On Periodicity Lemma for Partial Words^{*}

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Abstract

We investigate the function L(h, p, q), called here the *threshold function*, related to periodicity of partial words (words with holes). The value L(h, p, q) is defined as the minimum length threshold which guarantees that a natural extension of the periodicity lemma is valid for partial words with hholes and (strong) periods p, q. We show how to evaluate the threshold function in $\mathcal{O}(\log p + \log q)$ time, which is an improvement upon the best previously known $\mathcal{O}(p+q)$ -time algorithm. In a series of papers, the formulae for the threshold function, in terms of p and q, were provided for each fixed $h \leq 7$. We demystify the generic structure of such formulae, and for each value h we express the threshold function in terms of a piecewise-linear function with $\mathcal{O}(h)$ pieces.

1 Introduction

Consider a word X of length |X| = n, with its positions numbered 0 through n - 1. We say that X has a period p if X[i] = X[i + p] for all $0 \le i < n - p$. In this case, the prefix P = X[0..p - 1] is called a string period of X. Our work can be seen as a part of the quest to extend Fine and Wilf's Periodicity Lemma [11], which is a ubiquitous tool of combinatorics on words, into partial words.

Lemma 1.1 (Periodicity Lemma [11]). If p, q are periods of a word X of length $|X| \ge p + q - \text{gcd}(p,q)$, then gcd(p,q) is also a period of X.

A partial word is a word over the alphabet $\Sigma \cup \{\diamondsuit\}$, where \diamondsuit denotes a hole (a don't care symbol). In what follows, by n we denote the length of the partial word and by h the number of holes. For $a, b \in \Sigma \cup \{\diamondsuit\}$, the relation of matching \approx is defined so that $a \approx b$ if a = b or either of these symbols is a hole. A (solid) word P of length p is a *string period* of a partial word X if $X[i] \approx P[i \mod p]$ for $0 \le i < n$. In this case, we say that the integer p is a *(strong) period* of X.

We aim to compute the optimal thresholds L(h, p, q) which make the following generalization of the periodicity lemma valid:

Lemma 1.2 (Periodicity Lemma for Partial Words). If X is a partial word with h holes with periods p, q and $|X| \ge L(h, p, q)$, then gcd(p, q) is also a period of X.

If $gcd(p,q) \in \{p,q\}$, then Lemma 1.2 trivially holds for each partial word X. Otherwise, as proved by Fine and Wilf [11], the threshold in Lemma 1.1 is known to be optimal, so L(0, p, q) = p + q - gcd(p, q). *Example* 1.3. L(1, 5, 7) = 12, because:

- each partial word of length at least 12 with one hole and periods 5, 7 has also period 1 = gcd(5, 7),
- the partial word ababaababa of length 11 has periods 5, 7 and does not have period 1.

As our main aim, we examine the values L(h, p, q) as a function of p, q for a given h. Closed-form formulae for $L(h, \cdot, \cdot)$ with $h \leq 7$ were given in [2, 5, 22]. In these cases, L(h, p, q) can be expressed using a constant number of functions linear in p, q, and gcd(p,q). We discover a common pattern in such formulae which lets us derive a closed-form formula for L(h, p, q) with arbitrary fixed h using a sequence of $\mathcal{O}(h)$ fractions. Our construction relies on the theory of continued fractions; we also apply this link to describe L(h, p, q) in terms of standard Sturmian words.

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	h		L(h, 5,	7)	1	exai										
	0		11				abal										
	1	12				abal											
	2			16 ababaababa $\Diamond \Diamond$ aba													
	3		19				aaaa										
	4		21				aba $\Diamond\Diamond$ ababaababa $\Diamond\Diamond$ aba										
	5			25			aaaa	abaaa	aa⊘a	ı⊘aa	i⊘aa	a⊘⊘	>aaaa	3			
<i>r</i>	i:	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
H(n, 5, 7)):	0	1	2	2	2	2	3	3	3	4	4	5	5	5	5	6

Table 1: The optimal non-unary partial words with periods 5,7 and h = 0, ..., 5 holes (of length L(h, 5, 7) - 1) and the values H(n, 5, 7) for n = 10, ..., 25.

As an intermediate step, we consider a dual holes function H(n, p, q), which gives the minimum number of holes h for which there is a partial word of length n with h holes and periods p, q which do not satisfy Lemma 1.2.

Example 1.4. We have H(11, 5, 7) = 1 because:

- $H(11, 5, 7) \ge 1$: due to the classic periodicity lemma, every solid word of length 11 with periods 5 and 7 has period $1 = \gcd(5, 7)$, and
- $H(11, 5, 7) \leq 1$: ababaababa \Diamond is non-unary, has one hole and periods 5, 7.

We have $H(12,5,7) \leq H(11,5,7) + 1 = 2$ since appending \diamond preserves periods. In fact H(12,5,7) = H(15,5,7) = 2. However, there is no non-unary partial word of length 16 with 2 holes and periods 5, 7, so L(2,5,7) = 16; see Table 1.

For a function f(n, p, q) monotone in n, we define its generalized inverse as:

$$f(h, p, q) = \min\{n : f(n, p, q) > h\}.$$

Observation 1.5. $L = \tilde{H}$.

As observed above, Lemma 1.2 becomes trivial if $p \mid q$. The case of $p \mid 2q$ is known to be special as well, but it has been fully described in [22]. Furthermore, it was shown in [5, 21] that the case of gcd(p,q) > 1 is easily reducible to that of gcd(p,q) = 1. We recall these existing results in Section 4, while in the other sections we assume that gcd(p,q) = 1 and p,q > 2.

Previous results The study of periods in partial words was initiated by Berstel and Boasson [2], who proved that L(1, p, q) = p + q. They also showed that the same bound holds for *weak* periods¹ p and q. Shur and Konovalova [22] developed exact formulae for L(2, p, q) and L(h, 2, q), and an upper bound for L(h, p, q). A formula for L(h, p, q) with small values h was shown by Blanchet-Sadri et al. [3], whereas for large h, Shur and Gamzova [21] proved that the optimal counterexamples of length L(h, p, q) - 1 belong to a very restricted class of *special arrangements*. The latter contribution leads to an $\mathcal{O}(p + q)$ -time algorithm for computing L(h, p, q). An alternative procedure with the same running time was shown by Blanchet-Sadri et al. [5], who also stated closed-form formulae for L(h, p, q) with $h \leq 7$. Weak periods were further considered in [4, 6, 23].

Other known extensions of the periodicity lemma include a variant with three [8] and an arbitrary number of specified periods [13, 24], the so-called new periodicity lemma [1, 10], a periodicity lemma for repetitions with morphisms [17], extensions into abelian [9] and k-abelian [14] periodicity, into abelian periodicity for partial words [7], into bidimensional words [18], and other variations [12, 19].

Our results First, we show how to compute L(h, p, q) using $\mathcal{O}(\log p + \log q)$ arithmetic operations, improving upon the state-of-the-art complexity $\mathcal{O}(p+q)$.

¹An integer p is a weak period of X if $X[i] \approx X[i+p]$ for all $0 \le i < n-p$.

Furthermore, for any fixed h in $\mathcal{O}(h \log h)$ time we can compute a compact description of the threshold function L(h, p, q). For the base case of p < q, gcd(p, q) = 1, and h , the representation ispiecewise linear in <math>p and q. More precisely, the interval [0, 1] can be split into $\mathcal{O}(h)$ subintervals I so that L(h, p, q) restricted to $\frac{p}{q} \in I$ is of the form $a \cdot p + b \cdot q + c$ for some integers a, b, c.

Overview of the paper We start by introducing two auxiliary functions H^s and H^d which correspond to two restricted families of partial words. Our first key step is to prove that the value H(n, p, q) is always equal to $H^s(n, p, q)$ or $H^d(n, p, q)$ and to characterize the arguments n for which either case holds. The final function L is then obtained as a combination of the generalized inverses L^s and L^d of H^s and H^d , respectively. Developing the closed-form formula for L^d requires considerable effort; this is where continued fractions arise.

2 Functions H^s and L^s

For relatively prime integers $p, q, 1 , and an integer <math>n \ge q$, let us define

$$H^{s}(n, p, q) = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor.$$

We shall prove that $H(n, p, q) \leq H^s(n, p, q)$ for a suitable range of lengths n.

Fine and Wilf [11] constructed a word of length p + q - 2 with periods p and q and without period 1. For given p, q we choose such a word $S_{p,q}$ and, we define a partial word $W_{p,q}$ as follows, setting $k = \lfloor q/p \rfloor$ (see Fig. 1):

$$W_{p,q} = (S_{p,q}[0..p-3]\diamondsuit)^k \cdot S_{p,q} \cdot (\diamondsuit S_{p,q}[q..q+p-3])^k$$

Example 2.1. For p = 5 and q = 7, we can take $S_{5,7} =$ ababaababa and

$$W_{5,7} = aba \diamondsuit \diamondsuit ababaababa \diamondsuit \diamondsuit aba.$$

This partial word has length 20 and 4 holes. Hence, $H(20,5,7) \le 4 = H^s(20,5,7)$ and $L(4,5,7) \ge 21$. In fact, these bounds are tight; see Table 1.

Intuitively, the partial word $W_{p,q}$ is an extension of $S_{p,q}$ preserving the period p, in which a small number of symbols is changed to holes to guarantee the periodicity with respect to q.

Lemma 2.2. The partial word $W_{p,q}$ has periods p and q.

Proof. Let $n = |W_{p,q}|$. It is easy to observe that p is a period of $W_{p,q}$. We now show that q is a period of $W_{p,q}$ as well. Let X and Y be the prefix and the suffix of $W_{p,q}$ of length $p \lfloor q/p \rfloor$ (so that $W_{p,q} = X \cdot S_{p,q} \cdot Y$). Note that $|X|, |Y| < q \leq |S_{p,q}|$.

