Permutations sorted by a finite and an infinite stack in series

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Abstract. We prove that the set of permutations sorted by a stack of depth $t \geq 3$ and an infinite stack in series has infinite basis, by constructing an infinite antichain. This answers an open question on identifying the point at which, in a sorting process with two stacks in series, the basis changes from finite to infinite.

Keywords: patterns, string processing algorithms, pattern avoiding permutations, sorting with stacks.

1 Introduction

A permutation is an arrangement of an ordered set of elements. Two permutations with same relative ordering are said to be order isomorphic, for example, 132 and 275 are order isomorphic as they have relative ordering ijk where i < k < j. A subpermutation of a permutation $p_1 \dots p_n$ is a word $p_{i_1} \dots p_{i_s}$ with $1 \le i_1 < \dots < i_s \le n$. A permutation p contains q if it has a subpermutation that is order isomorphic to q. For example, 512634 contains 231 since the subpermutation 563 is order isomorphic to 231. A permutation that does not contain q is said to avoid q. Let S_n denote the set of permutations of $\{1, \dots, n\}$ and let $S^\infty = \bigcup_{n \in \mathbb{N}_+} S_n$. The set of all permutations in S^∞ which avoid every permutation in $\mathscr{B} \subseteq S^\infty$ is denoted $Av(\mathscr{B})$. A set of permutations is a pattern avoidance class if it equals $Av(\mathscr{B})$ for some $\mathscr{B} \subseteq S^\infty$. A set $\mathscr{B} = \{q_1, q_2, \dots\} \subseteq S^\infty$ is an antichain if no q_i contains q_j for any $i \ne j$. An antichain \mathscr{B} is a basis for a pattern avoidance class \mathscr{C} if $\mathscr{C} = Av(\mathscr{B})$.

Sorting mechanisms are natural sources of pattern avoidance classes, since (in general) if a permutation cannot be sorted then neither can any permutation containing it. Knuth characterised the set of permutations that can be sorted by a single pass through an infinite stack as the set of permutations that avoid 231 [11]. Since then many variants of the problem have been studied, for example [1,2,3,4,5,6,7,8,9,13,14,15,16,17,18]. The set of permutations sortable by a stack of depth 2 and an infinite stack in series has a basis of 20 permutations [7], while for two infinite stacks in series there is no finite basis [12]. For systems of a finite stack of depth 3 or more and infinite stack in series, it was not known whether the basis was finite or infinite.

Here we show that for depth 3 or more the basis is infinite. We identify an infinite antichain belonging to the basis of the set of permutations sortable by a stack of depth 3 and an infinite stack in series. A simple lemma then implies the result for depth 4 or more. A computer search by the authors ([10]) yielded 8194 basis permutations of lengths up to 13 (see Table 1; basis permutations are listed at https://github.com/gohyoongkuan/stackSorting-3). The antichain used to prove our theorem was found by examining this data and looking for patterns that could be arbitrarily extended.

Permutation length	Number of sortable permutations	Number of basis elements
5	120	0
6	711	9
7	4700	83
8	33039	169
9	239800	345
10	1769019	638
11	13160748	1069
12	98371244	1980
13	737463276	3901

Table 1. Number of basis elements for $S(3,\infty)$ of length up to 13

2 Preliminaries

The notation \mathbb{N} denotes the non-negative integers $\{0, 1, 2, \dots\}$ and \mathbb{N}_+ the positive integers $\{1, 2, \dots\}$.

Let M_t denote the machine consisting of a stack, R, of depth $t \in \mathbb{N}_+$ and infinite stack, L, in series as in Fig. 1. A sorting process is the process of moving entries of a permutation from right to left from the input to stack R, then to stack L, then to the output, in some order. Each item must pass through both stacks, and at all times stack R may contain no more than t items (so if at some point stack R holds t items, the next input item cannot enter until an item is moved from R to L).

