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Nonlinear Eigenproblems in Image Processing and Computer Vision

Springer

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ISSN 2191-6586 ISSN 2191-6594 (electronic)
Advances in Computer Vision and Pattern Recognition
ISBN 978-3-319-75846-6 ISBN 978-3-319-75847-3 (eBook)
https://doi.org/10.1007/978-3-319-75847-3

Library of Congress Control Number: 2018933476

Printed on acid-free paper

This Springer imprint is published by the registered company Springer International Publishing AG part of Springer Nature
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

To my wife Mor.

Preface

What are Nonlinear Eigenproblems and Why are They Important?

A vast majority of real-world phenomena are nonlinear. Therefore, linear modeling serves essentially as a rough approximation and in some cases offers only shallow understanding of the underlying problem. This holds not only for modeling physical processes but also for algorithms, such as in image processing and computer vision, which attempt to process and understand the physical world, based on various 2D and 3D sensors.

Linear algorithms and transforms have reached tremendous achievements in signal and image processing. This was accomplished by very effective tools such as Fourier and Laplace transforms and based on the broad and deep mathematical foundations of linear algebra. Thus, very strong theories have been established over the years and efficient numerical methods have been developed.

Unfortunately, linear algorithms have their limitations. This is especially evident in signal and image processing, where standard linear methods such as Fourier analysis are rarely used in modern algorithms. A main reason is that images and many other signals have nonstationary statistics and include discontinuities (or edges) in the data. Therefore, standard assumptions of global statistics and smoothness of the data do not apply. The common practice in the image processing field is to develop nonlinear algorithms. These can roughly be divided into several main approaches:

- **Local linearization**. Applying a linear operator which is adaptive and changes spatially in the image domain. Examples: adaptive Wiener filtering [1, 2], bilateral filtering [3], and nonlocal means [4].
- **Hybrid linear–nonlinear**. Performing successive iterations of linear processing followed by simple nonlinear functions (such as thresholding, sign, and sigmoid functions). This branch includes many popular algorithms such as wavelet thresholding [5], dictionary and sparse representation approaches [6–8], and recently, deep convolutional neural networks [9, 10].
- **Spectral methods**. In this branch, one often constructs a data-driven graph and then performs linear processing using spectral graph theory [11], where the graph Laplacian is most commonly used. Examples: graph cuts [12], diffusion maps [13], and random-walker segmentation [14].
- **Convex modeling**. Algorithms based on convex optimization with non-quadratic functionals, such as total variation [15–17]. More details are given on this branch in this book.
- **Kernel-based**. Applying directly nonlinear kernel operators such as median, rank filers [18], and morphological filtering [19].

In this book, we take a fresh look at nonlinear processing through nonlinear eigenvalue analysis. This is still a somewhat unorthodox approach, since eigenvalue analysis is traditionally concerned with linear operators. We show how one-homogeneous convex functionals induce operators which are nonlinear and can be analyzed within an eigenvalue framework. The book has three essential parts. First, mathematical background is provided along with a summary of some classical variational algorithms for vision (Chaps. 2–3). The second part (Chaps. 4–7) focuses on the foundations and applications of the new multiscale representation based on nonlinear eigenproblems. In the last part (Chaps. 8–11), new numerical techniques for finding nonlinear eigenfunctions are discussed along with promising research directions beyond the convex case. These approaches may be valuable also for scientific computations and for better understanding of scientific problems beyond the scope of image processing.

In the following, we present in more details the intuition and motivation for formulating nonlinear transforms. We begin with the classical Fourier transform and its associated operator and energy. We ask how these concepts can be generalized in the nonlinear case? This can give the reader the flavor of topics discussed in this book and the approach taken to address them.

Basic Intuition and Examples

Fourier and the Dirichlet Energy

Let us first look at the classical case of the linear eigenvalue problem

$$Lu = \lambda u, \tag{1}$$

where L is a bounded linear operator and u is some function in a Hilbert space. For u admitting (1), we refer to as an eigenfunction, where λ is the corresponding eigenvalue.

In this book, we investigate nonlinear operators which can be formed based on regularizers. This can be done either by deriving the variational derivative of the regularizer (its subdifferential in the general convex nonsmooth case) or by using the proximal operator. Let us examine the standard quadratic regularizer, frequent in physics, and the basis of Tikhonov regularization, the Dirichlet energy

$$J_D = \frac{1}{2}\int |\nabla u|^2 dx. \tag{2}$$

The variational derivative is

$$\partial_u J_D = -\Delta u,$$

with Δ the Laplacian. The corresponding eigenvalue problem is

$$-\Delta u = \lambda u, \tag{3}$$

where Fourier basis yields the set of eigenfunctions (with appropriate boundary conditions).

