# On Periodicity Lemma for Partial Words* 

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#### Abstract

We investigate the function $L(h, p, q)$, called here the threshold function, related to periodicity of partial words (words with holes). The value $L(h, p, q)$ is defined as the minimum length threshold which guarantees that a natural extension of the periodicity lemma is valid for partial words with $h$ holes and (strong) periods $p, q$. We show how to evaluate the threshold function in $\mathcal{O}(\log p+\log q)$ time, which is an improvement upon the best previously known $\mathcal{O}(p+q)$-time algorithm. In a series of papers, the formulae for the threshold function, in terms of $p$ and $q$, were provided for each fixed $h \leq 7$. We demystify the generic structure of such formulae, and for each value $h$ we express the threshold function in terms of a piecewise-linear function with $\mathcal{O}(h)$ pieces.


## 1 Introduction

Consider a word $X$ of length $|X|=n$, with its positions numbered 0 through $n-1$. We say that $X$ has a period $p$ if $X[i]=X[i+p]$ for all $0 \leq i<n-p$. In this case, the prefix $P=X[0 . . p-1]$ is called a string period of $X$. Our work can be seen as a part of the quest to extend Fine and Wilf's Periodicity Lemma [11], which is a ubiquitous tool of combinatorics on words, into partial words.

Lemma 1.1 (Periodicity Lemma [11]). If $p, q$ are periods of a word $X$ of length $|X| \geq p+q-\operatorname{gcd}(p, q)$, then $\operatorname{gcd}(p, q)$ is also a period of $X$.

A partial word is a word over the alphabet $\Sigma \cup\{\diamond\}$, where $\diamond$ denotes a hole (a don't care symbol). In what follows, by $n$ we denote the length of the partial word and by $h$ the number of holes. For $a, b \in \Sigma \cup\{\diamond\}$, the relation of matching $\approx$ is defined so that $a \approx b$ if $a=b$ or either of these symbols is a hole. A (solid) word $P$ of length $p$ is a string period of a partial word $X$ if $X[i] \approx P[i \bmod p]$ for $0 \leq i<n$. In this case, we say that the integer $p$ is a (strong) period of $X$.

We aim to compute the optimal thresholds $L(h, p, q)$ which make the following generalization of the periodicity lemma valid:

Lemma 1.2 (Periodicity Lemma for Partial Words). If $X$ is a partial word with $h$ holes with periods $p, q$ and $|X| \geq L(h, p, q)$, then $\operatorname{gcd}(p, q)$ is also a period of $X$.

If $\operatorname{gcd}(p, q) \in\{p, q\}$, then Lemma 1.2 trivially holds for each partial word $X$. Otherwise, as proved by Fine and Wilf [11], the threshold in Lemma 1.1 is known to be optimal, so $L(0, p, q)=p+q-\operatorname{gcd}(p, q)$.
Example 1.3. $L(1,5,7)=12$, because:

- each partial word of length at least 12 with one hole and periods 5,7 has also period $1=\operatorname{gcd}(5,7)$,
- the partial word ababaababa $\diamond$ of length 11 has periods 5,7 and does not have period 1.

As our main aim, we examine the values $L(h, p, q)$ as a function of $p, q$ for a given $h$. Closed-form formulae for $L(h, \cdot, \cdot)$ with $h \leq 7$ were given in $[2,5,22]$. In these cases, $L(h, p, q)$ can be expressed using a constant number of functions linear in $p, q$, and $\operatorname{gcd}(p, q)$. We discover a common pattern in such formulae which lets us derive a closed-form formula for $L(h, p, q)$ with arbitrary fixed $h$ using a sequence of $\mathcal{O}(h)$ fractions. Our construction relies on the theory of continued fractions; we also apply this link to describe $L(h, p, q)$ in terms of standard Sturmian words.
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| $h$ | $L(h, 5,7)$ | example of length $L(h, 5,7)-1$ |
| :---: | :---: | :---: |
| 0 | 11 | ababaababa |
| 1 | 12 | ababaababa $\diamond$ |
| 2 | 16 | ababaababa $\downarrow$ \aba |
| 3 | 19 | aaaabaaaa $\$ a $\$ aa $\$ aaa  \hline 4 & 21 & aba $\downarrow \diamond$ ababaababa $\downarrow$ 人 ${ }^{\text {aba }}$ |
| 5 | 25 |  |

$$
\begin{array}{rcccccccccccccccc}
n: & 10 & \mathbf{1 1} & \mathbf{1 2} & 13 & 14 & 15 & \mathbf{1 6} & 17 & 18 & \mathbf{1 9} & 20 & \mathbf{2 1} & 22 & 23 & 24 & \mathbf{2 5} \\
\hline H(n, 5,7): & 0 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 6
\end{array}
$$

Table 1: The optimal non-unary partial words with periods 5,7 and $h=0, \ldots, 5$ holes (of length $L(h, 5,7)-1)$ and the values $H(n, 5,7)$ for $n=10, \ldots, 25$.

As an intermediate step, we consider a dual holes function $H(n, p, q)$, which gives the minimum number of holes $h$ for which there is a partial word of length $n$ with $h$ holes and periods $p, q$ which do not satisfy Lemma 1.2.
Example 1.4. We have $H(11,5,7)=1$ because:

- $H(11,5,7) \geq 1$ : due to the classic periodicity lemma, every solid word of length 11 with periods 5 and 7 has period $1=\operatorname{gcd}(5,7)$, and
- $H(11,5,7) \leq 1$ : ababaababa $\diamond$ is non-unary, has one hole and periods 5,7 .

We have $H(12,5,7) \leq H(11,5,7)+1=2$ since appending $\diamond$ preserves periods. In fact $H(12,5,7)=$ $H(15,5,7)=2$. However, there is no non-unary partial word of length 16 with 2 holes and periods 5,7 , so $L(2,5,7)=16$; see Table 1 .
For a function $f(n, p, q)$ monotone in $n$, we define its generalized inverse as:

$$
\widetilde{f}(h, p, q)=\min \{n: f(n, p, q)>h\} .
$$

Observation 1.5. $L=\widetilde{H}$.
As observed above, Lemma 1.2 becomes trivial if $p \mid q$. The case of $p \mid 2 q$ is known to be special as well, but it has been fully described in [22]. Furthermore, it was shown in [5, 21] that the case of $\operatorname{gcd}(p, q)>1$ is easily reducible to that of $\operatorname{gcd}(p, q)=1$. We recall these existing results in Section 4, while in the other sections we assume that $\operatorname{gcd}(p, q)=1$ and $p, q>2$.

Previous results The study of periods in partial words was initiated by Berstel and Boasson [2], who proved that $L(1, p, q)=p+q$. They also showed that the same bound holds for weak periods ${ }^{1} p$ and $q$. Shur and Konovalova [22] developed exact formulae for $L(2, p, q)$ and $L(h, 2, q)$, and an upper bound for $L(h, p, q)$. A formula for $L(h, p, q)$ with small values $h$ was shown by Blanchet-Sadri et al. [3], whereas for large $h$, Shur and Gamzova [21] proved that the optimal counterexamples of length $L(h, p, q)-1$ belong to a very restricted class of special arrangements. The latter contribution leads to an $\mathcal{O}(p+q)$-time algorithm for computing $L(h, p, q)$. An alternative procedure with the same running time was shown by Blanchet-Sadri et al. [5], who also stated closed-form formulae for $L(h, p, q)$ with $h \leq 7$. Weak periods were further considered in $[4,6,23]$.