A permutation $\alpha = a_1 a_2 \dots a_n$ is in $S(t, \infty)$ if it can be sorted to $123 \dots n$ using M_t . For example, $243651 \in S(t, \infty)$ for $t \geq 3$ since it can be sorted using the following process: place 2, 4 into stack R, move 4, 3, 2 across to stack L, place 6, 5, 1 into stack R, then output 1, 2, 3, 4, 5, 6. Note $243651 \notin S(2, \infty)$ by [7].

The following lemmas will be used to prove our main result.

Lemma 1. Let $\alpha = a_1 a_2 \dots a_n \in S(t, \infty)$ for $t \in \mathbb{N}_+$. If i < j and $a_i < a_j$ then in any sorting process that sorts α , if both a_i and a_j appear together in stack L then a_i must be above a_j .

Proof. If a_j is above a_i in stack L then the permutation will fail to be sorted. \Box

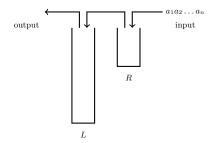


Fig. 1. A stack R of depth t and an infinite stack L in series

Lemma 2. Let $\alpha = a_1 a_2 \dots a_n \in S(t, \infty)$ for $t \geq 3$ and suppose $1 \leq i < j < k \leq n$ with $a_i a_j a_k$ order-isomorphic to 132. Then in any sorting process that sorts α , a_i, a_j, a_k do not appear together in stack R.

Proof. If a_i, a_j, a_k appear together in stack R, we must move a_k then a_j onto stack L before we can move a_i , but this means a_j, a_k violate Lemma 1.

Lemma 3. Let $\alpha = a_1 a_2 \dots a_n \in S(t, \infty)$ for $t \geq 3$ and $1 \leq i_1 < i_2 < \dots < i_6 \leq n$ with $a_{i_1} a_{i_2} \dots a_{i_6}$ order isomorphic to 243651. Then in any sorting process that sorts α , at some step of the process a_{i_4} and a_{i_5} appear together in stack R.

Proof. For simplicity let us write $a_{i_1} = 2$, $a_{i_2} = 4$, $a_{i_3} = 3$, $a_{i_4} = 6$, $a_{i_5} = 5$, $a_{i_6} = 1$. Before 6 is input, 2, 3, 4 are in the two stacks in one of the following configurations:

- 1. 2, 4, 3 are all in stack R. In this case we violate Lemma 2.
- 2. two items are in stack R and one is in stack L. In this case by Lemma 1 we cannot move 6 to stack L, so 6 must placed and kept in stack R. If t=3 stack R is now full, so 5 cannot move into the system, and if $t \geq 4$, when 5 is input we violate Lemma 2.
- 3. one item, say a, is in stack R and two items are in stack L. In this case we cannot move 6,5 into stack L by Lemma 1 so they remain in stack R on top of a, violating Lemma 2.
- 4. stack R is empty. In this case, 2, 3, 4 must be placed in stack L in order, else we violate Lemma 1. We cannot place 6, 5 into stack L until it is empty, so they must both stay in stack R until 4 is output.

In particular, the last case is the only possibility and in this case a_{i_4}, a_{i_5} appear in stack R together.

Lemma 4. Let $\alpha = a_1 a_2 \dots a_n \in S(t, \infty)$ for $t \geq 3$ and suppose $1 \leq i_1 < i_2 < \dots < i_5 \leq n$ with $a_{i_1} a_{i_2} \dots a_{i_5}$ order-isomorphic to 32514. Then, in any sorting process that sorts α , if a_{i_1}, a_{i_2} appear together in stack R, then at some step in the process a_{i_3}, a_{i_4} appear together in stack L.

Proof. For simplicity let us write $a_{i_1} = 3$, $a_{i_2} = 2$, $a_{i_3} = 5$, $a_{i_4} = 1$, $a_{i_5} = 4$. Figure 2 indicates the possible ways to sort these entries, and in the case that 2,3 appear together in stack ${\cal R}$ we see that 4,5 must appear in stack ${\cal L}$ together at some later point.