Another classical result in this context is the relation to the Rayleigh quotient,

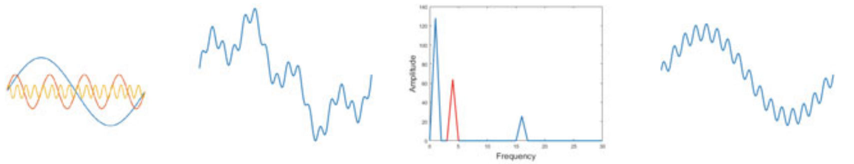
$$R(u) = \frac{\|\nabla u\|^2_{L^2}}{\|u\|^2_{L^2}}.$$

Taking the inner product with respect to u from both sides of (3), we get $\langle -\Delta u, u\rangle = \lambda \langle u, u\rangle$. Using integration by parts (or the divergence theorem), we have $\langle -\Delta u, u\rangle = \|\nabla u\|^2_{L^2}$ and thus for any eigenfunction u we obtain

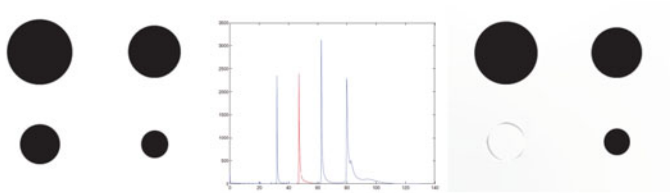
$$\lambda = R(u).$$

Moreover, if we seek to minimize $R(u)$, with respect to u, we can formulate a constrained minimization problem, subject to $\|u\|^2_{L^2} = 1$, and see (using Lagrange multipliers) that eigenfunctions are extremal functions of the Rayleigh quotient (hence, also its global minimizer is an eigenfunction).

From the above discussion, we see the connection between a quadratic regularizer (J_D) and the related linear operator ($-\Delta$). We notice that the corresponding eigenvalue problem induces a multiscale representation (Fourier) and that eigenfunctions are local minimizers of the Rayleigh quotient. The remarkable recent findings are that these relations generalize very nicely to the nonlinear case. We will now look at the total variation (TV) regularizer, which induces a nonlinear eigenvalue problem.



(a) Components of input (b) Input (c) Fourier spectrum (d) Fourier band-stop filtered



(e) Input (f) TV Spectrum (g) TV band-stop filtered

Fig. 1 Example of multiscale representation and filtering based on linear and nonlinear eigenfunctions. In the first row, three sine functions of different frequencies (left) are combined and serve as the input signal. A classical Fourier transform shows the spectrum containing three numerical delta functions. Filtering out the middle frequency (band-stop or notch filter) yields the result on the right. On the bottom row, we show a similar phenomenon based on the TV transform, where the input is a combination of four eigenfunctions in the shape of disks. Based on the TV transform, explained in Chap. 5, we get a spectrum with four numerical deltas, corresponding to each disk (from small to large radius). One can filter out the second smallest disk in an analog manner to Fourier, receiving the result on the right, preserving perfectly the three other disks, with some marginal errors due to grid discretization effects

The Total Variation Eigenvalue Problem

First, we would like to introduce the typical nonlinear eigenvalue problem associated with a bounded nonlinear operator T,

$$Tu = \lambda u. \tag{4}$$

This is a straightforward analog of (1) and will be investigated throughout this book, mainly in the context of convex optimization. Further generalizations to this equation will also be briefly addressed.

The L^1-type regularizer, which is analog to the Dirichlet energy, is the total variation functional defined by

$$J_{TV} = \int |\nabla u| dx. \tag{5}$$

Here, we use the simple (strong-sense) formulation. TV is very frequent in regularization of image processing problems, such as denoising, 3D reconstruction, stereo, and optical flow estimations. Chapter 3 gives more details on the properties of this functional. Based on the variational derivative of TV, $\partial_u J_{TV}$, we reach the following nonlinear eigenvalue problem:

$$-\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \lambda u, \tag{6}$$

which is the analog of (3). For those less familiar with variational methods, we give the basic background in Sect. 1.3 on how the operator on the left side of (6) was derived. This operator is often referred to as the 1-Laplacian. We also note that this writing is somewhat informal (a subgradient inclusion is more precise, and the derivatives should be understood in the distributional sense).