Let us start by showing that $W_{p,q}[i] \approx W_{p,q}[i+q]$ for $0 \leq i < n-q$. First, suppose that $W_{p,q}[i]$ is contained in X. The claim is obvious if $i \mod p \geq p-2$, because in this case we have $W_{p,q}[i] = \Diamond$. Otherwise

$$W_{p,q}[i] = S_{p,q}[i \mod p] \stackrel{(1)}{=} S_{p,q}[i \mod p+q] \stackrel{(2)}{=} S_{p,q}[i+q-\lfloor \frac{q}{p} \rfloor p] = W_{p,q}[i+q],$$

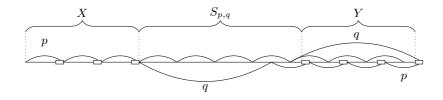


Figure 1: The structure of the partial word $W_{p,q} \diamond \diamond = X \cdot S_{p,q} \cdot Y \diamond \diamond$ for $\lfloor q/p \rfloor = 3$. Tiny rectangles correspond to two holes $\diamond \diamond$. We have $|X| = |Y| = p \lfloor q/p \rfloor = 3p$ and $|W_{p,q}| = p + q + 2p \lfloor q/p \rfloor - 2 = q + 7p - 2$. There are $4 \cdot \lfloor q/p \rfloor = 12$ holes.

where (1) follows from the fact that $S_{p,q}$ has period q and $i \mod p < p-2$, and (2) from the fact that $S_{p,q}$ has period p. By symmetry of our construction, we also have $W_{p,q}[i] \approx W_{p,q}[i+q]$ if $W_{p,q}[i+q]$ is contained in Y. In the remaining case, $W_{p,q}[i]$ and $W_{p,q}[i+q]$ are both contained in $S_{p,q}$, which yields $W_{p,q}[i+q] = W_{p,q}[i]$.

Next, we claim that $W_{p,q}[i] \approx W_{p,q}[i+kq]$ for every $k \geq 2$. Observe that $W_{p,q}[i+q], \ldots, W_{p,q}[i+(k-1)q]$ are contained in $S_{p,q}$ and thus they are equal solid symbols. Hence,

$$W_{p,q}[i] \approx W_{p,q}[i+q] = \dots = W_{p,q}[i+(k-1)q] \approx W_{p,q}[i+kq].$$

The intermediate symbols are solid, so this implies $W_{p,q}[i] \approx W_{p,q}[i+kq]$, as claimed. Consequently, q is indeed a period of $W_{p,q}$.

We use the word $S_{p,q}$ and the partial word $W_{p,q} \diamond \diamond$ to show that H^s is an upper bound for H for all intermediate lengths n $(|S_{p,q}| \leq n \leq |W_{p,q} \diamond \diamond|)$.

Lemma 2.3. Let $1 be relatively prime integers. For each length <math>p+q-2 \le n \le p+q+2p \lfloor q/p \rfloor$, we have $H(n, p, q) \le H^s(n, p, q)$.

Proof. We extend $S_{p,q}$ to $W_{p,q} \diamond \diamond$ symbol by symbol, first prepending the characters before $S_{p,q}$, and then appending the characters after $S_{p,q}$. By Lemma 2.2, the resulting partial word has periods p and q because it is contained in $W_{p,q} \diamond \diamond$. Moreover, it is not unary because it contains $S_{p,q}$.

A hole is added at the first two iterations among every p iterations. Hence, the total number of holes is as claimed:

$$\left\lceil \frac{n-|S_{p,q}|}{p} \right\rceil + \left\lceil \frac{n-|S_{p,q}|-1}{p} \right\rceil = \left\lfloor \frac{n-q+1}{p} \right\rfloor + \left\lfloor \frac{n-q}{p} \right\rfloor = H^s(n,p,q),$$

because $\left\lceil \frac{x}{p} \right\rceil = \left\lfloor \frac{x+p-1}{p} \right\rfloor$ for every integer x.

Finally, the function $L^s = \widetilde{H^s}$ is very simple and easily computable.

Lemma 2.4. If $h \ge 0$ is an integer, then $L^s(h, p, q) = \left\lceil \frac{h+1}{2} \right\rceil p + q - (h+1) \mod 2$.

Proof. We have to determine the smallest n such that $\left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor = h+1$. There are two cases, depending on parity of h:

Case 1: h = 2k. In this case $\left\lfloor \frac{n-q}{p} \right\rfloor = k$ and $\left\lfloor \frac{n-q+1}{p} \right\rfloor = k+1$. Hence, n-q+1 = p(k+1), i.e., $n = p(k+1) + q - 1 = \left\lceil \frac{h+1}{2} \right\rceil p + q - (h+1) \mod 2$.

Case 2: h = 2k + 1. In this case $\left\lfloor \frac{n-q}{p} \right\rfloor = k+1$ and $\left\lfloor \frac{n-q+1}{p} \right\rfloor = k+1$. Hence, n-q = p(k+1), i.e., $n = p(k+1) + q = \left\lceil \frac{h+1}{2} \right\rceil p + q - (h+1) \mod 2$.

3 Functions H^d and L^d

In this section, we study a family of partial words corresponding to the *special arrangements* introduced in [21]. For relatively prime integers p, q > 1, we say that a partial word S of length $n \ge \max(p, q)$ is (p, q)-special if it has a position l such that for each position i:

$$S[i] = \begin{cases} \mathsf{a} & \text{if } p \nmid (l-i) \text{ and } q \nmid (l-i), \\ \mathsf{b} & \text{if } p \mid (l-i) \text{ and } q \mid (l-i), \\ \diamondsuit & \text{otherwise.} \end{cases}$$

Let $H^d(n, p, q)$ be the minimum number of holes in a (p, q)-special partial word of length n.

Fact 3.1. For each $n \ge \max(p,q)$, we have $H(n,p,q) \le H^d(n,p,q)$.

Proof. Observe that every (p, q)-special partial word has periods p and q. However, due to p, q > 1, it does not have period 1 = gcd(p, q).

	0																				
$\widetilde{G}(h,5,7)$	5	7	10	14	15	20	21	25	28	30	40	42	45	49	50	55	56	60	63	65	75
$L^{d}(h, 5, 7)$	10	12	15	19	21	25	28	30	34	35	45	47	50	54	56	60	63	65	69	70	80

Table 2: Functions \widetilde{G} and L^d for p = 5, q = 7, and $h = 0, \ldots, 20$. By Lemma 3.3, we have, for example, $L^d(8,5,7) = \max(\widetilde{G}(0,5,7) + \widetilde{G}(8,5,7), \ldots, \widetilde{G}(4,5,7) + \widetilde{G}(4,5,7)) = \max(5+28, 7+25, 10+21, 14+20, 15+15) = 34.$

To derive a formula for $H^d(n, p, q)$, let us introduce an auxiliary function G, which counts integers $i \in \{1, \ldots, n\}$ that are multiples of p or of q but not both:

$$G(n, p, q) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor - 2 \left\lfloor \frac{n}{pq} \right\rfloor.$$

The function H^d can be characterized using G, while the generalized inverse $L^d = \widetilde{H^d}$ admits a dual characterization in terms of \widetilde{G} ; see also Table 2.

Lemma 3.3. Let p, q > 1 be relatively prime integers.

- (a) If $n \ge \max(p,q)$, then $H^d(n,p,q) = \min_{l=0}^{n-1} (G(l,p,q) + G(n-l-1,p,q)).$
- (b) If $h \ge 0$, then $L^d(h, p, q) = \max_{k=0}^h \left(\widetilde{G}(k, p, q) + \widetilde{G}(h k, p, q) \right)$.

Proof. Let S be a (p,q)-special partial word of length n with h holes, k of which are located to the left of position l. Observe that k = G(l, p, q) (so $l + 1 \leq \tilde{G}(k, p, q)$) and h - k = G(n - l - 1, p, q) (so $n - l \leq \tilde{G}(h - k, p, q)$). Hence, h = G(l, p, q) + G(n - l - 1, p, q) and $n + 1 \leq \tilde{G}(k, p, q) + \tilde{G}(h - k, p, q)$. The claimed equalities follow from the fact that these bounds can be attained for each l and k, respectively.

4 Characterizations of *H* and *L*

Shur and Gamzova in [21] proved that $H(n, p, q) = H^d(n, p, q)$ for $n \ge 3q + p$. In this section, we give a complete characterization of H in terms of H^d and H^s , and we derive an analogous characterization of L in terms of L^d and L^s . Our proof is based on a graph-theoretic approach similar to that in [5].

Let us define the (n, p, q)-graph $\mathbf{G} = (V, E)$ as an undirected graph with vertices $V = \{0, \ldots, n-1\}$. The vertices i and j are connected if and only if $p \mid (j - i)$ or $q \mid (j - i)$. Observe that H(n, p, q) is the minimum size of a vertex separator in \mathbf{G} , i.e., the minimum number of vertices to be removed from \mathbf{G} so that the resulting graph is no longer connected; see Fig. 2.

We say that an edge (i, j) of the (n, p, q)-graph is a *p*-edge if $p \mid (j - i)$ and a *q*-edge if $q \mid (j - i)$. The set of all nodes giving the same remainder modulo p (modulo q) is called a *p*-class (*q*-class, respectively). Each *p*-class and each *q*-class forms a clique in the (n, p, q)-graph.

Fact 4.1 (see [5]). Let 1 be relatively prime integers. If <math>n < pq, then $H^d(n, p, q)$ is the minimal degree of a vertex in the (n, p, q)-graph.

Proof. Observe that vertex number l has G(l, p, q) neighbors i < l and G(n - l - 1, p, q) neighbors i > l. Consequently, by Lemma 3.3, $H^d(n, p, q) = \min_{l=0}^{n-1} (G(l, p, q) + G(n - l - 1, p, q)) = \min_{l=0}^{n-1} \deg_{\mathbf{G}}(l)$.