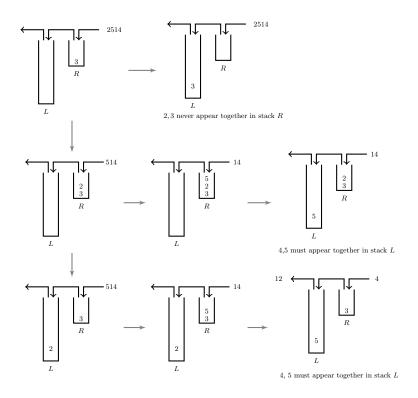


Fig. 2. Sorting 32514

Lemma 5. Let $\alpha = a_1 a_2 \dots a_n \in S(t, \infty)$ for $t \geq 3$ and suppose $1 \leq i_1 < i_2 < \infty$ $\cdots < i_5 \le n$ with $a_{i_1}a_{i_2} \dots a_{i_5}$ order-isomorphic to 32541. Then, in any sorting process that sorts α , if a_{i_1} , a_{i_2} appear together in stack L, then at the step that a_{i_1} is output,

- 1. a_{i_3}, a_{i_4} are both in stack R, and
- 2. if a_k is in stack L then $k < i_2$.

Proof. For simplicity let us write $a_{i_1} = 3$, $a_{i_2} = 2$, $a_{i_3} = 5$, $a_{i_4} = 4$, $a_{i_5} = 1$, and $\alpha = u_0 3u_1 2u_2 5u_3 4u_4 1u_5$. Figure 3 indicates the possible ways to sort these entries. In the case that 2,3 appear in stack R together, Lemma 1 ensures 2,3 do not appear together in stack L. In the other case, before 3 is moved into stack L, any tokens in stack L come from u_0u_1 . Thus when 3 is output the only tokens in stack L will be a_k with $k < i_2$. Lemma 1 ensures that 4,5 are not placed on top of 3 in stack L, so that the step that 3 is output they sit together in stack R. \square

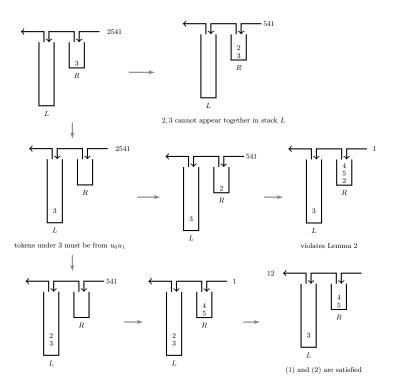


Fig. 3. Sorting 32541

3 An infinite antichain

We use the following notation. If $\alpha = a_1 \dots a_n$ is a permutation of $12 \dots n$ and $m \in \mathbb{Z}$ then let α_m be the permutation obtained by adding m to each entry of α . For example $(1\ 2\ 3)_4 = 5\ 6\ 7$ and $13_6 = 19$.

We construct a family of permutations $\mathscr{G} = \{G_i \mid i \in \mathbb{N}\}$ as follows. Define

$$P = 2 \ 4 \ 3 \ 7 \ 6 \ 1$$

$$x_j = (10 \ 5 \ 9)_{6j}$$

$$y_j = (13 \ 12 \ 8)_{6j}$$

$$S_i = (14 \ 15 \ 11)_{6i}$$

$$G_i = P \ x_0 \ y_0 \ x_1 \ y_1 \ \dots \ x_i \ y_i \ S_i$$

The first three terms are

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\begin{array}{l} G_0=2\ 4\ 3\ 7\ 6\ 1\ (10\ 5\ 9)\ (13\ 12\ 8)\ 14\ 15\ 11,\\ G_1=2\ 4\ 3\ 7\ 6\ 1\ (10\ 5\ 9)\ (13\ 12\ 8)\ (16\ 11\ 15)\ (19\ 18\ 14)\ 20\ 21\ 17,\\ G_2=P\ (10\ 5\ 9)\ (13\ 12\ 8)\ (16\ 11\ 15)\ (19\ 18\ 14)\ (22\ 17\ 21)(25\ 24\ 20)\ 26\ 27\ 23. \end{array}
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A diagram of G_2 is shown in Figure 4 which shows the general pattern.

