of size at most k and for graph H we find all minimal vertex covers S_1, S_2, \cdots, S_m , each of size at most k. Notice that m is at most 2^k . We know that if $G \cong H$ then $G[S] \cong H[S_i]$ for some $1 \le i \le m$. We try all isomorphisms from the minimal vertex cover S of G to each minimal vertex cover S_i of G. Suppose $G[S] \cong H[S_i]$ via G. We need to extend this isomorphism from the independent set $G \setminus S$ to G to G to extend this isomorphisms between G[S] to G to G to G to G to expect at most G isomorphisms between G[S] to G to G to G to G to G to G to expect the isomorphisms in parallel.

For each isomorphism φ , we need to check whether this φ can be extended to an isomorphism between $G \setminus S$ to $H \setminus S_i$. We partition the vertices of the graph $G \setminus S$ into at most 2^k sets (called 'types') based on their neighborhood in S. For

each $U \subseteq S$ let $T_G(U,S) = \{u \in G \setminus S \mid N(u) = U\}$. It is not hard to see that $G \cong H$ if and only if there is a minimal vertex cover S_i of H and an isomorphism φ from G[S] to $H[S_i]$ such that for each $U \subseteq S$, $|T_G(U,S)| = |T_H(\varphi(U),S_i)|$. The problem of testing whether G is isomorphic to H reduces to counting the number of vertices in each type. We represent each type using an n-length binary string, where i^{th} entry is one if v_i belongs to that type and zero otherwise. Since the BIT COUNT³ problem is in TC^0 , counting the number of vertices in a type can be implemented using a TC^0 circuit. In summary, for each S_i and each isomorphism between G[S] and $H[S_i]$, and for each $U \subseteq S$ we check whether $|T_G(U,S)| = |T_H(\varphi(U),S_i)|$. This completes the proof.

Corollary 2. The graph isomorphism problem is in Para-TC⁰ when parameterized by the distance to clique.

Proof. We apply Theorem 4 to the complements of the input graphs. \Box

Corollary 3. The graph isomorphism problem parameterized by the size of a twin-cover is in Para-TC⁰.

Proof. To find the twin-cover, we first remove all the twin edges and then compute a vertex cover of size at most k in the resulting graph as was done in [23]. The first step runs through all edges and deletes an edge if it is a twin edge. Next it finds a vertex cover in the resulting graph which can be done in Para-AC⁰ [5]. Thus, computing all the twin-covers can be done in Para-AC⁰.

Now we describe the process of testing isomorphism. The idea for testing isomorphism of the input graphs parameterized by the size of a twin-cover is similar to that in the proof of Theorem 4. Let S_1 be a fixed twin-cover in G_1 and S_2 be a twin-cover in G_2 of same size. The algorithm processes all such (S_1, S_2) pairs in parallel. First fix an isomorphism (say σ) from S_1 to S_2 and try to extend it to $G_1 \setminus S_1$ to $G_2 \setminus S_2$. Again, all such isomorphisms are processed in parallel. We know that the graph $G \setminus S$ obtained by removing a twin-cover S from G is a disjoint collection of cliques. Any two vertices in a clique C have same neighbourhood in G i.e., if $u, v \in C$ then N[u] = N[v]. Thus, the 'type' of a clique is completely determined by the neighbourhood of any of the vertices in the vertex deletion set, and the size of the clique. Formally, with respect to the isomorphism σ , a clique C_{g_1} in $G_1 \setminus S_1$ and a clique C_{g_2} in $G_2 \setminus S_2$ have same type if 1 $|V(C_{g_1})| = |V(C_{g_2})|$ and 2 $|V(C_{g_1})| = |V(C_{g_2})|$. The algorithm needs to check that the number of cliques in each type is same in both the graphs. This problem can again be reduced to instances of the BIT COUNT problem.

It is easy to see that, the above process can be implemented in Para- TC^0 . \Box

5 Logspace **GI** Algorithms for Bounded Distance to Graph Classes

In this Section, we show that for fixed k GI for graphs in $\mathcal{G} + kv$ is in L if the colored GI for graphs in \mathcal{G} is in L where \mathcal{G} is a graph class. From this result we

³ Counting the number one's in n length binary string.

obtain that GI for cographs + kv and interval + kv graphs is in L. Note that these results are not in the parameterized complexity theory framework. The proof of the following theorem given in appendix.

Theorem 5. Let k be a fixed and \mathcal{G} be a class of graphs. Suppose the problem of deciding if a given graph is in \mathcal{G} and the colored graph isomorphism problem for graphs in \mathcal{G} is in L. Then the graph isomorphism problem for graphs from $\mathcal{G} + kv$ is in L.

Suppose graph class \mathcal{G} is define as in Definition 6. It is not hard to see the problem of deciding if a graph G is in a class \mathcal{G} characterized by finitely many forbidden induced subgraphs is in logspace (See Lemma 2 in the Appendix). The proof of the next corollary follows from Lemma 2 and Theorem 5.

Corollary 4. Let the graph class \mathcal{G} be characterized by finitely many forbidden induced subgraphs $\mathcal{H} = \{H_1, H_2, \cdots, H_l\}$ with $|V(H_i)| \leq d$ for all $1 \leq i \leq l$ where d is a constant. The graph isomorphism problem for graphs with bounded vertex deletion from \mathcal{G} is in L provided the colored graph isomorphism problem for graphs from \mathcal{G} is in L.