Let $\mathbf{G} = (V, E)$ be the (n, p, q)-graph. For each $i \in \{0, \ldots, p-1\}$ let C_i be the *p*-class containing the vertex *i*; see Fig. 2. We slightly abuse the notation and use arbitrary integers for indexing the *p*-classes: $C_i = C_{i \mod p}$ for $i \in \mathbb{Z}$. We denote by E_i the set of *q*-edges of the form (j, j + q) for $j \in C_i$. Let us start with two auxiliary facts.

Fact 4.2. Let 1 be relatively prime integers.

- (a) For $j \in \{0, ..., p-1\}$, we have $|E_j| = \left|\frac{n-j-q}{p}\right|$.
- (b) $H^{s}(n, p, q) = |E_{p-1}| + |E_{p-2}| = \min_{i \neq j} (|E_{i}| + |E_{j}|).$

Figure 2: The structure of the (20, 5, 7)-graph. Each 5-clique C_i is actually a clique; same applies for the vertical 7-cliques. The 5-clique C_2 is repeated to show the cyclicity. The set $U = \{3, 4, 5, 7\}$ of encircled vertices is a minimum-size vertex separator. It corresponds to the partial word $W_{5,7}$ from Example 2.1: holes of $W_{5,7}$ are located at positions $i \in U$, the positions i with $W_{5,7}[i] = b$ form a connected component $(C_1 \cup C_4) \setminus U$, while the positions *i* with $W_{5,7}[i] = a$ form a connected component $(C_0 \cup C_2 \cup C_3) \setminus U$.

Proof. Let i = kp + j, where $0 \le j < p$. There is a q-edge (i, i + q) if and only if

$$kp + j + q \le n - 1$$
, so $k \le \left\lfloor \frac{n - 1 - j - q}{p} \right\rfloor$.

The number of such values of k is $\left\lfloor \frac{n-1-j-q}{p} \right\rfloor + 1 = \left\lceil \frac{n-j-q}{p} \right\rceil$. As for the second statement of the fact, we have:

$$|E_j| \ge |E_{p-1}| = \left\lceil \frac{n-p+1-q}{p} \right\rceil = \left\lfloor \frac{n-q}{p} \right\rfloor$$

for $0 \le j < p$ and, similarly, $|E_j| \ge |E_{p-2}| = \left\lfloor \frac{n-q+1}{p} \right\rfloor$ for $0 \le j < p-1$.

Fact 4.3. Let U be a vertex separator in the (n, p, q)-graph $\mathbf{G} = (V, E)$ and let $\mathbf{G}' = \mathbf{G} \setminus U$. One can color the vertices of G in two colors so that every edge in \mathbf{G}' and every p-class in \mathbf{G} is monochromatic, but \mathbf{G}' is not monochromatic.

Proof. Recall that each p-class C_i is a clique in **G**, so $C_i \setminus U$ is still a clique in **G**'. We distinguish a connected component M of \mathbf{G}' and color the vertices of C_i depending on whether $C_i \setminus U \subseteq M$. It is easy to verify that this coloring satisfies the claimed conditions.

The following lemma provides lower bounds on H(n, p, q).

Lemma 4.4. Let 1 be relatively prime integers.

- (1) If n < 2q, then $H(n, p, q) \ge H^s(n, p, q)$.
- (2) If $p \ge 3$ and $n \ge 2q$, then $H(n, p, q) \ge \min(H^d(n, p, q), H^s(n, p, q))$.
- (3) If $p \ge 5$ and $n \ge 4q$, then $H(n, p, q) \ge \min(H^d(n, p, q), H^s(n, p, q) + 1)$.

Proof. Let U be a minimum-size vertex separator of hte (n, p, q)-graph $\mathbf{G} = (V, E)$; recall that |U| =H(n, p, q). Let us fix a coloring of **G** using colors $\{A, B\}$ satisfying Fact 4.3; without loss of generality we assume that the number of p-classes with color A is at least the number of p-classes with color B. We have the following two cases.

Case a: Exactly one *p*-class has color *B*. Let C_j be the unique *p*-class with color *B*. By definition, the edges in $E_{j-q} \cup E_j$ are bichromatic. If n < 2q, then all the q-edges form a matching in **G**. In particular, in order to disconnect **G**, we need to remove at least one endpoint of each edge in $E_{j-q} \cup E_j$. Hence, $H(n, p, q) \ge |E_{j-q}| + |E_j| \ge H^s(n, p, q)$, where the second inequality follows from Fact 4.2(b). This concludes the proof of (1) in this case.

Now assume that $n \ge 2q$ and $p \ge 3$. We will show that $H(n, p, q) \ge H^d(n, p, q)$ holds in this case. Consider any q-class D and let k be its size; we have $k \ge \lfloor \frac{n}{q} \rfloor \ge 2$. In this q-class, every p-th element has color B. Let $\#_A(D)$ and $\#_B(D)$ denote the number of vertices in D colored with A and B, respectively. Then:

$$\#_B(D) \le \left\lceil \frac{k}{p} \right\rceil \le \left\lceil \frac{k}{3} \right\rceil = \left\lfloor \frac{k+2}{3} \right\rfloor \le \left\lfloor \frac{2k}{3} \right\rfloor = k - \left\lceil \frac{k}{3} \right\rceil \le k - \left\lceil \frac{k}{p} \right\rceil \le \#_A(D).$$

The set U contains all B-colored vertices or all A-colored vertices of every q-class D, as otherwise there would be a non-monochromatic edge in \mathbf{G}' connecting two vertices of $D \setminus U$, contradicting Fact 4.3. At least one vertex of \mathbf{G}' is B-colored, so in at least one q-class, U must contain all A-colored vertices; assume that this is the q-class D_0 . Consequently,

$$\begin{split} |U| &= \sum_{D:q\text{-class}} |U \cap D| \geq \#_A(D_0) + \sum_{D:q\text{-class}, \ D \neq D_0} \#_B(D) = \\ |D_0| &+ \sum_{D:q\text{-class}} \#_B(D) - 2\#_B(D_0) = |D_0| + |C_j| - 2|D_0 \cap C_j| \geq H^d(n, p, q). \end{split}$$

The last inequality follows from Fact 4.1. This concludes (2) and (3) in this case.

Case b: There are at least two *p*-classes with each color. In particular, $p \ge 4$. We consider two subcases based on the colors c_i of classes C_i . In each case we will show that H(n, p, q) is bounded from below by $H^s(n, p, q)$ or $H^s(n, p, q) + 1$.

First, suppose that there is exactly one *p*-class C_i such that $c_i = A$ and $c_{i+q} = B$. Equivalently, there is exactly one *p*-class C_j such that $c_j = B$ and $c_{j+q} = A$. Since there are at least two *p*-classes with each color, $c_{i+2q} = B$ and $c_{j+2q} = A$, so $C_{i+q} \neq C_j$ and $C_{j+q} \neq C_i$. This means that $E_i \cup E_j$ forms a bichromatic matching in **G**. Consequently,

$$H(n, p, q) \ge |E_i| + |E_j| \ge H^s(n, p, q).$$

This concludes the proof of (1) and (2) in the current subcase.

For the proof of (3), observe that $p \ge 5$ and the choice of A as the more frequent color yields that $c_{j+3q} = A$, so C_{j+2q} is distinct from C_i . Hence, we can extend the matching $E_i \cup E_j$ with an edge (x, y) where $x = (j-q) \mod p \in C_{j-q}$ and $y = x + 3q \in C_{j+2q}$. This edge exists because $n \ge 4q > p + 3q > y$. It forms a matching with $E_i \cup E_j$ because no edge in $E_i \cup E_j$ is incident to C_{j+2q} , while the only edges incident to C_{j-q} could be the edges in E_i provided that $C_i = C_{j-2q}$. However, x < q, so x is not an endpoint of any edge in E_{j-2q} . This concludes the proof of (3) in the current subcase.

Let us proceed to the second subcase. Let C_i, C_j be two distinct *p*-classes such that $c_i = c_j = A$ and $c_{i+q} = c_{j+q} = B$. It is easy to see that $E_i \cup E_j$ forms a bichromatic matching in **G**, so $H(n, p, q) \ge |E_i| + |E_j| \ge H^s(n, p, q)$. Thus, it remains to prove (3) in this subcase.

If $p \ge 5$, then there is a third *p*-class C_k with color *A*. Moreover, we may choose *k* so that $c_{k-q} = B$ and $c_k = A$. We extend $E_i \cup E_j$ with an edge (x, y) where $x = (k-q) \mod p \in C_{k-q}$ and $y = x+q \in C_k$. This edge exists because n > q + p > y. It forms a matching with $E_i \cup E_j$ because no edge in $E_i \cup E_j$ is incident to C_k , while the only edges incident to C_{k-q} might be the edges in E_i or E_j provided that i = k - 2q or j = k - 2q. However, x < q, so *x* is not an endpoint of any edge in E_{k-2q} . This concludes the proof of (3) in the current subcase, and the proof of the entire lemma.