Other known extensions of the periodicity lemma include a variant with three [8] and an arbitrary number of specified periods $[13,24]$, the so-called new periodicity lemma [1, 10], a periodicity lemma for repetitions with morphisms [17], extensions into abelian [9] and $k$-abelian [14] periodicity, into abelian periodicity for partial words [7], into bidimensional words [18], and other variations [12, 19].

Our results First, we show how to compute $L(h, p, q)$ using $\mathcal{O}(\log p+\log q)$ arithmetic operations, improving upon the state-of-the-art complexity $\mathcal{O}(p+q)$.

[^0]Furthermore, for any fixed $h$ in $\mathcal{O}(h \log h)$ time we can compute a compact description of the threshold function $L(h, p, q)$. For the base case of $p<q, \operatorname{gcd}(p, q)=1$, and $h<p+q-2$, the representation is piecewise linear in $p$ and $q$. More precisely, the interval $[0,1]$ can be split into $\mathcal{O}(h)$ subintervals $I$ so that $L(h, p, q)$ restricted to $\frac{p}{q} \in I$ is of the form $a \cdot p+b \cdot q+c$ for some integers $a, b, c$.

Overview of the paper We start by introducing two auxiliary functions $H^{s}$ and $H^{d}$ which correspond to two restricted families of partial words. Our first key step is to prove that the value $H(n, p, q)$ is always equal to $H^{s}(n, p, q)$ or $H^{d}(n, p, q)$ and to characterize the arguments $n$ for which either case holds. The final function $L$ is then obtained as a combination of the generalized inverses $L^{s}$ and $L^{d}$ of $H^{s}$ and $H^{d}$, respectively. Developing the closed-form formula for $L^{d}$ requires considerable effort; this is where continued fractions arise.

## 2 Functions $H^{s}$ and $L^{s}$

For relatively prime integers $p, q, 1<p<q$, and an integer $n \geq q$, let us define

$$
H^{s}(n, p, q)=\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n-q+1}{p}\right\rfloor .
$$

We shall prove that $H(n, p, q) \leq H^{s}(n, p, q)$ for a suitable range of lengths $n$.
Fine and Wilf [11] constructed a word of length $p+q-2$ with periods $p$ and $q$ and without period 1. For given $p, q$ we choose such a word $S_{p, q}$ and, we define a partial word $W_{p, q}$ as follows, setting $k=\lfloor q / p\rfloor$ (see Fig. 1):

$$
W_{p, q}=\left(S_{p, q}[0 . . p-3] \diamond \diamond\right)^{k} \cdot S_{p, q} \cdot\left(\diamond \diamond S_{p, q}[q . . q+p-3]\right)^{k} .
$$

Example 2.1. For $p=5$ and $q=7$, we can take $S_{5,7}=$ ababaababa and

$$
W_{5,7}=\text { aba } \diamond \diamond \text { ababaababa } \diamond \diamond \text { aba. }
$$

This partial word has length 20 and 4 holes. Hence, $H(20,5,7) \leq 4=H^{s}(20,5,7)$ and $L(4,5,7) \geq 21$. In fact, these bounds are tight; see Table 1.

Intuitively, the partial word $W_{p, q}$ is an extension of $S_{p, q}$ preserving the period $p$, in which a small number of symbols is changed to holes to guarantee the periodicity with respect to $q$.

Lemma 2.2. The partial word $W_{p, q}$ has periods $p$ and $q$.
Proof. Let $n=\left|W_{p, q}\right|$. It is easy to observe that $p$ is a period of $W_{p, q}$. We now show that $q$ is a period of $W_{p, q}$ as well. Let $X$ and $Y$ be the prefix and the suffix of $W_{p, q}$ of length $p\lfloor q / p\rfloor$ (so that $\left.W_{p, q}=X \cdot S_{p, q} \cdot Y\right)$. Note that $|X|,|Y|<q \leq\left|S_{p, q}\right|$.

Let us start by showing that $W_{p, q}[i] \approx W_{p, q}[i+q]$ for $0 \leq i<n-q$. First, suppose that $W_{p, q}[i]$ is contained in $X$. The claim is obvious if $i \bmod p \geq p-2$, because in this case we have $W_{p, q}[i]=\diamond$. Otherwise

$$
W_{p, q}[i]=S_{p, q}[i \bmod p] \stackrel{(1)}{=} S_{p, q}[i \bmod p+q] \stackrel{(2)}{=} S_{p, q}\left[i+q-\left\lfloor\frac{q}{p}\right\rfloor p\right]=W_{p, q}[i+q],
$$



Figure 1: The structure of the partial word $W_{p, q} \diamond \diamond=X \cdot S_{p, q} \cdot Y \diamond \diamond$ for $\lfloor q / p\rfloor=3$. Tiny rectangles correspond to two holes $\diamond \diamond$. We have $|X|=|Y|=p\lfloor q / p\rfloor=3 p$ and $\left|W_{p, q}\right|=p+q+2 p\lfloor q / p\rfloor-2=$ $q+7 p-2$. There are $4 \cdot\lfloor q / p\rfloor=12$ holes.
where (1) follows from the fact that $S_{p, q}$ has period $q$ and $i \bmod p<p-2$, and (2) from the fact that $S_{p, q}$ has period $p$. By symmetry of our construction, we also have $W_{p, q}[i] \approx W_{p, q}[i+q]$ if $W_{p, q}[i+q]$ is contained in $Y$. In the remaining case, $W_{p, q}[i]$ and $W_{p, q}[i+q]$ are both contained in $S_{p, q}$, which yields $W_{p, q}[i+q]=W_{p, q}[i]$.

Next, we claim that $W_{p, q}[i] \approx W_{p, q}[i+k q]$ for every $k \geq 2$. Observe that $W_{p, q}[i+q], \ldots, W_{p, q}[i+$ $(k-1) q]$ are contained in $S_{p, q}$ and thus they are equal solid symbols. Hence,

$$
W_{p, q}[i] \approx W_{p, q}[i+q]=\cdots=W_{p, q}[i+(k-1) q] \approx W_{p, q}[i+k q] .
$$

The intermediate symbols are solid, so this implies $W_{p, q}[i] \approx W_{p, q}[i+k q]$, as claimed. Consequently, $q$ is indeed a period of $W_{p, q}$.

We use the word $S_{p, q}$ and the partial word $W_{p, q} \diamond \diamond$ to show that $H^{s}$ is an upper bound for $H$ for all intermediate lengths $n\left(\left|S_{p, q}\right| \leq n \leq\left|W_{p, q} \diamond \diamond\right|\right)$.

Lemma 2.3. Let $1<p<q$ be relatively prime integers. For each length $p+q-2 \leq n \leq p+q+2 p\lfloor q / p\rfloor$, we have $H(n, p, q) \leq H^{s}(n, p, q)$.
Proof. We extend $S_{p, q}$ to $W_{p, q} \diamond \diamond$ symbol by symbol, first prepending the characters before $S_{p, q}$, and then appending the characters after $S_{p, q}$. By Lemma 2.2, the resulting partial word has periods $p$ and $q$ because it is contained in $W_{p, q} \diamond \diamond$. Moreover, it is not unary because it contains $S_{p, q}$.