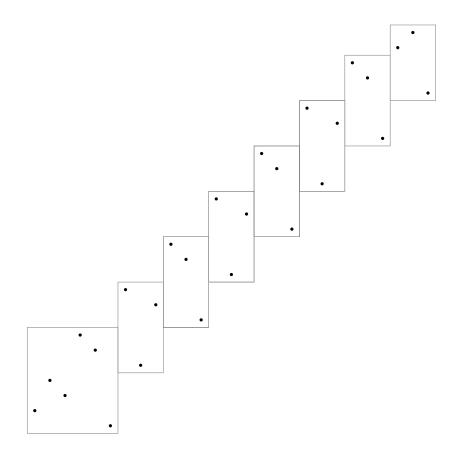


Fig. 4. Diagram of the permutation $G_2=2\ 4\ 3\ 7\ 6\ 1\ x_0\ y_0\ x_1\ y_1\ x_2\ y_2\ 26\ 27\ 23$

We will prove that each G_i is an element of the basis of $S(3,\infty)$ for all $i\in\mathbb{N}$. Note that if we define x_{-1},y_{-1} to be empty, $G_{-1}=243761895$ is also an element of the basis. We noticed this and G_0 had a particular pattern which we could extend using x_jy_j . However, we exclude G_{-1} from our antichain to make the proofs simpler.

Proposition 6. The permutation $G_i \notin S(3, \infty)$ for all $i \in \mathbb{N}$.

Proof. Suppose for contradiction that G_i can be sorted by some sorting process. Since P is order isomorphic to 243651, by Lemma 3 in any sorting process 7,6 appear together in stack R. Next, 7 6 10 5 9 is order isomorphic to 32514 so by Lemma 4 since 7,6 appear together in stack R we must have that 10,9 appear together in stack L at some point in the process.

Now consider $x_j y_j = (10\ 5\ 9\ 13\ 12\ 8)_{6j}$, and assume that $10_{6j}, 9_{6j}$ both appear in stack L together. Since $(10\ 9\ 13\ 12\ 8)_{6j}$ is order isomorphic to 32541 by Lemma 5 $13_{6j}, 12_{6j}$ must be placed together in stack R and stay there until 10_{6j} is output.

Next consider $y_j x_{j+1} = (13\ 12\ 8\ 16\ 11\ 15)_{6j}$, and assume that $13_{6j}, 12_{6j}$ both appear in stack R together. Then since $(13\ 12\ 16\ 11\ 15)_{6j}$ is order isomorphic to 32514 by Lemma 4 we have $16_{6j}, 15_{6j}$ appear together in stack L. Note that $16_{6j}, 15_{6j} = 10_{6(j+1)}, 9_{6(j+1)}$, so putting the above observations together we see that for all $0 \le j \le i$ we have $10_{6j}, 9_{6j}$ both appear in stack L together and $13_{6j}, 12_{6j}$ appear together in stack R and stay there until 10_{6j} is output.

Now we consider the suffix

$$x_i y_i S_i = (10 \ 5 \ 9 \ 13 \ 12 \ 8 \ 14 \ 15 \ 11)_{6i}$$

where 10_{6i} , 9_{6i} are together in stack L. Lemma 5 tells us not only that 13_{6i} , 12_{6i} appear together in stack R and stay there until 10_{6i} is output, but that anything sitting underneath 10_{6i} in stack L comes before 9_{6i} in G_i , so in particular 14_{6i} , 15_{6i} are not underneath 10_{6i} . All possible processes to sort x_iy_iS are shown in Fig. 5. All possible sorting moves fail, which means G_i cannot be sorted.