Lemma 4.5. If $n \ge q$, then

$$\left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{n}{p} \right\rfloor - 2 \left\lceil \frac{n}{pq} \right\rceil \le H^d(n, p, q) \le \left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor - 1.$$

Proof. Recall that $H^d(n, p, q) = \min_{l=0}^{n-1} (G(l, p, q) + G(n - l - 1, p, q))$ due to Lemma 3.3. The first part of the claim holds because for $0 \le l < n$ we have:

$$\begin{split} G(l,p,q) + G(n-l-1,p,q) &= \left\lfloor \frac{l}{p} \right\rfloor + \left\lfloor \frac{l}{q} \right\rfloor - 2 \left\lfloor \frac{l}{pq} \right\rfloor + \left\lfloor \frac{n-l-1}{p} \right\rfloor + \left\lfloor \frac{n-l-1}{q} \right\rfloor - 2 \left\lfloor \frac{n-l-1}{pq} \right\rfloor = \\ & \left\lceil \frac{l-p+1}{p} \right\rceil + \left\lceil \frac{l-q+1}{q} \right\rceil - 2 \left\lfloor \frac{l}{pq} \right\rfloor + \left\lceil \frac{n-l-p}{p} \right\rceil + \left\lfloor \frac{n-l-q}{q} \right\rfloor - 2 \left\lfloor \frac{n-l-1}{pq} \right\rfloor \ge \\ & \left\lceil \frac{n-2p+1}{p} \right\rceil + \left\lceil \frac{n-2q+1}{q} \right\rceil - 2 \left\lfloor \frac{n-1}{pq} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor - 1 + \left\lfloor \frac{n}{q} \right\rfloor - 1 - 2 \left\lceil \frac{n}{pq} \right\rceil + 2 = \left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{n}{p} \right\rfloor - 2 \left\lceil \frac{n}{pq} \right\rceil. \end{split}$$

As for the second part, due to $n \ge q$ we have:

$$\begin{aligned} H^{d}(n,p,q) \geq G(q-1,p,q) + G(n-q,p,q) &= \left\lfloor \frac{q-1}{p} \right\rfloor + \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-q}{q} \right\rfloor - 2 \left\lfloor \frac{n-q}{pq} \right\rfloor \leq \\ & \left\lfloor \frac{q-1}{p} \right\rfloor + \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor - 1, \end{aligned}$$
 hich completes he proof.

which completes he proof.

Theorem 4.6. Let p and q be relatively prime integers such that $2 . For each integer <math>n \ge p+q-2$, we have

$$H(n,p,q) = \begin{cases} H^s(n,p,q) & \text{if } n \leq q + p \lceil \frac{q}{p} \rceil - 1 \text{ or } 3q \leq n \leq q + 3p - 1, \\ H^d(n,p,q) & \text{otherwise.} \end{cases}$$

Moreover, for each integer $h \ge 0$:

$$L(h, p, q) = \begin{cases} L^s(h, p, q) & \text{if } \frac{q}{p} > \left\lceil \frac{h}{2} \right\rceil \text{ or } (h = 4 \text{ and } \frac{q}{p} < \frac{3}{2}) \\ L^d(h, p, q) & \text{otherwise.} \end{cases}$$

Proof. First, we prove the claim concerning H by analyzing several cases.

Case 0. $pq \leq n$.

By Fact 3.1, we have $H(n, p, q) \leq H^d(n, p, q)$. Moreover, using Lemma 4.5, we obtain:

$$\begin{split} H^s(n,p,q) &= \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor \geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-pq}{p} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor + \left\lfloor \frac{q(p-2)+2}{p} \right\rfloor \geq \\ &\geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor - p + \left\lfloor \frac{q-1}{p} \right\rfloor + \left\lfloor \frac{(p+1)(p-2)+2}{p} \right\rfloor = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor + \left\lfloor \frac{p^2-p-p^2}{p} \right\rfloor = \\ &= \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor - 1 \geq H^d(n,p,q). \end{split}$$

Finally, Lemma 4.4(2) yields that $H(n, p, q) \geq \min(H^d(n, p, q), H^s(n, p, q)) = H^d(n, p, q)$, which completes the proof.

Henceforth we assume that n < pq.

Case 1. $p + q - 1 \le n < 2q$.

We get $H(n, p, q) = H^s(n, p, q)$ directly from Lemma 4.4(1) and Lemma 2.3.

Case 2. $2q \le n \le q + \left\lceil \frac{q}{p} \right\rceil p - 1$.

Note that $n \leq q + \left\lceil \frac{q}{p} \right\rceil p - 1 = p + q + \left\lfloor \frac{q}{p} \right\rfloor p - 1 , so <math>H(n, p, q) \leq H^s(n, p, q)$ due to Lemma 2.3. Moreover, using Lemma 4.5, we obtain:

$$H^{s}(n,p,q) = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor \leq \left\lfloor \frac{\left\lceil \frac{q}{p} \right\rceil p-1}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor = \left\lceil \frac{q}{p} \right\rceil - 1 + \left\lfloor \frac{n-q+1}{p} \right\rfloor = \left\lfloor \frac{q-1}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor \leq \left\lfloor \frac{n}{p} \right\rfloor \leq \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor - 2 \leq H^{d}(n,p,q).$$

Finally, Lemma 4.4(2) yields that $H(n, p, q) \geq \min(H^d(n, p, q), H^s(n, p, q)) = H^s(n, p, q)$, which completes the proof.

Case 3. $q + \left\lceil \frac{q}{p} \right\rceil p - 1 \le n < 3q$.

By Fact 3.1, we have $H(n, p, q) \leq H^d(n, p, q)$. Moreover, using Lemma 4.5, we obtain:

$$H^{s}(n,p,q) = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor \geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{\left\lceil \frac{q}{p} \right\rceil p}{p} \right\rfloor = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lceil \frac{q}{p} \right\rceil = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor + 1 + \left\lfloor \frac{n}{q} \right\rfloor - 2 \geq H^{d}(n,p,q).$$

Finally, Lemma 4.4(2) yields that $H(n, p, q) \ge \min(H^d(n, p, q), H^s(n, p, q)) = H^d(n, p, q)$, which completes the proof.

Case 4. $3q \le n \le 3p + q - 1$.

Note that $n \leq 3p + q - 1 , so <math>H(n, p, q) \leq H^s(n, p, q)$ due to Lemma 2.3. Moreover, using Lemma 4.5, we obtain:

$$\begin{split} H^s(n,p,q) &= \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor \leq \left\lfloor \frac{3p-1}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor = 2 + \left\lfloor \frac{n-q+1}{p} \right\rfloor \leq 1 + \left\lfloor \frac{q-1}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor \leq \\ &\leq 1 + \left\lfloor \frac{n}{p} \right\rfloor = 3 + \left\lfloor \frac{n}{p} \right\rfloor - 2 \leq \left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{n}{p} \right\rfloor - 2 \leq H^d(n,p,q). \end{split}$$

Finally, Lemma 4.4(2) yields that $H(n, p, q) \ge \min(H^d(n, p, q), H^s(n, p, q)) = H^s(n, p, q)$, which completes the proof.

Case 5. $\max(3q, 3p + q - 1) \le n < 4q$ and p < q < 2p.

By Fact 3.1, we have $H(n, p, q) \leq H^d(n, p, q)$. Moreover, using Lemma 4.5, we obtain:

$$H^{s}(n,p,q) = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-q+1}{p} \right\rfloor \geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{3p}{p} \right\rfloor = \left\lfloor \frac{n-q}{p} \right\rfloor + 3 = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor - 1 \geq H^{d}(n,p,q).$$

Finally, Lemma 4.4(2) yields that $H(n, p, q) \ge \min(H^d(n, p, q), H^s(n, p, q)) = H^d(n, p, q)$, which completes the proof.

Case 6. $3q \leq n$ and q > 2p.

By Fact 3.1, we have $H(n, p, q) \leq H^d(n, p, q)$. Moreover, using Lemma 4.5, we obtain:

$$\begin{aligned} H^{s}(n,p,q) &\geq 2\left\lfloor \frac{n-q}{p} \right\rfloor \geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-3q}{p} \right\rfloor + 2\left\lfloor \frac{q}{p} \right\rfloor \geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-3q}{q} \right\rfloor + 2\left\lfloor \frac{q}{p} \right\rfloor \geq \\ &\geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor - 3 + \left\lfloor \frac{q-1}{p} \right\rfloor + 2 = \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor - 1 \geq H^{d}(n,p,q). \end{aligned}$$

Finally, Lemma 4.4(2) yields that $H(n, p, q) \ge \min(H^d(n, p, q), H^s(n, p, q)) = H^d(n, p, q)$, which completes the proof.

Case 7. $4q \le n$ and $p \ge 5$ (and q < 2p).

By Fact 3.1, we have $H(n, p, q) \leq H^d(n, p, q)$. Moreover, using Lemma 4.5, we obtain:

$$\begin{aligned} H^{s}(n,p,q) &= \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n+1-q}{p} \right\rfloor \geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-2q}{p} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor \geq \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n-2q}{q} \right\rfloor + \left\lfloor \frac{q-1}{p} \right\rfloor = \\ &= \left\lfloor \frac{n-q}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor - 2 + \left\lfloor \frac{q-1}{p} \right\rfloor \geq H^{d}(n,p,q) - 1. \end{aligned}$$

Finally, Lemma 4.4(3) yields that $H(n, p, q) \ge \min(H^d(n, p, q), H^s(n, p, q) + 1) = H^d(n, p, q)$, which completes the proof.

The only remaining case, that $4q \le n$ and p < 5, is a subcase of Case 0. This completes the proof of the formula for H(n, p, q).

The characterization of L(h, p, q) is relatively easy to derive from that of H(n, p, q). Recall that L, L^s , and L^d are generalized inverses of H, H^s , and H^d , respectively. Note that Cases 1. and 2. yield $H(n, p, q) = H^s(n, p, q)$ for $n \le q + p \lceil \frac{q}{p} \rceil - 1$ while Case 3. additionally implies

$$H^{d}(q + p\lceil \frac{q}{p} \rceil - 1, p, q) = H^{s}(q + p\lceil \frac{q}{p} \rceil - 1, p, q) = 2\lceil \frac{q}{p} \rceil - 1.$$

Consequently, $L(h, p, q) = L^s(h, p, q)$ if $h < 2\lceil \frac{q}{p} \rceil - 1$, i.e., if $\frac{q}{p} > \lceil \frac{h}{2} \rceil$. Moreover, if $\frac{3}{2} < \frac{q}{p}$, then 3q > q + 3p - 1, and therefore $H(n, p, q) = H^d(n, p, q)$ for $n \ge q + p\lceil \frac{q}{p} \rceil - 1$ due to Cases 0. and 5.-7. Hence, if $h \ge 2\lceil \frac{q}{p} \rceil - 1$, i.e., $\frac{3}{2} < \frac{q}{p} < \lceil \frac{h}{2} \rceil$, then $L(h, p, q) = L^d(h, p, q)$. Now, it suffices to consider the case of $\frac{q}{p} < \frac{3}{2} < \lceil \frac{h}{2} \rceil$. Then, by Cases 5., 7., and 0., $H(n, p, q) = L^d(n, p, q)$.