A hole is added at the first two iterations among every $p$ iterations. Hence, the total number of holes is as claimed:

$$
\left\lceil\frac{n-\left|S_{p, q}\right|}{p}\right\rceil+\left\lceil\frac{n-\left|S_{p, q}\right|-1}{p}\right\rceil=\left\lfloor\frac{n-q+1}{p}\right\rfloor+\left\lfloor\frac{n-q}{p}\right\rfloor=H^{s}(n, p, q),
$$

because $\left\lceil\frac{x}{p}\right\rceil=\left\lfloor\frac{x+p-1}{p}\right\rfloor$ for every integer $x$.
Finally, the function $L^{s}=\widetilde{H^{s}}$ is very simple and easily computable.
Lemma 2.4. If $h \geq 0$ is an integer, then $L^{s}(h, p, q)=\left\lceil\frac{h+1}{2}\right\rceil p+q-(h+1) \bmod 2$.
Proof. We have to determine the smallest $n$ such that $\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n-q+1}{p}\right\rfloor=h+1$. There are two cases, depending on parity of $h$ :

Case 1: $h=2 k$. In this case $\left\lfloor\frac{n-q}{p}\right\rfloor=k$ and $\left\lfloor\frac{n-q+1}{p}\right\rfloor=k+1$. Hence, $n-q+1=p(k+1)$, i.e., $n=p(k+1)+q-1=\left\lceil\frac{h+1}{2}\right\rceil p+q-(h+1) \bmod 2$.

Case 2: $h=2 k+1$. In this case $\left\lfloor\frac{n-q}{p}\right\rfloor=k+1$ and $\left\lfloor\frac{n-q+1}{p}\right\rfloor=k+1$. Hence, $n-q=p(k+1)$, i.e., $n=p(k+1)+q=\left\lceil\frac{h+1}{2}\right\rceil p+q-(h+1) \bmod 2$.

## 3 Functions $H^{d}$ and $L^{d}$

In this section, we study a family of partial words corresponding to the special arrangements introduced in [21]. For relatively prime integers $p, q>1$, we say that a partial word $S$ of length $n \geq \max (p, q)$ is $(p, q)$-special if it has a position $l$ such that for each position $i$ :

$$
S[i]= \begin{cases}\mathrm{a} & \text { if } p \nmid(l-i) \text { and } q \nmid(l-i), \\ \mathrm{b} & \text { if } p \mid(l-i) \text { and } q \mid(l-i), \\ \diamond & \text { otherwise. }\end{cases}
$$

Let $H^{d}(n, p, q)$ be the minimum number of holes in a $(p, q)$-special partial word of length $n$.
Fact 3.1. For each $n \geq \max (p, q)$, we have $H(n, p, q) \leq H^{d}(n, p, q)$.
Proof. Observe that every $(p, q)$-special partial word has periods $p$ and $q$. However, due to $p, q>1$, it does not have period $1=\operatorname{gcd}(p, q)$.

| $h$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\widetilde{G}(h, 5,7)$ | 5 | 7 | 10 | 14 | 15 | 20 | 21 | 25 | 28 | 30 | 40 | 42 | 45 | 49 | 50 | 55 | 56 | 60 | 63 | 65 | 75 |
| $L^{d}(h, 5,7)$ | 10 | 12 | 15 | 19 | 21 | 25 | 28 | 30 | 34 | 35 | 45 | 47 | 50 | 54 | 56 | 60 | 63 | 65 | 69 | 70 | 80 |

Table 2: Functions $\widetilde{G}$ and $L^{d}$ for $p=5, q=7$, and $h=0, \ldots, 20$. By Lemma 3.3, we have, for example, $L^{d}(8,5,7)=\max (\widetilde{G}(0,5,7)+\widetilde{G}(8,5,7), \ldots, \widetilde{G}(4,5,7)+\widetilde{G}(4,5,7))=\max (5+28,7+25,10+21,14+$ $20,15+15)=34$.

Example 3.2. The partial word aaaabaaaa $\diamond$ a $\diamond$ aa $\diamond$ aaa is $(5,7)$-special (with $l=4$ ), so $H(18,5,7) \leq$ $H^{d}(18,5,7) \leq 3$ and $L(3,5,7) \geq 19$. In fact, these bounds are tight; see Table 1.

To derive a formula for $H^{d}(n, p, q)$, let us introduce an auxiliary function $G$, which counts integers $i \in\{1, \ldots, n\}$ that are multiples of $p$ or of $q$ but not both:

$$
G(n, p, q)=\left\lfloor\frac{n}{p}\right\rfloor+\left\lfloor\frac{n}{q}\right\rfloor-2\left\lfloor\frac{n}{p q}\right\rfloor .
$$

The function $H^{d}$ can be characterized using $G$, while the generalized inverse $L^{d}=\widetilde{H^{d}}$ admits a dual characterization in terms of $\widetilde{G}$; see also Table 2.
Lemma 3.3. Let $p, q>1$ be relatively prime integers.
(a) If $n \geq \max (p, q)$, then $H^{d}(n, p, q)=\min _{l=0}^{n-1}(G(l, p, q)+G(n-l-1, p, q))$.
(b) If $h \geq 0$, then $L^{d}(h, p, q)=\max _{k=0}^{h}(\widetilde{G}(k, p, q)+\widetilde{G}(h-k, p, q))$.

Proof. Let $S$ be a $(p, q)$-special partial word of length $n$ with $h$ holes, $k$ of which are located to the left of position $l$. Observe that $k=G(l, p, q)$ (so $l+1 \leq \widetilde{G}(k, p, q)$ ) and $h-k=G(n-l-1, p, q)$ (so $n-l \leq \widetilde{G}(h-k, p, q))$. Hence, $h=G(l, p, q)+G(n-l-1, p, q)$ and $n+1 \leq \widetilde{G}(k, p, q)+\widetilde{G}(h-k, p, q)$. The claimed equalities follow from the fact that these bounds can be attained for each $l$ and $k$, respectively.

## 4 Characterizations of $H$ and $L$

Shur and Gamzova in [21] proved that $H(n, p, q)=H^{d}(n, p, q)$ for $n \geq 3 q+p$. In this section, we give a complete characterization of $H$ in terms of $H^{d}$ and $H^{s}$, and we derive an analogous characterization of $L$ in terms of $L^{d}$ and $L^{s}$. Our proof is based on a graph-theoretic approach similar to that in [5].

Let us define the $(n, p, q)$-graph $\mathbf{G}=(V, E)$ as an undirected graph with vertices $V=\{0, \ldots, n-1\}$. The vertices $i$ and $j$ are connected if and only if $p \mid(j-i)$ or $q \mid(j-i)$. Observe that $H(n, p, q)$ is the minimum size of a vertex separator in $\mathbf{G}$, i.e., the minimum number of vertices to be removed from $\mathbf{G}$ so that the resulting graph is no longer connected; see Fig. 2.