The idea of the preceding proof can be summarised informally as follows. The prefix P forces 7,6 to be together in stack R, then Lemmas 4 and 5 alternately imply that the $10_{6j}, 9_{6j}$ terms of x_j must be in stack L and the $13_{6j}, 12_{6j}$ terms of y_j must be in stack R. When we reach the suffix S_i the fact that certain entries are forced to be in a particular stack means we are unable to sort the final terms. We now show that if a single entry is removed from G_i , we can choose to place the $10_{6j}, 9_{6j}$ terms in stack R and $13_{6j}, 12_{6j}$ terms in stack L, which allows the suffix to be sorted.

Lemma 7. Let $0 \le j \le i$. If stack R contains one or both of $10_{6j}, 9_{6j}$ in ascending order, and $y_j \dots y_i S_i$ is to be input as in Fig. 6, then there is a sorting procedure to output all remaining entries in order.

Proof. For j < i move 13_{6j} , 12_{6j} into stack L, output 8_{6j} , 9_{6j} , 10_{6j} , move $16_{6j} = 10_{6(j+1)}$ into stack R, output $11_{6j} = 5_{6(j+1)}$, output 13_{6j} , 12_{6j} from stack L and input $15_{6j} = 9_{6(j+1)}$ so that the configuration has the same form as Fig. 6 with j incremented by 1.

For j=i the remaining input is $(13\ 12\ 8\ 14\ 15\ 11)_{6j}$. Put $13_{6i}, 12_{6i}$ in stack L in order, output $8_{6i}, 9_{6i}, 10_{6i}$, put $14_{6i}, 15_{6i}$ in stack R and output $11_{6i}, 12_{6i}, 13_{6i}$, move 15_{6i} into stack L and output 14_{6i} then 15_{6i} .

If one of 9_{6j} , 10_{6j} is missing, use the same procedure ignoring the missing entry.

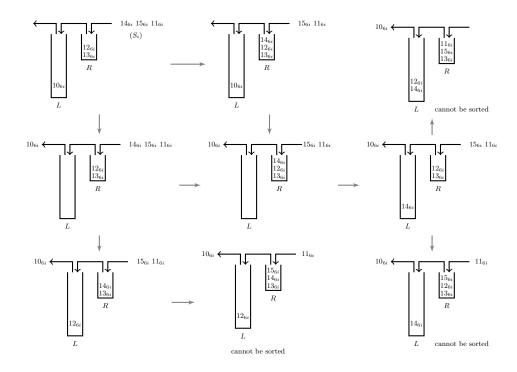


Fig. 5. All possible ways to sort $x_i y_i S$

Lemma 8. Let $0 \le j \le i$. If stack L contains one or both of 12_{6j} , 13_{6j} in ascending order, and $x_{j+1} \dots S_i$ (or just S_i if j = i) is to be input as in Fig. 7, then there is a sorting procedure to output all remaining entries in order.

Proof. If j < i move $10_{6(j+1)}$ into stack R, output $5_{6(j+1)}$, 12_{6j} , 13_{6j} , move $9_{6(j+1)}$ to stack R to reach the configuration in Fig. 6, which we can sort by Lemma 7. If j = i then the remaining input is just $S_i = (14\ 15\ 11)_{6i}$: move 14_{6i} , 15_{6i} to stack R, then output all entries.

If one of $12_{6j}, 13_{6j}$ is missing, use the same procedure ignoring the missing entry. \Box

Proposition 9. Let G'_i be a permutation obtained by removing a single entry from G_i . Then $G'_i \in S(3,\infty)$.

Proof. We give a deterministic procedure to sort G'_i . There are three cases depending on from where the entry is removed.

Term removed from P. Let P' be the factor P with one entry removed. We claim that there is a sorting sequence for $P'x_0$ which outputs the smallest six items in order and leaves 10, 9 in stack R. To show this we simply consider all cases.