Now, it suffices to consider the case of $\frac{q}{p} < \frac{3}{2} < \lfloor \frac{n}{2} \rfloor$. Then, by Cases 5., 7., and 0., $H(n, p, q) = H^d(n, p, q)$ for $n \ge q+3p-1$. Case 4. additionally yields $H^d(q+3p-1, p, q) = H^s(q+3p-1, p, q) = 5$, so $L(h, p, q) = L^d(h, p, q)$ if $h \ge 5$. Moreover, by Case 4., $H(n, p, q) = H^d(n, p, q)$ for $q+p-1 \le n < 3q$, so $L(3, p, q) = L^d(3, p, q)$ due to $H^s(3q, p, q) \ge 4$. Finally, we note that Case 3. yields $H(n, p, q) = H^s(n, p, q)$ for $3q \le n \le 3p+q-1$, so $L(4, p, q) = L^s(4, p, q)$ due to $H(3q-1, p, q) \le H^s(3q-1, p, q) \le 4$.

The remaining cases have already been well understood:

Fact 4.7 ([21, 5]). If p, q > 1 are integers such that $gcd(p,q) \notin \{p,q\}$, then

$$L(h, p, q) = \operatorname{gcd}(p, q) \cdot L\left(h, \frac{p}{\operatorname{gcd}(p,q)}, \frac{q}{\operatorname{gcd}(p,q)}\right)$$

Fact 4.8 ([22]). If q, h are integers such that q > 2, $2 \nmid q$, and $h \ge 0$, then

$$L(h, 2, q) = (2p+1) \left\lfloor \frac{h}{p} \right\rfloor + h \mod p.$$

The results above lead to our first algorithm for computing L(h, p, q).

Corollary 4.9. Given integers p, q > 1 such that $gcd(p,q) \notin \{p,q\}$ and an integer $h \ge 0$, the value L(h, p, q) can be computed in $\mathcal{O}(h + \log p + \log q)$ time.

Proof. First, we apply Fact 4.7 to reduce the computation to L(h, p', q') such that gcd(p', q') = 1 and, without loss of generality, 1 < p' < q'. This takes $\mathcal{O}(\log p + \log q)$ time. If p' = 2, we use Fact 4.8, while for p' > 2 we rely on the characterization of Theorem 4.6, using Lemmas 2.4 and 3.3 for computing L^s and L^d , respectively. The values $\tilde{G}(h', p', q')$ form a sorted sequence of multiples of p' and q', but not of p'q'. Hence, it takes $\mathcal{O}(h)$ time to generate them for $0 \le h' \le h$. The overall running time is $\mathcal{O}(h + \log p + \log q)$.

5 Faster Algorithm for Evaluating L

A more efficient algorithm for evaluating L relies on the theory of continued fractions; we refer to [15] and [20] for a self-contained yet compact introduction. A finite continued fraction is a sequence $[\gamma_0; \gamma_1, \ldots, \gamma_m]$, where $\gamma_0, m \in \mathbb{Z}_{\geq 0}$ and $\gamma_i \in \mathbb{Z}_{\geq 1}$ for $1 \leq i \leq m$. We associate it with the following rational number:

$$[\gamma_0; \gamma_1, \dots, \gamma_m] = \gamma_0 + \frac{1}{\gamma_1 + \frac{1}{\ddots + \frac{1}{\gamma_m}}}.$$

Depending on the parity of m, we distinguish odd and even continued fractions. Often, an improper continued fraction $[;] = \frac{1}{0}$ is also introduced and assumed to be odd. Each positive rational number has exactly two representations as a continued fraction, one as an even continued fraction, and one as an odd continued fraction. For example, $\frac{5}{7} = [0; 1, 2, 2] = [0; 1, 2, 1, 1]$.

Consider a continued fraction $[\gamma_0; \gamma_1, \ldots, \gamma_m]$. Its *convergents* are continued fractions of the form $[\gamma_0; \gamma_1, \ldots, \gamma_{m'}]$ for $0 \le m' < m$, and $[;] = \frac{1}{0}$. The *semiconvergents* also include continued fractions of the form $[\gamma_0; \gamma_1, \ldots, \gamma_{m'-1}, \gamma'_{m'}]$, where $0 \le m' \le m$ and $0 < \gamma'_{m'} < \gamma_{m'}$. The two continued fractions representing a positive rational number have the same semiconvergents.

Example 5.1. The semiconvergents of $[0; 1, 2, 2] = \frac{5}{7} = [0; 1, 2, 1, 1]$ are $[;] = \frac{1}{0}, [0;] = \frac{0}{1}, [0;1] = \frac{1}{1}, [0;1,1] = \frac{1}{2}, [0;1,2] = \frac{2}{3}, \text{ and } [0;1,2,1] = \frac{3}{4}.$

Semiconvergents of $\frac{p}{q}$ can be generated using the (slow) continued fraction algorithm, which produces a sequence of Farey pairs $(\frac{a}{b}, \frac{c}{d})$ such that $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$.

Algorithm 1: Farey process for a rational number $\frac{p}{q} > 0$
$\left(\frac{a}{b}, \frac{c}{d}\right) := \left(\frac{0}{1}, \frac{1}{0}\right);$
while true do
Report a Farey pair $(\frac{a}{b}, \frac{c}{d});$
Report a Farey pair $(\frac{a}{b}, \frac{c}{d});$ if $\frac{a+c}{b+d} < \frac{p}{q}$ then $\frac{a}{b} := \frac{a+c}{b+d};$
else if $\frac{a+c}{b+d} = \frac{p}{q}$ then break;
else $\frac{c}{d} := \frac{a+c}{b+d};$

Example 5.2. For $\frac{p}{q} = \frac{5}{7}$, the Farey pairs are $(\frac{0}{1}, \frac{1}{0}) \rightsquigarrow (\frac{0}{1}, \frac{1}{1}) \rightsquigarrow (\frac{1}{2}, \frac{1}{1}) \rightsquigarrow (\frac{2}{3}, \frac{1}{1}) \rightsquigarrow (\frac{2}{3}, \frac{3}{4})$. The process terminates at $\frac{2+3}{3+4} = \frac{5}{7}$.

Consider the set $\mathcal{F} = \{\frac{a}{b} : a, b \in \mathbb{Z}_{\geq 0}, \operatorname{gcd}(a, b) = 1\}$ of reduced fractions (including $\frac{1}{0}$). We denote $\mathcal{F}_k = \{\frac{a}{b} \in \mathcal{F} : a + b \leq k\}$ and, for each $x \in \mathbb{R}_+$:

Left_k(x) = max{a $\in \mathcal{F}_k : a \leq x$ } and Right_k(x) = min{a $\in \mathcal{F}_k : a \geq x$ }.

We say that $\frac{a}{b} < x$ is a best left approximation of x if $\frac{a}{b} = \text{Left}_k(x)$ for some $k \in \mathbb{Z}_{\geq 0}$. Similarly, $\frac{c}{d} > x$ is a best right approximation of x if $\frac{c}{d} = \text{Right}_k(x)$.

Example 5.3. We have $\mathcal{F}_7 = (\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \frac{4}{3}, \frac{3}{2}, \frac{2}{1}, \frac{5}{2}, \frac{3}{1}, \frac{4}{1}, \frac{5}{1}, \frac{6}{1}, \frac{1}{0})$. Here, Left₇($\frac{5}{7}$) = $\frac{2}{3}$ and Right₇($\frac{5}{7}$) = $\frac{3}{4}$ are best approximations of $\frac{5}{7}$.

We heavily rely on the following extensive characterization of semiconvergents:

Fact 5.4 ([15], [25, Theorem 3.3], [20, Theorem 2]). Let $\frac{p}{q} \in \mathcal{F} \setminus \{\frac{1}{0}, \frac{0}{1}\}$. The following conditions are equivalent for reduced fractions $\frac{a}{b} < \frac{p}{q}$:

- (a) the Farey process for $\frac{p}{q}$ generates a pair $(\frac{a}{b}, \frac{c}{d})$ for some $\frac{c}{d} \in \mathcal{F}$,
- (b) $\frac{a}{b}$ is an even semiconvergent of $\frac{p}{q}$,
- (c) $\frac{a}{b}$ is a best left approximation of $\frac{p}{q}$,

(d) $b = \left| \frac{aq}{p} \right| + 1$ and $aq \mod p > iq \mod p$ for $0 \le i < a$.

By symmetry, the following conditions are equivalent for reduced fractions $\frac{c}{d} > \frac{p}{q}$:

- (a) the Farey process for $\frac{p}{q}$ generates a pair $(\frac{a}{b}, \frac{c}{d})$ for some $\frac{a}{b} \in \mathcal{F}$,
- (b) $\frac{c}{d}$ is an odd semiconvergent of $\frac{p}{q}$,
- (c) $\frac{c}{d}$ is a best right approximation of $\frac{p}{q}$,
- (d) $c = \left| \frac{dp}{q} \right| + 1$ and $dp \mod q > ip \mod q$ for $0 \le i < d$.

Example 5.5. For $\frac{p}{q} = \frac{5}{7}$, the prefix maxima of $(iq \mod p)_{i=0}^{p-1} = (0, 2, 4, 1, 3)$ are attained for i = 0, 1, 2 (numerators of $\frac{0}{1}, \frac{1}{2}, \frac{2}{3}$) while the prefix maxima of $(ip \mod q)_{i=0}^{q-1} = (0, 5, 3, 1, 6, 4, 2)$ are attained for i = 0, 1, 4 (denominators $\frac{1}{0}, \frac{1}{1}, \frac{3}{4}$).