We say that an edge $(i, j)$ of the $(n, p, q)$-graph is a $p$-edge if $p \mid(j-i)$ and a $q$-edge if $q \mid(j-i)$. The set of all nodes giving the same remainder modulo $p$ (modulo $q$ ) is called a $p$-class ( $q$-class, respectively). Each $p$-class and each $q$-class forms a clique in the ( $n, p, q$ )-graph.
Fact 4.1 (see [5]). Let $1<p<q$ be relatively prime integers. If $n<p q$, then $H^{d}(n, p, q)$ is the minimal degree of a vertex in the $(n, p, q)$-graph.
Proof. Observe that vertex number $l$ has $G(l, p, q)$ neighbors $i<l$ and $G(n-l-1, p, q)$ neighbors $i>l$. Consequently, by Lemma 3.3, $H^{d}(n, p, q)=\min _{l=0}^{n-1}(G(l, p, q)+G(n-l-1, p, q))=\min _{l=0}^{n-1} \operatorname{deg}_{\mathbf{G}}(l)$.
Let $\mathbf{G}=(V, E)$ be the $(n, p, q)$-graph. For each $i \in\{0, \ldots, p-1\}$ let $C_{i}$ be the $p$-class containing the vertex $i$; see Fig. 2. We slightly abuse the notation and use arbitrary integers for indexing the $p$-classes: $C_{i}=C_{i \bmod p}$ for $i \in \mathbb{Z}$. We denote by $E_{i}$ the set of $q$-edges of the form $(j, j+q)$ for $j \in C_{i}$. Let us start with two auxiliary facts.

Fact 4.2. Let $1<p<q$ be relatively prime integers.
(a) For $j \in\{0, \ldots, p-1\}$, we have $\left|E_{j}\right|=\left\lceil\frac{n-j-q}{p}\right\rceil$.
(b) $H^{s}(n, p, q)=\left|E_{p-1}\right|+\left|E_{p-2}\right|=\min _{i \neq j}\left(\left|E_{i}\right|+\left|E_{j}\right|\right)$.


Figure 2: The structure of the $(20,5,7)$-graph. Each 5 -clique $C_{i}$ is actually a clique; same applies for the vertical 7 -cliques. The 5 -clique $C_{2}$ is repeated to show the cyclicity. The set $U=\{3,4,5,7\}$ of encircled vertices is a minimum-size vertex separator. It corresponds to the partial word $W_{5,7}$ from Example 2.1: holes of $W_{5,7}$ are located at positions $i \in U$, the positions $i$ with $W_{5,7}[i]=\mathrm{b}$ form a connected component $\left(C_{1} \cup C_{4}\right) \backslash U$, while the positions $i$ with $W_{5,7}[i]=$ a form a connected component $\left(C_{0} \cup C_{2} \cup C_{3}\right) \backslash U$.

Proof. Let $i=k p+j$, where $0 \leq j<p$. There is a $q$-edge $(i, i+q)$ if and only if

$$
k p+j+q \leq n-1, \text { so } k \leq\left\lfloor\frac{n-1-j-q}{p}\right\rfloor .
$$

The number of such values of $k$ is $\left\lfloor\frac{n-1-j-q}{p}\right\rfloor+1=\left\lceil\frac{n-j-q}{p}\right\rceil$.
As for the second statement of the fact, we have:

$$
\left|E_{j}\right| \geq\left|E_{p-1}\right|=\left\lceil\frac{n-p+1-q}{p}\right\rceil=\left\lfloor\frac{n-q}{p}\right\rfloor
$$

for $0 \leq j<p$ and, similarly, $\left|E_{j}\right| \geq\left|E_{p-2}\right|=\left\lfloor\frac{n-q+1}{p}\right\rfloor$ for $0 \leq j<p-1$.
Fact 4.3. Let $U$ be a vertex separator in the $(n, p, q)$-graph $\mathbf{G}=(V, E)$ and let $\mathbf{G}^{\prime}=\mathbf{G} \backslash U$. One can color the vertices of $\mathbf{G}$ in two colors so that every edge in $\mathbf{G}^{\prime}$ and every p-class in $\mathbf{G}$ is monochromatic, but $\mathbf{G}^{\prime}$ is not monochromatic.

Proof. Recall that each $p$-class $C_{i}$ is a clique in $\mathbf{G}$, so $C_{i} \backslash U$ is still a clique in $\mathbf{G}^{\prime}$. We distinguish a connected component $M$ of $\mathbf{G}^{\prime}$ and color the vertices of $C_{i}$ depending on whether $C_{i} \backslash U \subseteq M$. It is easy to verify that this coloring satisfies the claimed conditions.

The following lemma provides lower bounds on $H(n, p, q)$.
Lemma 4.4. Let $1<p<q$ be relatively prime integers.
(1) If $n<2 q$, then $H(n, p, q) \geq H^{s}(n, p, q)$.
(2) If $p \geq 3$ and $n \geq 2 q$, then $H(n, p, q) \geq \min \left(H^{d}(n, p, q), H^{s}(n, p, q)\right)$.
(3) If $p \geq 5$ and $n \geq 4 q$, then $H(n, p, q) \geq \min \left(H^{d}(n, p, q), H^{s}(n, p, q)+1\right)$.

Proof. Let $U$ be a minimum-size vertex separator of hte $(n, p, q)$-graph $\mathbf{G}=(V, E)$; recall that $|U|=$ $H(n, p, q)$. Let us fix a coloring of $\mathbf{G}$ using colors $\{A, B\}$ satisfying Fact 4.3 ; without loss of generality we assume that the number of $p$-classes with color $A$ is at least the number of $p$-classes with color $B$. We have the following two cases.

Case a: Exactly one $p$-class has color $B$. Let $C_{j}$ be the unique $p$-class with color $B$. By definition, the edges in $E_{j-q} \cup E_{j}$ are bichromatic. If $n<2 q$, then all the $q$-edges form a matching in $\mathbf{G}$. In particular, in order to disconnect $\mathbf{G}$, we need to remove at least one endpoint of each edge in $E_{j-q} \cup E_{j}$. Hence, $H(n, p, q) \geq\left|E_{j-q}\right|+\left|E_{j}\right| \geq H^{s}(n, p, q)$, where the second inequality follows from Fact $4.2(\mathrm{~b})$. This concludes the proof of (1) in this case.

Now assume that $n \geq 2 q$ and $p \geq 3$. We will show that $H(n, p, q) \geq H^{d}(n, p, q)$ holds in this case. Consider any $q$-class $D$ and let $k$ be its size; we have $k \geq\left\lfloor\frac{n}{q}\right\rfloor \geq 2$. In this $q$-class, every $p$-th element has color $B$. Let $\#_{A}(D)$ and $\#_{B}(D)$ denote the number of vertices in $D$ colored with $A$ and $B$, respectively. Then:

$$
\#_{B}(D) \leq\left\lceil\frac{k}{p}\right\rceil \leq\left\lceil\frac{k}{3}\right\rceil=\left\lfloor\frac{k+2}{3}\right\rfloor \leq\left\lfloor\frac{2 k}{3}\right\rfloor=k-\left\lceil\frac{k}{3}\right\rceil \leq k-\left\lceil\frac{k}{p}\right\rceil \leq \#_{A}(D)
$$

The set $U$ contains all $B$-colored vertices or all $A$-colored vertices of every $q$-class $D$, as otherwise there would be a non-monochromatic edge in $\mathbf{G}^{\prime}$ connecting two vertices of $D \backslash U$, contradicting Fact 4.3. At least one vertex of $\mathbf{G}^{\prime}$ is $B$-colored, so in at least one $q$-class, $U$ must contain all $A$-colored vertices; assume that this is the $q$-class $D_{0}$. Consequently,

$$
\begin{aligned}
|U|= & \sum_{D: q \text {-class }}|U \cap D| \geq \#_{A}\left(D_{0}\right)+\sum_{D: q \text {-class, } D \neq D_{0}} \#_{B}(D)= \\
& \left|D_{0}\right|+\sum_{D: q \text {-class }} \#_{B}(D)-2 \#_{B}\left(D_{0}\right)=\left|D_{0}\right|+\left|C_{j}\right|-2\left|D_{0} \cap C_{j}\right| \geq H^{d}(n, p, q) .
\end{aligned}
$$

The last inequality follows from Fact 4.1. This concludes (2) and (3) in this case.
Case b: There are at least two $p$-classes with each color. In particular, $p \geq 4$. We consider two subcases based on the colors $c_{i}$ of classes $C_{i}$. In each case we will show that $H(n, p, q)$ is bounded from below by $H^{s}(n, p, q)$ or $H^{s}(n, p, q)+1$.