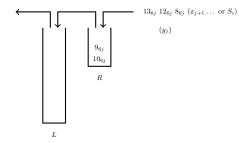


Fig. 6. A sortable configuration

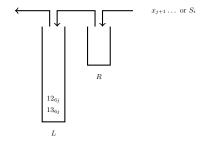


Fig. 7. Another sortable configuration

- 1. If 1 is removed, 2,4,3 can be output in order, then 7,6 placed in stack L, 10 in stack R, then 5,6,7 output, and 9 placed on top of 10 in stack R.
- 2. If 2, 3, or 4 are removed, write P' = ab761 with $a, b \in \{2, 3, 4\}$. Place a, b in stack R, move 7, 6 into stack L, output 1, then output a, b in the correct order, then move 10 into stack R, output 5, 6, 7 and move 9 into stack R.
- 3. If 6 or 7 is removed, write P' = 243a1 with $a \in \{7,6\}$. Place 4, 3, 2 in stack L in order, move a into stack R, output 1 then 2, 3, 4, then move a into stack L, move 10 into stack R, output 5, a and move 9 into stack R.

Thus after inputting $P'x_0$ we have the configuration shown in Fig. 6 with j = 0, which we can sort by Lemma 7.

Term removed from $x_s, 0 \le s \le i$.

Input P leaving 6, 7 in stack R, which brings us to the configuration in Fig. 8 with j=0. Now assume we have input $P \dots x_{j-1}y_{j-1}$ with $j \leq s$ (note the convention that x_{-1}, y_{-1} are empty) and the configuration is as in Fig. 8.

If j < s we can input $x_j y_j$ into the stacks to arrive at the same configuration with j incremented by 1, as follows: move 10_{6j} to stack L, output 5_{6j} , $6_{6j} = 12_{6(j-1)}$, $7_{6j} = 13_{6(j-1)}$, move 9_{6j} to stack L, move 13_{6j} , 12_{6j} to stack R, output 8_{6j} , 9_{6j} , 10_{6j} .

If j = s, we proceed as follows:

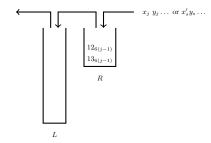


Fig. 8. Configuration after $P \dots x_{j-1} y_{j-1}$ is input

- 1. If 5_{6s} removed, output $6_{6s} = 12_{6(s-1)}$, $7_{6s} = 12_{6(s-1)}$, move 9_{6s} , 10_{6s} to stack R, to reach the configuration in Fig. 6 with j = s. From here the remaining entries can be sorted by Lemma 7.
- 2. If 10_{6s} is removed, output 5_{6s} , 6_{6s} , 7_{6s} and place 9_{6s} in stack R, to reach the configuration in Fig. 6 with j=s and 10_{6s} missing. From here the remaining entries can be sorted Lemma 7.
- 3. If 9_{6s} is removed, move 6_{6s} to stack L, move 10_{6s} on top of 7_{6s} in stack R, output 5_{6s} , 6_{6s} , move 13_{6s} , 12_{6s} into L, then output 8_{6s} , 10_{6s} . This gives the configuration in Fig. 7 with j=s. From here the remaining entries can be sorted by Lemma 8.

Term removed from $y_s, 0 \le s \le i$ or S_i . Input Px_0 to reach the configuration in Fig. 9 with j = 0: move 2, 3, 4 into stack L, 7, 6 to R, output 1, 2, 3, 4, move 10 into L, output 5, 6, 7 then move 9 into L.

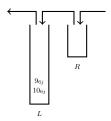


Fig. 9. Configuration after $Px_0y_0 \ldots x_j$ is input

Now suppose we have input $Px_0y_0 \ldots x_j$ to reach the configuration in Fig. 9. If no entry is removed from y_j and j < i then we can input y_jx_{j+1} to return to the configuration in Fig. 9 with j incremented by 1 as follows: move 13_{6j} , 12_{6j} to stack R, output 8_{6j} , 9_{6j} , 10_{6j} , move $10_{6(j+1)}$ to L, output $5_{6(j+1)} = 11_{6j}$, 12_{6j} , 13_{6j} , then move $9_{6(j+1)}$ to stack L.