Due to Fact 5.4, the best approximations can be efficiently computed using the *fast* continued fraction algorithm; see [20].

Corollary 5.6. Given $\frac{p}{q} \in \mathcal{F}$ and a positive integer $k, 1 \leq k < p+q$, the values $\operatorname{Left}_k(\frac{p}{q})$ and $\operatorname{Right}_k(\frac{p}{q})$ can be computed in $\mathcal{O}(\log k)$ time.

Next, we characterize the function L^d .

Lemma 5.7. Let p, q > 2 be relatively prime integers and let $h . If <math>\frac{a}{b} = \text{Left}_{h+3}(\frac{p}{q})$ and $\frac{c}{d} = \text{Right}_{h+3}(\frac{p}{q})$, then, assuming G(-1, p, q) = 0:

$$L^{d}(h, p, q) = \begin{cases} \widetilde{G}(a+b-2, p, q) + \widetilde{G}(c+d-2, p, q) & \text{if } a+b+c+d = h+4, \\ \widetilde{G}(h+2, p, q) & \text{otherwise.} \end{cases}$$

Proof. Let us start with a special case of $\frac{a}{b} = \frac{0}{1}$. Then $\frac{c}{d} = \frac{1}{h+2}$, so q > (h+2)p and $\widetilde{G}(k, p, q) = (k+1)p$ for $k \le h+1$. Consequently, by Lemma 3.3,

$$L^{d}(h, p, q) = \min_{k=0}^{h} \left(\widetilde{G}(k, p, q) + \widetilde{G}(h - k, p, q) \right) = (h + 2)p.$$

Due to a+b+c+d = 0+1+1+h+2 = h+4, this is equal to the claimed value of $\widetilde{G}(-1, p, q) + \widetilde{G}(h+1, p, q) = 0 + (h+2)p$. Symmetrically, the lemma holds if $\frac{c}{d} = \frac{1}{0}$. Thus, below we assume $\frac{1}{h+2} < \frac{p}{q} < \frac{h+2}{1}$.

By Fact 5.4, $\frac{a+c}{b+d}$ is a best (left or right) approximation of $\frac{p}{q}$, so $\max(a+b,c+d) \le h+3 < a+b+c+d$. Moreover,

$$G(aq, p, q) = \left\lfloor \frac{aq}{p} \right\rfloor + \left\lfloor \frac{aq}{q} \right\rfloor = b - 1 + a \quad \text{and} \quad G(dp, p, q) = \left\lfloor \frac{dp}{p} \right\rfloor + \left\lfloor \frac{dp}{q} \right\rfloor = d + c - 1.$$

so $\widetilde{G}(a+b-2,p,q)+\widetilde{G}(c+d-2,p,q)=aq+dp$

First, suppose that a + b + c + d < h + 4. Assume without loss of generality that $\widetilde{G}(h + 2, p, q) = \alpha p$ is a multiple of p. Note that $d < \alpha < b + d$ due to

$$G((b+d)p, p, q) = b + d + \left\lfloor \frac{(b+d)p}{q} \right\rfloor \ge a + b + c + d - 1 \ge h + 4.$$

Consequently, Fact 5.4 yields $\alpha p \mod q < dp \mod q$. Hence

$$G((\alpha - d)p, p, q) = (\alpha - d) + \left\lfloor \frac{\alpha p - dp}{q} \right\rfloor = (\alpha - d) + \left\lfloor \frac{\alpha p}{q} \right\rfloor + \left\lfloor \frac{-dp}{q} \right\rfloor = h + 3 - c - d,$$

and therefore

$$L^{d}(h, p, q) \ge \widetilde{G}(h + 2 - c - d, p, q) + \widetilde{G}(c + d - 2, p, q) = (\alpha - d)p + dp = \widetilde{G}(h + 2, p, q).$$

On the other hand, Lemma 4.5 yields $H^d(\alpha p, p, q) \ge G(\alpha p, p, q) - 2 = h + 1$, so $L^d(h, p, q) \le \widetilde{G}(h + 2, p, q)$.

Finally, suppose that a + b + c + d = h + 4. Lemma 3.3 immediately yields $L^d(h, p, q) \ge \widetilde{G}(a + b - 2, p, q) + \widetilde{G}(c + d - 2, p, q) = aq + dp$. For the proof of the inverse inequality, let us take k such that $L^d(h, p, q) = \widetilde{G}(k, p, q) + \widetilde{G}(h - k, p, q)$, and define $x = \widetilde{G}(k, p, q)$ and $y = \widetilde{G}(h - k, p, q)$. Consequently,

$$a+b+c+d = h+4 = \left\lfloor \frac{x-1}{p} \right\rfloor + \left\lfloor \frac{x-1}{q} \right\rfloor + \left\lfloor \frac{y-1}{p} \right\rfloor + \left\lfloor \frac{y-1}{q} \right\rfloor + 4 = \left\lceil \frac{x}{p} \right\rceil + \left\lceil \frac{y}{p} \right\rceil + \left\lceil \frac{x}{q} \right\rceil + \left\lceil \frac{y}{q} \right\rceil \ge \left\lceil \frac{x+y}{p} \right\rceil + \left\lceil \frac{x+y}{q} \right\rceil \ge \left\lceil \frac{aq+dp}{p} \right\rceil + \left\lceil \frac{aq+dp}{q} \right\rceil = d + \left\lceil \frac{aq}{p} \right\rceil + a + \left\lceil \frac{dp}{q} \right\rceil = d + b + a + c.$$

Each intermediate inequality must therefore be an equality, so we conclude that

$$\left\lceil \frac{x}{p} \right\rceil + \left\lceil \frac{y}{p} \right\rceil = \left\lceil \frac{x+y}{p} \right\rceil = \left\lceil \frac{aq+dp}{p} \right\rceil = b+d \text{ and } \left\lceil \frac{x}{q} \right\rceil + \left\lceil \frac{y}{q} \right\rceil = \left\lceil \frac{x+y}{q} \right\rceil = \left\lceil \frac{aq+dp}{q} \right\rceil = a+c.$$

If $p \mid x$ and $p \mid y$, then $\frac{x+y}{p} = b + d$, so $\left\lceil \frac{(b+d)p}{q} \right\rceil = a + c$. Hence $\frac{a+c}{b+d} \ge \frac{p}{q}$, and consequently $\frac{a+c}{b+d}$ is either a right semiconvergent of $\frac{p}{q}$ or is equal to $\frac{p}{q}$. In both cases, Fact 5.4 implies $(-(b+d)p) \mod q < \min((-x) \mod q, (-y) \mod q)$. This lets us derive a contradiction:

$$\left\lceil \frac{x}{q} \right\rceil + \left\lceil \frac{y}{q} \right\rceil = \frac{x+y+(-x) \mod q+(-y) \mod q}{q} > \frac{(b+d)p+2((-(b+d)p) \mod q)}{q} \ge \left\lceil \frac{(b+d)p}{q} \right\rceil.$$

Symmetrically, $q \mid x$ and $q \mid y$ yields an analogous contradiction.

Thus, without loss of generality we may assume $p \mid x$ and $q \mid y$. However, the conditions $x+y \ge aq+dp$ and $\left\lceil \frac{x+y}{p} \right\rceil = \left\lceil \frac{aq+dp}{p} \right\rceil$ yield $(-y) \mod p = (-(x+y)) \mod p \le (-(aq+dp)) \mod p = (-dp) \mod p$. By Fact 5.4, this implies y = dp. Symmetrically, x = aq. Thus, $L^d(h, p, q) = aq + dp$, as claimed.

Lemma 5.7 applies to $h ; the following fact lets us deal with <math>h \ge p + q - 3$. It appeared in [5], but we provide an alternative proof for completeness.

Fact 5.8 ([5, Theorem 4]). Let p, q be relatively prime positive integers. For each $h \ge 0$, we have

$$L^{d}(h, p, q) = L^{d}(h \mod (p+q-2), p, q) + \left\lfloor \frac{h}{p+q-2} \right\rfloor \cdot pq$$

Moreover, $L^d(p+q-3, p, q) = pq$.

Proof. First, note that $\widetilde{G}(k, p, q) + \widetilde{G}(p + q - 3 - k, p, q) = pq$ holds for $0 \le k \le p + q - 3$. Hence, $L^{d}(p+q-3, p, q) = pq$ holds as claimed due to Lemma 3.3.

For the first part of the statement, it suffices to prove that $H^d(n + pq, p, q) = H^d(n, p, q) + p + q - 2$ for each $n \ge q$. The function G satisfies an analogous equality, so Lemma 3.3 immediately yields $H^{d}(n+pq, p, q) \leq p+q+2+H^{d}(n, p, q)$. The other inequality also follows from Lemma 3.3 unless each optimum value l for n + pq satisfies $n \le l < pq$. However, for such l (and q < n < pq), we have

$$\begin{aligned} G(l,p,q) + G(n+pq-l-1,p,q) &= \left\lfloor \frac{l}{p} \right\rfloor + \left\lfloor \frac{l}{q} \right\rfloor + \left\lfloor \frac{n+pq-l-1}{p} \right\rfloor + \left\lfloor \frac{n+pq-l-1}{p} \right\rfloor \geq \\ & \left\lfloor \frac{n+pq}{p} \right\rfloor - 1 + \left\lfloor \frac{n+pq}{q} \right\rfloor - 1 = G(n+pq,p,q) + G(0,p,q), \end{aligned}$$
contradiction. This concludes the proof.

a contradiction. This concludes the proof.

Theorem 5.9. Given integers $p, q \ge 1$ such that $gcd(p,q) \notin \{p,q\}$ and an integer $h \ge 0$, the value L(h, p, q) can be computed in $\mathcal{O}(\log p + \log q)$ time.