First, suppose that there is exactly one $p$-class $C_{i}$ such that $c_{i}=A$ and $c_{i+q}=B$. Equivalently, there is exactly one $p$-class $C_{j}$ such that $c_{j}=B$ and $c_{j+q}=A$. Since there are at least two $p$-classes with each color, $c_{i+2 q}=B$ and $c_{j+2 q}=A$, so $C_{i+q} \neq C_{j}$ and $C_{j+q} \neq C_{i}$. This means that $E_{i} \cup E_{j}$ forms a bichromatic matching in G. Consequently,

$$
H(n, p, q) \geq\left|E_{i}\right|+\left|E_{j}\right| \geq H^{s}(n, p, q)
$$

This concludes the proof of (1) and (2) in the current subcase.
For the proof of (3), observe that $p \geq 5$ and the choice of $A$ as the more frequent color yields that $c_{j+3 q}=A$, so $C_{j+2 q}$ is distinct from $C_{i}$. Hence, we can extend the matching $E_{i} \cup E_{j}$ with an edge $(x, y)$ where $x=(j-q) \bmod p \in C_{j-q}$ and $y=x+3 q \in C_{j+2 q}$. This edge exists because $n \geq 4 q>p+3 q>y$. It forms a matching with $E_{i} \cup E_{j}$ because no edge in $E_{i} \cup E_{j}$ is incident to $C_{j+2 q}$, while the only edges incident to $C_{j-q}$ could be the edges in $E_{i}$ provided that $C_{i}=C_{j-2 q}$. However, $x<q$, so $x$ is not an endpoint of any edge in $E_{j-2 q}$. This concludes the proof of (3) in the current subcase.

Let us proceed to the second subcase. Let $C_{i}, C_{j}$ be two distinct $p$-classes such that $c_{i}=c_{j}=A$ and $c_{i+q}=c_{j+q}=B$. It is easy to see that $E_{i} \cup E_{j}$ forms a bichromatic matching in $\mathbf{G}$, so $H(n, p, q) \geq$ $\left|E_{i}\right|+\left|E_{j}\right| \geq H^{s}(n, p, q)$. Thus, it remains to prove (3) in this subcase.

If $p \geq 5$, then there is a third $p$-class $C_{k}$ with color $A$. Moreover, we may choose $k$ so that $c_{k-q}=B$ and $c_{k}=A$. We extend $E_{i} \cup E_{j}$ with an edge $(x, y)$ where $x=(k-q) \bmod p \in C_{k-q}$ and $y=x+q \in C_{k}$. This edge exists because $n>q+p>y$. It forms a matching with $E_{i} \cup E_{j}$ because no edge in $E_{i} \cup E_{j}$ is incident to $C_{k}$, while the only edges incident to $C_{k-q}$ might be the edges in $E_{i}$ or $E_{j}$ provided that $i=k-2 q$ or $j=k-2 q$. However, $x<q$, so $x$ is not an endpoint of any edge in $E_{k-2 q}$. This concludes the proof of (3) in the current subcase, and the proof of the entire lemma.

Lemma 4.5. If $n \geq q$, then

$$
\left\lfloor\frac{n}{q}\right\rfloor+\left\lfloor\frac{n}{p}\right\rfloor-2\left\lceil\frac{n}{p q}\right\rceil \leq H^{d}(n, p, q) \leq\left\lfloor\frac{n}{q}\right\rfloor+\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{q-1}{p}\right\rfloor-1 .
$$

Proof. Recall that $H^{d}(n, p, q)=\min _{l=0}^{n-1}(G(l, p, q)+G(n-l-1, p, q))$ due to Lemma 3.3. The first part of the claim holds because for $0 \leq l<n$ we have:

$$
\begin{gathered}
G(l, p, q)+G(n-l-1, p, q)=\left\lfloor\frac{l}{p}\right\rfloor+\left\lfloor\frac{l}{q}\right\rfloor-2\left\lfloor\frac{l}{p q}\right\rfloor+\left\lfloor\frac{n-l-1}{p}\right\rfloor+\left\lfloor\frac{n-l-1}{q}\right\rfloor-2\left\lfloor\frac{n-l-1}{p q}\right\rfloor= \\
\left\lceil\frac{l-p+1}{p}\right\rceil+\left\lceil\frac{l-q+1}{q}\right\rceil-2\left\lfloor\frac{l}{p q}\right\rfloor+\left\lceil\frac{n-l-p}{p}\right\rceil+\left\lfloor\frac{n-l-q}{q}\right\rfloor-2\left\lfloor\frac{n-l-1}{p q}\right\rfloor \geq \\
\left\lceil\frac{n-2 p+1}{p}\right\rceil+\left\lceil\frac{n-2 q+1}{q}\right\rceil-2\left\lfloor\frac{n-1}{p q}\right\rfloor=\left\lfloor\frac{n}{p}\right\rfloor-1+\left\lfloor\frac{n}{q}\right\rfloor-1-2\left\lceil\frac{n}{p q}\right\rceil+2=\left\lfloor\frac{n}{q}\right\rfloor+\left\lfloor\frac{n}{p}\right\rfloor-2\left\lceil\frac{n}{p q}\right\rceil .
\end{gathered}
$$

As for the second part, due to $n \geq q$ we have:

$$
\begin{aligned}
& H^{d}(n, p, q) \geq G(q-1, p, q)+G(n-q, p, q)=\left\lfloor\frac{q-1}{p}\right\rfloor+\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n-q}{q}\right\rfloor-2\left\lfloor\frac{n-q}{p q}\right\rfloor \leq \\
&\left\lfloor\frac{q-1}{p}\right\rfloor+\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n}{q}\right\rfloor-1
\end{aligned}
$$

which completes he proof.
Theorem 4.6. Let $p$ and $q$ be relatively prime integers such that $2<p<q$. For each integer $n \geq p+q-2$, we have

$$
H(n, p, q)= \begin{cases}H^{s}(n, p, q) & \text { if } n \leq q+p\left\lceil\frac{q}{p}\right\rceil-1 \text { or } 3 q \leq n \leq q+3 p-1 \\ H^{d}(n, p, q) & \text { otherwise }\end{cases}
$$

Moreover, for each integer $h \geq 0$ :

$$
L(h, p, q)= \begin{cases}L^{s}(h, p, q) & \text { if } \frac{q}{p}>\left\lceil\frac{h}{2}\right\rceil \text { or }\left(h=4 \text { and } \frac{q}{p}<\frac{3}{2}\right) \\ L^{d}(h, p, q) & \text { otherwise. }\end{cases}
$$

Proof. First, we prove the claim concerning $H$ by analyzing several cases.