Let u be an indicator function of a convex set C in $\mathbb{R}^2$. A fascinating result by Andreau et al. [20] is that u is an eigenfunction, in the sense of (6), if C admits the following simple geometric condition:

$$(\text{maximal curvature on } \partial \mathrm{C}) \leq \frac{Per(C)}{|C|},$$

where ∂C is the boundary of C, $Per(C)$ is its perimeter and $|C|$ is its area. Checking this condition for the case of a disk of radius r (curvature $1/r$, $Per(C) = 2\pi r$, $|C| = \pi r^2$), we observe that any disk admits this condition. It essentially means that convex characteristics sets with smooth enough boundaries are TV eigenfunctions. Moreover, it was established that the eigenvalue is precisely the perimeter to area ratio

$$\lambda = \frac{Per(C)}{|C|}.$$

An alternative way to derive λ for any u is to follow the computation described earlier for the linear case. That is, we can take the inner product with respect to u of (6) and get

$$\lambda = \frac{J_{TV}(u)}{\|u\|_{L^2}^2}.$$

Indeed, for the specific case of a characteristic set, we obtain the equalities $Per(C) = J_{TV}(u)$ and $|C| = \|u\|^2_{L^2}$. The last equation is actually a generalized Rayleigh quotient, where one can show that eigenfunctions are extremal points, as in the linear case.

A natural question to ask is, can we also generalize a multiscale representation, based on TV eigenfunctions, in an analog manner to Fourier? If so, what are the properties and qualities of this representation? how well can it represent images? This book tries to address this fundamental question. A suggestion for a spectral representation suggested by the author and colleagues [21–23] is the spectral TV representation or the TV transform. A detailed explanation of the formalism is given in Chap. 5.

In the multiscale representation of the TV transform, scales (in the sense of eigenvalues) can be well separated, amplified, or attenuated, in a similar manner to classical Fourier filtering. In Fig. 1, we show a toy example of *TV band-stop filtering* along a comparison to standard Fourier filtering.

Graphs and Physics

Nonlinear eigenvalue problems related to TV are highly useful also for classification and learning, using graph data structures. The Cheeger constant is an isoperimetric value which essentially measures the degree of the dominant "bottleneck" in the graph. Perimeter of a set on a graph is directly related to the total variation on graphs. It was established that finding this bottleneck, or Cheeger cut, which is a NP-hard problem, can be well approximated by solving the 1-Laplacian eigenvalue problem

$$\Delta_1 u = \lambda \mathrm{sign}(u),$$

where $\Delta_1 u$ is the 1-Laplacian on the graph. In Chap. 8, it will be explained how this can be used for segmentation and classification. Also, numerical algorithms for solving such problems will be discussed in Chap. 7. Finally, we will briefly mention the more general double-nonlinear eigenvalue problem

$$T(u) = \lambda Q(u),$$

where both T and Q can be nonlinear operators. In the case of $T = -\Delta$ (a linear operator) and nonlinear Q, there are several physical problems, such as soliton waves, which are modeled by solutions to these types of eigenvalue problems.

What is Covered in This Book?

This book first presents some of the basic mathematical notions, which are needed for later chapters. An effort is made to make the book self-contained so it is accessible to many disciplines. We then outline briefly the use of variational and flow-based methods to solve many image processing and computer vision algorithms (Chap. 3).

As total variation is an important functional, which is used throughout this book, we present its properties in more details (Chap. 4). We then define the concept of nonlinear eigenfunctions related to convex functionals and state some of the properties known today (an area still under active research, Chap. 5).

We proceed by going into a fundamental concept presented in this book of the spectral framework for one-homogeneous functionals. We show how eigenfunctions appear naturally in gradient descent and variational methods and that a spectral decomposition can be used for new representations of signals. The concept and motivation are discussed, as well as current theory on the subject. Applications of using this framework for denoising, texture processing, and image fusion are presented (Chaps. 6–7).

In the following chapter, we go deeper into the nonlinear eigenvalue problem and propose new ways to solve it using special flows which converge to eigenfunctions (Chap. 8).

We then go to graph-based and nonlocal methods, where a TV eigenvalue analysis gives rise to strong segmentation, clustering, and classification algorithms (Chap. 9).

Next, we present a new direction of how the nonlinear spectral concept can be generalized beyond the convex case, based on pixel decay analysis (Chap. 10). We are thus able to construct a spectral representation with different nonlinear denoisers and get different eigenmodes.

Relations to other image processing branches, such as wavelets and dictionary based, are discussed (Chap. 11). We conclude with the current open problems and outline future directions for the development of theory and applications related to nonlinear eigenvalue problems. In the appendix, we summarize some standard discretization and convex optimization methods, which are used to implement numerically such methods.

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Acknowledgements

The author would like to thank the following people for their advice, help, and contributions, which made the writing of this book possible: Martin Burger, Michael Moeller, Jean-Francois Aujol, Martin Benning, Nicolas Papadakis, Tomer Michaeli, Ester Hait, Tal Feld, Shai Biton, Ido Cohen, Raz Nossek, Dikla Horesh, Oren Katzir, Alona Baruhov, and Nati Daniel.

The author acknowledges support by the Israel Science Foundation (Grant No. 718/15).

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