Proof. We proceed as in the proof of Corollary 4.9, except that we apply Fact 5.8 and Lemma 5.7 to compute $L^d(h, p, q)$. Fact 5.8 reduces the problem to determining $L^d(h', p, q)$, where $h' = h \mod (p+q-1)$ 2). We use Corollary 5.6 to compute $\operatorname{Left}_{h'+3}(\frac{p}{q})$ and $\operatorname{Right}_{h'+3}(\frac{p}{q})$ in $\mathcal{O}(\log h')$ time. The values $\widetilde{G}(r, p, q)$ can be determined in $\mathcal{O}(\log r)$ time using binary search (restricted to multiples of p or q). The overall running time for $L^d(h, p, q)$ is $\mathcal{O}(\log h') = \mathcal{O}(\log p + \log q)$, so for L(h, p, q) it is also $\mathcal{O}(\log p + \log q)$.

Closed-Form Formula for $L(h, \cdot, \cdot)$ 6

In this section we show how to compute a compact representation of the function $L(h, \cdot, \cdot)$ in $\mathcal{O}(h \log h)$ time. We start with such representations for \widetilde{G} and L^d .

Assume that $h . For <math>0 < i \le h + 4$, let us define fractions

$$l_i = \frac{i-1}{h+4-i}, \quad m_i = \frac{i}{h+4-i}$$

called the h-special points and the h-middle points, respectively. Now, The function G can be expressed as follows (see Fig. 3):

Lemma 6.1. If gcd(p,q) = 1 and h , then

$$\widetilde{G}(h+2,p,q) = \begin{cases} (h+4-i) \cdot p & \text{if } l_i \leq \frac{p}{q} \leq m_i, \\ i \cdot q & \text{if } m_i \leq \frac{p}{q} \leq l_{i+1} \end{cases}$$

Proof. Note that $\widetilde{G}(h+2,p,q) = n$ is equivalent to $G(n-1,p,q) \leq h+2 < G(n,p,q)$. Additionally, observe that $\widetilde{G}(h+2, p, q)$ is a multiple of p or q. We have two cases.

$$\frac{\mathbf{0}}{\mathbf{10}} \stackrel{10p}{\longleftarrow} \frac{\mathbf{1}}{\mathbf{10}} \stackrel{q}{\longrightarrow} \frac{\mathbf{1}}{\mathbf{9}} \stackrel{9p}{\longleftarrow} \frac{2}{\mathbf{9}} \stackrel{2q}{\longrightarrow} \frac{\mathbf{2}}{\mathbf{8}} \stackrel{8p}{\longleftarrow} \frac{3}{\mathbf{8}} \stackrel{3q}{\longrightarrow} \frac{\mathbf{3}}{\mathbf{7}} \stackrel{7p}{\longleftarrow} \frac{4}{\mathbf{7}} \stackrel{4q}{\longrightarrow} \frac{4}{\mathbf{6}} \stackrel{6p}{\longleftarrow} \frac{5}{\mathbf{6}} \stackrel{5q}{\longrightarrow} \frac{\mathbf{5}}{\mathbf{5}}$$
$$\frac{\mathbf{0}}{\mathbf{13}} \stackrel{13p}{\longleftarrow} \frac{\mathbf{1}}{\mathbf{13}} \stackrel{q}{\longrightarrow} \frac{\mathbf{1}}{\mathbf{12}} \stackrel{2q}{\longleftarrow} \frac{2}{\mathbf{12}} \stackrel{2q}{\longrightarrow} \frac{\mathbf{2}}{\mathbf{11}} \stackrel{13p}{\longleftarrow} \frac{3}{\mathbf{10}} \stackrel{10p}{\longleftarrow} \frac{4}{\mathbf{10}} \stackrel{4q}{\longrightarrow} \frac{\mathbf{4}}{\mathbf{9}} \stackrel{9p}{\longleftarrow} \frac{5}{\mathbf{9}} \stackrel{5q}{\longrightarrow} \frac{\mathbf{5}}{\mathbf{8}} \stackrel{8p}{\longleftarrow} \frac{6}{\mathbf{7}} \stackrel{q}{\longleftarrow} \frac{\mathbf{7}}{\mathbf{7}} \stackrel{7}{\longleftarrow} \frac{7}{\mathbf{7}}$$

Figure 3: Graphical representations of the closed-form formulae for $\tilde{G}(9, p, q)$ (above) and $\tilde{G}(12, p, q)$ (below) for p < q: partitions of [0, 1] into intervals w.r.t. p/q and linear functions of p and q for each interval. The respective special points are shown in **bold**.

Case 1: The condition $G(h+2, p, q) = j \cdot q$ for $j \in \mathbb{Z}_{>0}$ is equivalent to:

$$\left\lfloor \frac{jq}{p} \right\rfloor + j \ge h + 3$$
 and $\left\lfloor \frac{jq-1}{p} \right\rfloor + j - 1 \le h + 2$,

i.e.,

 $\left\lfloor \frac{jq}{p} \right\rfloor \ge h+3-j$ and $\left\lceil \frac{jq}{p} \right\rceil = \left\lfloor \frac{jq-1}{p} \right\rfloor + 1 \le h+4-j.$

In other words, we have $h + 3 - j \leq \frac{jq}{p} \leq h + 4 - j$, i.e.,

$$m_j = \frac{j}{h+4-j} \le \frac{p}{q} \le \frac{j}{h+3-j} = l_{j+1}$$

Case 2: The condition $\widetilde{G}(h+2, p, q) = j \cdot p$ for $j \in \mathbb{Z}_{>0}$ is equivalent to:

$$\left\lfloor \frac{jp}{q} \right\rfloor + j \ge h + 3$$
 and $\left\lfloor \frac{jp-1}{q} \right\rfloor + j - 1 \le h + 2,$

i.e.,

$$\left\lfloor \frac{jp}{q} \right\rfloor \ge h+3-j$$
 and $\left\lceil \frac{jq}{p} \right\rceil = \left\lfloor \frac{jp-1}{q} \right\rfloor + 1 \le h+4-j.$

In other words, we have $h + 3 - j \leq \frac{jp}{q} \leq h + 4 - j$, i.e.,

$$l_{h+4-j} = \frac{h+3-j}{j} \le \frac{p}{q} \le \frac{h+4-j}{j} = m_{h+4-j}.$$

The family of intervals $[m_i, l_{i+1}]$ and $[l_i, m_i]$ has the property that any two distinct intervals in this family have disjoint interiors. Hence, the values of $\widetilde{G}(h, p, q)$ are as claimed.

Combined with Lemma 5.7, Lemma 6.1 yields a closed-form formula for L^d . Note that for each i, we have $l_i \leq \text{Left}_{h+3}(m_i) \leq m_i \leq \text{Right}_{h+3}(m_i) \leq l_{i+1}$, but none of the inequalities is strict in general. In particular, $\text{Left}_{h+3}(m_i) = m_i = \text{Right}_{h+3}(m_i)$ if gcd(i, h + 4 - i) > 1.

Corollary 6.2. Let p, q be relatively prime positive integers and let $h \leq p+q-3$ be a non-negative integer. Suppose that $l_i \leq \frac{p}{q} \leq l_{i+1}$ and define reduced fractions $\frac{a_i}{b_i} = \text{Left}_{h+3}(m_i)$ and $\frac{c_i}{d_i} = \text{Right}_{h+3}(m_i)$. Then:

$$L^{d}(h, p, q) = \begin{cases} (h+4-i) \cdot p & \text{if } l_{i} \leq \frac{p}{q} \leq \frac{a_{i}}{b_{i}}, \\ a_{i}q + d_{i}p & \text{if } \frac{a_{i}}{b_{i}} < \frac{p}{q} < \frac{c_{i}}{d_{i}}, \\ i \cdot q & \text{if } \frac{c_{i}}{d_{i}} \leq \frac{p}{a} \leq l_{i+1} \end{cases}$$

Proof. First, observe that for h = p + q - 3, we have $\frac{p}{q} = l_{p+1}$ and $m_p < \frac{p}{q} < m_{p+1}$, so $\frac{c_p}{d_p} \le \frac{p}{q} \le \frac{a_{p+1}}{b_{p+1}}$. As claimed, $L^d(h, p, q) = (h + 4 - (p+1)) \cdot p = p \cdot q$.

Below, we assume $h . Let <math>\frac{a}{b} = \text{Left}_{h+3}(\frac{p}{q})$ and $\frac{c}{d} = \text{Right}_{h+3}(\frac{p}{q})$. We shall prove that a + b + c + d = h + 4 if and only if $\frac{a_i}{b_i} < \frac{p}{q} < \frac{c_i}{d_i}$ for some *i*.

First, suppose that a + b + c + d = h + 4. This means that $\frac{a+c}{b+d} \in \mathcal{F}_{h+4} \setminus \mathcal{F}_{h+3}$, so $\frac{a+c}{b+d} = m_i$ for some i, and therefore $\frac{a}{b} = \frac{a_i}{b_i}$ and $\frac{c}{d} = \frac{c_i}{d_i}$. Consequently, $\frac{a_i}{b_i} < \frac{p}{q} < \frac{c_i}{d_i}$. In the other direction, $\frac{a_i}{b_i} < \frac{p}{q} < \frac{c_i}{d_i}$ implies $\frac{a}{b} = \frac{a_i}{b_i}$ and $\frac{c}{d} = \frac{c_i}{d_i}$, so $\frac{a}{b} < m_i < \frac{c}{d}$. By Fact 5.4, this yields $a + b + c + d \le h + 4$. Moreover, $\frac{a+c}{b+d} \notin \mathcal{F}_{h+3}$, so a + b + c + d = h + 4. Since $G(a_iq, p, q) = a_i + b_i - 1$ and $G(d_ip, p, q) = c_i + d_i - 1$ by Fact 5.4, we have $a_iq + d_ip = a_i + b_i$.