Case 0. $p q \leq n$.
By Fact 3.1, we have $H(n, p, q) \leq H^{d}(n, p, q)$. Moreover, using Lemma 4.5, we obtain:

$$
\begin{aligned}
& H^{s}(n, p, q)=\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n-q+1}{p}\right\rfloor \geq\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n-p q}{p}\right\rfloor+\left\lfloor\frac{q-1}{p}\right\rfloor+\left\lfloor\frac{q(p-2)+2}{p}\right\rfloor \geq \\
& \geq \geq\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n}{q}\right\rfloor-p+\left\lfloor\frac{q-1}{p}\right\rfloor+\left\lfloor\frac{(p+1)(p-2)+2}{p}\right\rfloor=\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n}{q}\right\rfloor+\left\lfloor\frac{q-1}{p}\right\rfloor+\left\lfloor\frac{p^{2}-p-p^{2}}{p}\right\rfloor= \\
& \\
& =\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n}{q}\right\rfloor+\left\lfloor\frac{q-1}{p}\right\rfloor-1 \geq H^{d}(n, p, q)
\end{aligned}
$$

Finally, Lemma $4.4(2)$ yields that $H(n, p, q) \geq \min \left(H^{d}(n, p, q), H^{s}(n, p, q)\right)=H^{d}(n, p, q)$, which completes the proof.

Henceforth we assume that $n<p q$.
Case 1. $p+q-1 \leq n<2 q$.
We get $H(n, p, q)=H^{s}(n, p, q)$ directly from Lemma 4.4(1) and Lemma 2.3.
Case 2. $2 q \leq n \leq q+\left\lceil\frac{q}{p}\right\rceil p-1$.
Note that $n \leq q+\left\lceil\frac{q}{p}\right\rceil p-1=p+q+\left\lfloor\frac{q}{p}\right\rfloor p-1<p+q+2 p\left\lfloor\frac{q}{p}\right\rfloor$, so $H(n, p, q) \leq H^{s}(n, p, q)$ due to Lemma 2.3. Moreover, using Lemma 4.5, we obtain:

$$
\begin{array}{r}
H^{s}(n, p, q)=\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n-q+1}{p}\right\rfloor \leq\left\lfloor\frac{\left\lceil\frac{q}{p}\right\rceil p-1}{p}\right\rfloor+\left\lfloor\frac{n-q+1}{p}\right\rfloor=\left\lceil\frac{q}{p}\right\rceil-1+\left\lfloor\frac{n-q+1}{p}\right\rfloor=\left\lfloor\frac{q-1}{p}\right\rfloor+\left\lfloor\frac{n-q+1}{p}\right\rfloor \leq \\
\left\lfloor\frac{n}{p}\right\rfloor \leq\left\lfloor\frac{n}{p}\right\rfloor+\left\lfloor\frac{n}{q}\right\rfloor-2 \leq H^{d}(n, p, q) .
\end{array}
$$

Finally, Lemma $4.4(2)$ yields that $H(n, p, q) \geq \min \left(H^{d}(n, p, q), H^{s}(n, p, q)\right)=H^{s}(n, p, q)$, which completes the proof.

Case 3. $q+\left\lceil\frac{q}{p}\right\rceil p-1 \leq n<3 q$.
By Fact 3.1, we have $H(n, p, q) \leq H^{d}(n, p, q)$. Moreover, using Lemma 4.5, we obtain:

$$
\begin{aligned}
H^{s}(n, p, q)=\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n-q+1}{p}\right\rfloor \geq\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{\left\lceil\frac{q}{p}\right\rceil p}{p}\right\rfloor= & \left\lfloor\frac{n-q}{p}\right\rfloor+\left\lceil\frac{q}{p}\right\rceil= \\
& \left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{q-1}{p}\right\rfloor+1+\left\lfloor\frac{n}{q}\right\rfloor-2 \geq H^{d}(n, p, q) .
\end{aligned}
$$

Finally, Lemma $4.4(2)$ yields that $H(n, p, q) \geq \min \left(H^{d}(n, p, q), H^{s}(n, p, q)\right)=H^{d}(n, p, q)$, which completes the proof.

Case 4. $3 q \leq n \leq 3 p+q-1$.
Note that $n \leq 3 p+q-1<p+q+2 p \leq p+q+2 p\left\lfloor\frac{q}{p}\right\rfloor$, so $H(n, p, q) \leq H^{s}(n, p, q)$ due to Lemma 2.3. Moreover, using Lemma 4.5, we obtain:

$$
\begin{aligned}
H^{s}(n, p, q)=\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n-q+1}{p}\right\rfloor \leq\left\lfloor\frac{3 p-1}{p}\right\rfloor & +\left\lfloor\frac{n-q+1}{p}\right\rfloor=2+\left\lfloor\frac{n-q+1}{p}\right\rfloor \leq 1+\left\lfloor\frac{q-1}{p}\right\rfloor+\left\lfloor\frac{n-q+1}{p}\right\rfloor \leq \\
& \leq 1+\left\lfloor\frac{n}{p}\right\rfloor=3+\left\lfloor\frac{n}{p}\right\rfloor-2 \leq\left\lfloor\frac{n}{q}\right\rfloor+\left\lfloor\frac{n}{p}\right\rfloor-2 \leq H^{d}(n, p, q)
\end{aligned}
$$

Finally, Lemma $4.4(2)$ yields that $H(n, p, q) \geq \min \left(H^{d}(n, p, q), H^{s}(n, p, q)\right)=H^{s}(n, p, q)$, which completes the proof.

Case 5. $\max (3 q, 3 p+q-1) \leq n<4 q$ and $p<q<2 p$.
By Fact 3.1, we have $H(n, p, q) \leq H^{d}(n, p, q)$. Moreover, using Lemma 4.5, we obtain:

$$
H^{s}(n, p, q)=\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n-q+1}{p}\right\rfloor \geq\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{3 p}{p}\right\rfloor=\left\lfloor\frac{n-q}{p}\right\rfloor+3=\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{q-1}{p}\right\rfloor+\left\lfloor\frac{n}{q}\right\rfloor-1 \geq H^{d}(n, p, q) .
$$

Finally, Lemma $4.4(2)$ yields that $H(n, p, q) \geq \min \left(H^{d}(n, p, q), H^{s}(n, p, q)\right)=H^{d}(n, p, q)$, which completes the proof.

Case 6. $3 q \leq n$ and $q>2 p$.
By Fact 3.1, we have $H(n, p, q) \leq H^{d}(n, p, q)$. Moreover, using Lemma 4.5, we obtain:

$$
\begin{aligned}
H^{s}(n, p, q) \geq 2\left\lfloor\frac{n-q}{p}\right\rfloor & \geq\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n-3 q}{p}\right\rfloor+2\left\lfloor\frac{q}{p}\right\rfloor \geq\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n-3 q}{q}\right\rfloor+2\left\lfloor\frac{q}{p}\right\rfloor \geq \\
& \geq\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n}{q}\right\rfloor-3+\left\lfloor\frac{q-1}{p}\right\rfloor+2=\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{q-1}{p}\right\rfloor+\left\lfloor\frac{n}{q}\right\rfloor-1 \geq H^{d}(n, p, q)
\end{aligned}
$$

Finally, Lemma $4.4(2)$ yields that $H(n, p, q) \geq \min \left(H^{d}(n, p, q), H^{s}(n, p, q)\right)=H^{d}(n, p, q)$, which completes the proof.