If j = s (y_s is removed):

- 1. If 8_{6s} is removed, output 9_{6s} , 10_{6s} , move 13_{6s} , 12_{6s} to stack L to reach the configuration in Fig. 7, from which the remaining entries can be sorted by Lemma 8.
- 2. If $b \in \{13_{6s}, 12_{6s}\}$ is removed, place b in stack R, output $8_{6s}, 9_{6s}, 10_{6s}$, move b to stack L to reach the configuration in Fig. 7 with one of $12_{6s}, 13_{6s}$ removed, from which the remaining entries can be sorted a by Lemma 8.

If j = i and the entry is removed from S_i , sort the remaining entries as follows:

- 1. If 11_{6i} is removed, place 13_{6i} , 12_{6i} into stack R, output 8_{6i} , 9_{6i} , 10_{6i} , then 12_{6i} , 13_{6i} , 14_{6i} , 15_{6i} .
- 2. If $b \in \{14_{6i}, 15_{6i}\}$ is removed, place $13_{6i}, 12_{6i}$ into stack R, output $8_{6i}, 9_{6i}, 10_{6i}$, move 12_{6i} into stack L, place b on top of 13_{6i} in stack R, output 11_{6i} then 12_{6i} , move b into stack L, output 13_{6i} then b.

Theorem 10. The set of permutations that can be sorted by a stack of depth 3 and an infinite stack in series has an infinite basis.

Proof. Proposition 6 shows that each G_i cannot be sorted, and Proposition 9 shows that no G_i can contain G_j for $j \neq i$ as a subpermutation since any subpermutation of G_i can be sorted. Thus $\mathscr{G} = \{G_i \mid i \in \mathbb{N}\}$ is an infinite antichain in the basis for $S(3, \infty)$.

4 From finite to infinitely based

Let \mathscr{B}_t be the basis for $S(t, \infty)$ for $t \in \mathbb{N}_+$. Modifying Lemma 1 in [7] for the sorting case, we have the following:

Lemma 11. If $\sigma \in \mathcal{B}_t$ has length n then either σ or $(213)_n \sigma$ belongs to \mathcal{B}_{t+1} .

Proof. If $\sigma \notin S(t+1,\infty)$ then since $\sigma \in \mathcal{B}_t$, deleting any entry gives a permutation in $S(t,\infty) \subseteq S(t+1,\infty)$, so $\sigma \in \mathcal{B}_{t+1}$. Else $\sigma \in S(t+1,\infty)$. In any sorting process for $(213)_n \sigma$ the entries $1_n, 2_n, 3_n$ cannot appear together in stack L, so at least one entry must remain in stack R which means we must sort σ with stack R of depth at most t, which is not possible, so $(213)_n \sigma$ cannot be sorted. If we remove an entry of the prefix then the two entries $a, b \in \{1_n, 2_n, 3_n\}$ can be placed in stack L in order, leaving stack R depth t+1 so the permutation can be sorted, and if an entry is removed from σ then since $\sigma \in \mathcal{B}_t$ it can be sorted with R having one space occupied.

Theorem 12. The set of permutations that can be sorted using a stack of depth $t \in \mathbb{N}_+$ and an infinite stack in series is finitely based if and only if $t \in \{1, 2\}$.

Proof. We have $|\mathscr{B}_1| = 1$ and $|\mathscr{B}_2| = 20$ [11,7]. Theorem 10 shows that \mathscr{B}_3 is infinite. Lemma 11 implies if \mathscr{B}_t is infinite then so is \mathscr{B}_{t+1} .

A small modification of Propositions 6 and 9 shows that for $t \geq 4$ the set $\mathcal{G}_t = \{G_{i,t}\}$, where $G_{i,t} = P(x_0y_0) \dots (x_iy_i)(14\ 15\ 16\ \dots\ 12_t\ 11)_{6i}$, is an explicit antichain in the basis of $S(t,\infty)$. Details can be seen in [10].

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