Since $G(a_iq, p, q) = a_i + b_i - 1$ and $G(d_ip, p, q) = c_i + d_i - 1$ by Fact 5.4, we have $a_iq + d_ip = \widetilde{G}(a_i + b_i - 2, p, q) + \widetilde{G}(c_i + d_i - 2, p, q)$. Now, Lemmas 5.7 and 6.1 yield the final formula.

Theorem 6.3. Let 2 be relatively prime and let <math>4 < h < p + q - 2. Suppose that $l_i \leq \frac{p}{q} \leq l_{i+1}$ and define reduced fractions $\frac{a_i}{b_i} = \text{Left}_{h+3}(m_i)$ and $\frac{c_i}{d_i} = \text{Right}_{h+3}(m_i)$. Then:

$$L(h, p, q) = \begin{cases} \left\lceil \frac{h+1}{2} \right\rceil p + q - (h+1) \mod 2 & \text{if } 0 < \frac{p}{q} < 1/\left\lceil \frac{h}{2} \right\rceil & \text{else} \\ (h+4-i) \cdot p & \text{if } l_i \leq \frac{p}{q} \leq \frac{a_i}{b_i}, \\ a_i q + d_i p & \text{if } \frac{a_i}{b_i} < \frac{p}{q} < \frac{c_i}{d_i}, \\ i \cdot q & \text{if } \frac{c_i}{d_i} \leq \frac{p}{q} \leq l_{i+1}. \end{cases}$$

This compact representation of L(h, p, q) (see Fig. 4 for an example) for a given h has size $\mathcal{O}(h)$ and can be computed in time $\mathcal{O}(h \log h)$.

Figure 4: Graphical representations of the closed-form formulae for L(7, p, q) (middle) and L(10, p, q) (below). Compared to $\tilde{G}(9, p, q)$ and $\tilde{G}(12, p, q)$, respectively, an initial subinterval and several middle subintervals are added. A general pattern for the left, middle, and right subintervals, is presented above. However, the left subinterval $(\frac{1}{5}, \frac{1}{4})$ within L(10, p, q) is an exception because is has been trimmed by the initial interval.

Proof. The formula follows from the formulae for L^s (Lemma 2.4) and L^d (Corollary 6.2) combined using Theorem 4.6. To compute the table for L efficiently, we determine $\frac{a_i}{b_i} = \text{Left}_{h+3}(m_i)$ and $\frac{c_i}{d_i} = \text{Right}_{h+3}(m_i)$ using Corollary 5.6.

7 Relation to Standard Sturmian Words

For a finite directive sequence $\gamma = (\gamma_1, \ldots, \gamma_m)$ of positive integers, a Sturmian word $\operatorname{St}(\gamma)$ is recursively defined as X_m , where $X_{-1} = q$, $X_0 = p$, and $X_i = X_{i-1}^{\gamma_i} X_{i-2}$ for $1 \le i \le m$; see [16, Chapter 2]. We classify directive sequences γ (and the Sturmian words $\operatorname{St}(\gamma)$) into even and odd based on the parity of m.

Observation 7.1. Odd Sturmian words of length at least 2 end with pq, while even Sturmian words of length at least 2 end with qp.

For a directive sequence $\gamma = (\gamma_1, \ldots, \gamma_m)$, we define $\operatorname{fr}(\gamma) = [0; \gamma_1, \ldots, \gamma_m]$.

Fact 7.2 ([16, Proposition 2.2.24]). If $fr(\gamma) = \frac{p}{q}$, then $St(\gamma)$ contains p characters q and q characters p.

Example 7.3. We have $\frac{5}{7} = [0; 1, 2, 2] = [0; 1, 2, 1, 1]$, so the Sturmian words with 5 q's and 7 p's are: St(1, 2, 2) = pqpqppqppqppq and St(1, 2, 1, 1) = pqpqppqppqpqp.

For relatively prime integers $1 , we define <math>\operatorname{St}_{p,q}$ as a Sturmian word with $\operatorname{fr}(\gamma) = \frac{p}{q}$. Note that we always have two possibilities for $\operatorname{St}_{p,q}$ (one odd and one even), but they differ in the last two positions only. In fact, the first p + q - 2 characters of $\operatorname{St}_{p,q}$ are closely related to the values $\widetilde{G}(i, p, q)$.

Fact 7.4 ([16, Proposition 2.2.15]). Let $1 be relatively prime integers. If <math>i \le p + q - 3$, then

$$\mathrm{St}_{p,q}[i] = \begin{cases} \mathsf{p} & \text{if } p \mid \widetilde{G}(i,p,q), \\ \mathsf{q} & \text{if } q \mid \widetilde{G}(i,p,q). \end{cases}$$

As a result, the values $\widetilde{G}(i,p,q)$ can be derived from $\operatorname{St}_{p,q};$ see Table 3.

Fact 7.5 ([16, Exercise 2.2.9]). $St(\gamma'_0, \ldots, \gamma'_{m'})$ is a prefix of $St(\gamma)$ if and only if $[0; \gamma'_0, \ldots, \gamma'_{m'}]$ is a semiconvergent of $fr(\gamma)$.

Example 7.6. The semiconvergents of $[0;1,2,2] = \frac{5}{7} = [0;1,2,1,1]$ are $[0;1,2,1] = \frac{3}{4}$, $[0;1,2] = \frac{2}{3}$, $[0;1,1] = \frac{1}{2}$, [0;1] = 1, $[0;] = \frac{0}{1}$ (and $\frac{1}{0}$). They correspond to the following Sturmian prefixes of St(1,2,2) = pqpqppqppqppqppqp: St(1,2,1) = pqpqppqp, St(1,2) = pqpqp, St(1,1) = pqp, St(1) = pq, and St() = p.

	0	1	2	3	4	5	6	7	8	9	10	11
$\sim^{\operatorname{St}_{p,q}[i]}$	р	q	р	q	р	р	q	р	q	р	p/q	q/p
$\widetilde{G}(i,p,q)$	p	q	2p	2q	3p	4p	3q	5p	4q			
$\widetilde{G}(i,p,q)$	5	7	10	14	15	20	21	25	28	30		

Table 3: The Sturmian words $\operatorname{St}_{p,q}$ for p = 5 and q = 7 and the corresponding values of $\widetilde{G}(i, p, q)$ for i .

Corollary 7.7. Consider a proper prefix P of Sturmian word $St(\gamma)$. Moreover, let $\frac{a}{b} = Left_{|P|}(fr(\gamma))$ and $\frac{c}{d} = Right_{|P|}(fr(\gamma))$. The longest even Sturmian prefix of P has length a + b, whereas the longest odd Sturmian prefix of P has length c + d.

Proof. By Fact 7.5, the longest even Sturmian prefix of P is the longest Sturmian word $\operatorname{St}(\gamma')$ such that $\frac{a'}{b'} := \operatorname{fr}(\gamma')$ is an even semiconvergent of $\operatorname{fr}(\gamma)$. Its length $a' + b' \leq |P|$ is largest possible, so by Fact 5.4 $\frac{a'}{b'}$ is the best left approximation of $\operatorname{fr}(\gamma)$ with $a' + b' \leq |P|$. This is precisely how $\frac{a}{b} = \operatorname{Left}_{|P|}(\operatorname{fr}(\gamma))$ is defined.

The proof for odd Sturmian prefixes is symmetric.

The following theorem can be seen as a restatement of Lemma 5.7 in terms of Sturmian words.

Theorem 7.8. Let $\operatorname{St}_{p,q}$ be a standard Sturmian word corresponding to $\frac{p}{q}$ and let $0 \leq h < p+q-3$. If $\operatorname{St}_{p,q}[0..h+3]$ is a Sturmian word, then $L^d(h,p,q) = \widetilde{G}(l-2,p,q) + \widetilde{G}(r-2,p,q)$, where l,r are the lengths of the longest proper Sturmian prefixes of $\operatorname{St}_{p,q}[0..h+3]$ of different parities, and $\widetilde{G}(-1,p,q) = 0$. Otherwise, $L^d(h,p,q) = \widetilde{G}(h+2,p,q)$.

Proof. To apply Lemma 5.7, we set $\frac{a}{b} = \text{Left}_{h+3}(\frac{p}{q} \text{ and } \frac{c}{d} = \text{Right}_{h+3}(\frac{p}{q})$. Observe that the mediant $\frac{a+c}{b+d}$ is a better approximation of $\frac{p}{q}$ than $\frac{a}{b}$ or $\frac{c}{d}$, and thus it is a semiconvergent of $\frac{p}{q}$. Thus, we always have $a+b+c+d \ge h+4$ and, by Fact 7.5, equality holds if and only if $\text{St}_{p,q}$ has a Sturmian prefix of length h+4. In other words, the case distinction here coincides with the one in Lemma 5.7. If a+b+c+d > h+4, then we have $L^d(h, p, q) = \tilde{G}(h+2, p, q)$. Otherwise, $L^d(h, p, q) = \tilde{G}(a+b-2, p, q) + \tilde{G}(c+d-2, p, q)$. However, due to Corollary 7.7, $\text{St}_{p,q}[0..a+b-1]$ is an even Sturmian word corresponding to (a, b), $\text{St}_{p,q}[0..c+d-1]$ is an odd Sturmian word corresponding to (c, d), and these are the longest Sturmian prefixes of $\text{St}_{p,q}[0..h+2]$ of each parity.

Example 7.9. Consider a word $St_{5,7}$ as in Table 3. The lengths of its proper even Sturmian prefixes are 2, 7, whereas the lengths of its proper odd Sturmian prefixes are 1, 3, 5. Hence, $L^d(7,5,7) = \tilde{G}(9,5,7) = 30$, since $St_{5,7}[0..10]$ is not a Sturmian word. Moreover, $L^d(8,5,7) = \tilde{G}(5,5,7) + \tilde{G}(3,5,7) = 20 + 14 = 34$, since $St_{5,7}[0..11] = St_{5,7}$ is a Sturmian word.

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