Case 7. $4 q \leq n$ and $p \geq 5$ (and $q<2 p$ ).
By Fact 3.1, we have $H(n, p, q) \leq H^{d}(n, p, q)$. Moreover, using Lemma 4.5, we obtain:

$$
\begin{aligned}
H^{s}(n, p, q)=\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n+1-q}{p}\right\rfloor \geq\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n-2 q}{p}\right\rfloor & +\left\lfloor\frac{q-1}{p}\right\rfloor \geq\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n-2 q}{q}\right\rfloor+\left\lfloor\frac{q-1}{p}\right\rfloor= \\
& =\left\lfloor\frac{n-q}{p}\right\rfloor+\left\lfloor\frac{n}{q}\right\rfloor-2+\left\lfloor\frac{q-1}{p}\right\rfloor \geq H^{d}(n, p, q)-1
\end{aligned}
$$

Finally, Lemma $4.4(3)$ yields that $H(n, p, q) \geq \min \left(H^{d}(n, p, q), H^{s}(n, p, q)+1\right)=H^{d}(n, p, q)$, which completes the proof.

The only remaining case, that $4 q \leq n$ and $p<5$, is a subcase of Case 0 . This completes the proof of the formula for $H(n, p, q)$.

The characterization of $L(h, p, q)$ is relatively easy to derive from that of $H(n, p, q)$. Recall that $L$, $L^{s}$, and $L^{d}$ are generalized inverses of $H, H^{s}$, and $H^{d}$, respectively. Note that Cases 1. and 2. yield $H(n, p, q)=H^{s}(n, p, q)$ for $n \leq q+p\left\lceil\frac{q}{p}\right\rceil-1$ while Case 3 . additionally implies

$$
H^{d}\left(q+p\left\lceil\frac{q}{p}\right\rceil-1, p, q\right)=H^{s}\left(q+p\left\lceil\frac{q}{p}\right\rceil-1, p, q\right)=2\left\lceil\frac{q}{p}\right\rceil-1 .
$$

Consequently, $L(h, p, q)=L^{s}(h, p, q)$ if $h<2\left\lceil\frac{q}{p}\right\rceil-1$, i.e., if $\frac{q}{p}>\left\lceil\frac{h}{2}\right\rceil$. Moreover, if $\frac{3}{2}<\frac{q}{p}$, then $3 q>q+3 p-1$, and therefore $H(n, p, q)=H^{d}(n, p, q)$ for $n \geq q+p\left\lceil\frac{q}{p}\right\rceil-1$ due to Cases 0 . and 5.-7. Hence, if $h \geq 2\left\lceil\frac{q}{p}\right\rceil-1$, i.e., $\frac{3}{2}<\frac{q}{p}<\left\lceil\frac{h}{2}\right\rceil$, then $L(h, p, q)=L^{d}(h, p, q)$.

Now, it suffices to consider the case of $\frac{q}{p}<\frac{3}{2}<\left\lceil\frac{h}{2}\right\rceil$. Then, by Cases 5., 7., and $0 ., H(n, p, q)=$ $H^{d}(n, p, q)$ for $n \geq q+3 p-1$. Case 4. additionally yields $H^{d}(q+3 p-1, p, q)=H^{s}(q+3 p-1, p, q)=5$, so $L(h, p, q)=L^{d}(h, p, q)$ if $h \geq 5$. Moreover, by Case 4., $H(n, p, q)=H^{d}(n, p, q)$ for $q+p-1 \leq n<3 q$, so $L(3, p, q)=L^{d}(3, p, q)$ due to $H^{s}(3 q, p, q) \geq 4$. Finally, we note that Case 3 . yields $H(n, p, q)=H^{s}(n, p, q)$ for $3 q \leq n \leq 3 p+q-1$, so $L(4, p, q)=L^{s}(4, p, q)$ due to $H(3 q-1, p, q) \leq H^{s}(3 q-1, p, q) \leq 4$.

The remaining cases have already been well understood:
Fact $4.7([21,5])$. If $p, q>1$ are integers such that $\operatorname{gcd}(p, q) \notin\{p, q\}$, then

$$
L(h, p, q)=\operatorname{gcd}(p, q) \cdot L\left(h, \frac{p}{\operatorname{gcd}(p, q)}, \frac{q}{\operatorname{gcd}(p, q)}\right) .
$$

Fact 4.8 ([22]). If $q, h$ are integers such that $q>2,2 \nmid q$, and $h \geq 0$, then

$$
L(h, 2, q)=(2 p+1)\left\lfloor\frac{h}{p}\right\rfloor+h \bmod p .
$$

The results above lead to our first algorithm for computing $L(h, p, q)$.
Corollary 4.9. Given integers $p, q>1$ such that $\operatorname{gcd}(p, q) \notin\{p, q\}$ and an integer $h \geq 0$, the value $L(h, p, q)$ can be computed in $\mathcal{O}(h+\log p+\log q)$ time.

Proof. First, we apply Fact 4.7 to reduce the computation to $L\left(h, p^{\prime}, q^{\prime}\right)$ such that $\operatorname{gcd}\left(p^{\prime}, q^{\prime}\right)=1$ and, without loss of generality, $1<p^{\prime}<q^{\prime}$. This takes $\mathcal{O}(\log p+\log q)$ time. If $p^{\prime}=2$, we use Fact 4.8 , while for $p^{\prime}>2$ we rely on the characterization of Theorem 4.6 , using Lemmas 2.4 and 3.3 for computing $L^{s}$ and $L^{d}$, respectively. The values $\widetilde{G}\left(h^{\prime}, p^{\prime}, q^{\prime}\right)$ form a sorted sequence of multiples of $p^{\prime}$ and $q^{\prime}$, but not of $p^{\prime} q^{\prime}$. Hence, it takes $\mathcal{O}(h)$ time to generate them for $0 \leq h^{\prime} \leq h$. The overall running time is $\mathcal{O}(h+\log p+\log q)$.

## 5 Faster Algorithm for Evaluating $L$

A more efficient algorithm for evaluating $L$ relies on the theory of continued fractions; we refer to [15] and [20] for a self-contained yet compact introduction. A finite continued fraction is a sequence [ $\gamma_{0} ; \gamma_{1}, \ldots, \gamma_{m}$ ], where $\gamma_{0}, m \in \mathbb{Z}_{\geq 0}$ and $\gamma_{i} \in \mathbb{Z}_{\geq 1}$ for $1 \leq i \leq m$. We associate it with the following rational number:

$$
\left[\gamma_{0} ; \gamma_{1}, \ldots, \gamma_{m}\right]=\gamma_{0}+\frac{1}{\gamma_{1}+\frac{1}{\ddots+\frac{1}{\gamma_{m}}}}
$$

Depending on the parity of $m$, we distinguish odd and even continued fractions. Often, an improper continued fraction $[;]=\frac{1}{0}$ is also introduced and assumed to be odd. Each positive rational number has exactly two representations as a continued fraction, one as an even continued fraction, and one as an odd continued fraction. For example, $\frac{5}{7}=[0 ; 1,2,2]=[0 ; 1,2,1,1]$.