Corollary 5. The graph isomorphism problem is in L for following graph classes: 1) distance to interval graphs 2) distance to cographs.

Proof. The proof of (1), follows from Theorem 5 and the logspace algorithm for colored GI for interval graphs (see [26]).

The proof of (2), follows from Corollary 4 and the logspace isomorphism algorithm for colored GI for cographs [24].

6 Conclusion

In this paper we showed that graph isomorphism problem is in Para-TC⁰ when parameterized by the vertex cover number of the input graphs. We also studied the parameterized complexity of graph isomorphism problem for the class of graphs $\mathcal G$ characterized by finitely many forbidden induced subgraphs. We showed that graph isomorphism problem is in Para-AC¹ for the graphs in $\mathcal G + kv$ if there is an AC¹ algorithm for colored-GI for the graph class $\mathcal G$. From this result, we show that GI parameterized by the distance to cographs is in Para-AC¹.

The following questions remain open. Can we get a parameterized logspace algorithm for GI parameterized by feedback vertex set number? Does the problem admit parameterized parallel algorithm? Elberfeld et al. [20] showed that GI is in logspace for graphs of bounded tree-width. In this paper, we showed that GI for some subclasses of bounded clique-width graphs is in L. It is an interesting open question to extend these results to bounded clique-width graphs.

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7 Appendix

Proof of Theorem 5

Proof. The idea behind this proof is similar to that of Theorem 1. Let G_1 and G_2 be the two input graphs. The logspace graph isomorphism algorithm for graphs in \mathcal{G} works via finding a vertex deletion set S_1 for G_1 of size at most k. Next we iterate over all vertex deletion sets S_2 of the same size. The idea is to fix an isomorphism from $G_1[S_1]$ to $G_2[S_2]$ and check if the isomorphism can be extended to an isomorphism of the input graphs. To store the vertex deletion sets we need $O(k \log n)$ space in the work-tape.

We first describe how to find a vertex deletion set of a graph G. Choose a set S of size at most k vertices from V(G) and test whether $G \setminus S \in \mathcal{G}$ by using the logspace algorithm (say \mathcal{A}_r) for deciding if an input graph is in \mathcal{G} . For every set S of size at most k from V(G), if the recognition algorithm says $G \setminus S \notin \mathcal{G}$ then algorithm can infer that $G \notin \mathcal{G} + kv$. If for any of the sets, the algorithm \mathcal{A}_r says $G \setminus S \in \mathcal{G}$ then algorithm outputs S as vertex deletion set. The iteration of over sets of size at most k can be easily implemented in logspace by using k counters. Therefore, the whole process can be executed in logspace.

Let S_1 and S_2 be the vertex deletion sets of G_1 and G_2 obtained using the above logspace procedure. Fix a bijection σ from S_1 to S_2 and test if it is an isomorphism form $G_1[S_1]$ to $G_2[S_2]$. If not we try the next S_2 in the lexicographic input order. Otherwise, we test if σ can be extended to an isomorphism of the input graphs. The map σ induces a coloring of the graphs $G_1' = G_1 \setminus S_1$ and $G_2' = G_2 \setminus S_2$. Two vertices in G_1' (G_2') get same color if they have the same neighbourhood in S_1 (S_2 respectively). A vertex u in G_1' and a vertex v in G_2' will get same color if $\sigma(N(u) \cap S_1) = N(v) \cap S_2$. It is easy to see that the resulting graphs have at most 2^k colors. Moreover, σ can be extended to an isomorphism of G_1 and G_2 if and only if the colored versions of G_1' and G_2' are isomorphic.

Computing if two vertices u and u' in G'_1 have same color amounts to searching their neighbourhood in S_1 . Since S_1 is in the work-tape this can be done in logspace. Similarly, by the fact that S_2 and σ are in the work-tape, checking $\sigma(N(u) \cap S_1) = N(v) \cap S_2$ can also be performed in logspace. Since we can compute the colors in logspace, testing if colored G'_1 and G'_2 can be done in logspace using the logspace isomorphism test of colored graphs in \mathcal{G} . This completes the description of the algorithm.

Lemma 2. Let \mathcal{G} be a class of graphs characterized by finitely many forbidden induced subgraphs $\mathcal{H} = \{H_1, H_2, \cdots, H_l\}$ with $|V(H_i)| \leq d$ for all $1 \leq i \leq l$ where d is a constant. There is a logspace algorithm that on input a graph G decides if $G \in \mathcal{G}$.

Proof. Given a graph G, the aim is to check whether $G \in \mathcal{G}$. For this it is enough to check for each $i \in \{1, 2, \cdots, l\}$ whether G contains H_i as an induced subgraph. The algorithm heavily uses the input order of the vertices of G. Let d_i be the number of vertices in H_i . To check if H_i appears as an induced subgraph of G, the algorithm picks vertices $v_1, v_2, \cdots, v_{d_i}$ from V(G) and checks if these vertices induces H_i in G. If not then the algorithm chooses a different set of G vertices according to the input order of G and repeats the same process. For each G in none of the G-sized subsets of G forms G it is easy to see that this algorithm can be implemented in logspace because at each step we need G and G space to store at most G vertices of G and constantly many counters.