Consider a continued fraction $\left[\gamma_{0} ; \gamma_{1}, \ldots, \gamma_{m}\right]$. Its convergents are continued fractions of the form $\left[\gamma_{0} ; \gamma_{1}, \ldots, \gamma_{m^{\prime}}\right]$ for $0 \leq m^{\prime}<m$, and $[;]=\frac{1}{0}$. The semiconvergents also include continued fractions of the form $\left[\gamma_{0} ; \gamma_{1}, \ldots, \gamma_{m^{\prime}-1}, \gamma_{m^{\prime}}^{\prime}\right]$, where $0 \leq m^{\prime} \leq m$ and $0<\gamma_{m^{\prime}}^{\prime}<\gamma_{m^{\prime}}$. The two continued fractions representing a positive rational number have the same semiconvergents.
Example 5.1. The semiconvergents of $[0 ; 1,2,2]=\frac{5}{7}=[0 ; 1,2,1,1]$ are $[;]=\frac{1}{0},[0 ;]=\frac{0}{1},[0 ; 1]=\frac{1}{1}$, $[0 ; 1,1]=\frac{1}{2},[0 ; 1,2]=\frac{2}{3}$, and $[0 ; 1,2,1]=\frac{3}{4}$.

Semiconvergents of $\frac{p}{q}$ can be generated using the (slow) continued fraction algorithm, which produces a sequence of Farey pairs $\left(\frac{a}{b}, \frac{c}{d}\right)$ such that $\frac{a}{b}<\frac{p}{q}<\frac{c}{d}$.

```
Algorithm 1: Farey process for a rational number \(\frac{p}{q}>0\)
    \(\left(\frac{a}{b}, \frac{c}{d}\right):=\left(\frac{0}{1}, \frac{1}{0}\right)\);
    while true do
        Report a Farey pair ( \(\frac{a}{b}, \frac{c}{d}\) );
        if \(\frac{a+c}{b+d}<\frac{p}{q}\) then \(\frac{a}{b}:=\frac{a+c}{b+d}\);
        else if \(\frac{a+c}{b+d}=\frac{p}{q}\) then break;
        else \(\frac{c}{d}:=\frac{a+c}{b+d}\);
```

Example 5.2. For $\frac{p}{q}=\frac{5}{7}$, the Farey pairs are $\left(\frac{0}{1}, \frac{1}{0}\right) \rightsquigarrow\left(\frac{0}{1}, \frac{1}{1}\right) \rightsquigarrow\left(\frac{1}{2}, \frac{1}{1}\right) \rightsquigarrow\left(\frac{2}{3}, \frac{1}{1}\right) \rightsquigarrow\left(\frac{2}{3}, \frac{3}{4}\right)$. The process terminates at $\frac{2+3}{3+4}=\frac{5}{7}$.

Consider the set $\mathcal{F}=\left\{\frac{a}{b}: a, b \in \mathbb{Z}_{\geq 0}, \operatorname{gcd}(a, b)=1\right\}$ of reduced fractions (including $\frac{1}{0}$ ). We denote $\mathcal{F}_{k}=\left\{\frac{a}{b} \in \mathcal{F}: a+b \leq k\right\}$ and, for each $x \in \mathbb{R}_{+}$:

$$
\operatorname{Left}_{k}(x)=\max \left\{a \in \mathcal{F}_{k}: a \leq x\right\} \quad \text { and } \quad \operatorname{Right}_{k}(x)=\min \left\{a \in \mathcal{F}_{k}: a \geq x\right\} .
$$

We say that $\frac{a}{b}<x$ is a best left approximation of $x$ if $\frac{a}{b}=\operatorname{Left}_{k}(x)$ for some $k \in \mathbb{Z}_{\geq 0}$. Similarly, $\frac{c}{d}>x$ is a best right approximation of $x$ if $\frac{c}{d}=\operatorname{Right}_{k}(x)$.
Example 5.3. We have $\mathcal{F}_{7}=\left(\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \frac{4}{3}, \frac{3}{2}, \frac{2}{1}, \frac{5}{2}, \frac{3}{1}, \frac{4}{1}, \frac{5}{1}, \frac{6}{1}, \frac{1}{0}\right)$. Here, $\operatorname{Left}_{7}\left(\frac{5}{7}\right)=\frac{2}{3}$ and $\operatorname{Right}_{7}\left(\frac{5}{7}\right)=\frac{3}{4}$ are best approximations of $\frac{5}{7}$.

We heavily rely on the following extensive characterization of semiconvergents:
Fact 5.4 ([15], [25, Theorem 3.3], [20, Theorem 2]). Let $\frac{p}{q} \in \mathcal{F} \backslash\left\{\frac{1}{0}, \frac{0}{1}\right\}$. The following conditions are equivalent for reduced fractions $\frac{a}{b}<\frac{p}{q}$ :
(a) the Farey process for $\frac{p}{q}$ generates a pair $\left(\frac{a}{b}, \frac{c}{d}\right)$ for some $\frac{c}{d} \in \mathcal{F}$,
(b) $\frac{a}{b}$ is an even semiconvergent of $\frac{p}{q}$,
(c) $\frac{a}{b}$ is a best left approximation of $\frac{p}{q}$,
(d) $b=\left\lfloor\frac{a q}{p}\right\rfloor+1$ and $a q \bmod p>i q \bmod p$ for $0 \leq i<a$.

By symmetry, the following conditions are equivalent for reduced fractions $\frac{c}{d}>\frac{p}{q}$ :
(a) the Farey process for $\frac{p}{q}$ generates a pair $\left(\frac{a}{b}, \frac{c}{d}\right)$ for some $\frac{a}{b} \in \mathcal{F}$,
(b) $\frac{c}{d}$ is an odd semiconvergent of $\frac{p}{q}$,
(c) $\frac{c}{d}$ is a best right approximation of $\frac{p}{q}$,
(d) $c=\left\lfloor\frac{d p}{q}\right\rfloor+1$ and $d p \bmod q>i p \bmod q$ for $0 \leq i<d$.

Example 5.5. For $\frac{p}{q}=\frac{5}{7}$, the prefix maxima of $(i q \bmod p)_{i=0}^{p-1}=(0,2,4,1,3)$ are attained for $i=0,1,2$ (numerators of $\frac{0}{1}, \frac{1}{2}, \frac{2}{3}$ ) while the prefix maxima of $(i p \bmod q)_{i=0}^{q-1}=(0,5,3,1,6,4,2)$ are attained for $i=0,1,4$ (denominators $\frac{1}{0}, \frac{1}{1}, \frac{3}{4}$ ).

Due to Fact 5.4, the best approximations can be efficiently computed using the fast continued fraction algorithm; see [20].
Corollary 5.6. Given $\frac{p}{q} \in \mathcal{F}$ and a positive integer $k, 1 \leq k<p+q$, the values $\operatorname{Left}_{k}\left(\frac{p}{q}\right)$ and $\operatorname{Right}_{k}\left(\frac{p}{q}\right)$ can be computed in $\mathcal{O}(\log k)$ time.

Next, we characterize the function $L^{d}$.
Lemma 5.7. Let $p, q>2$ be relatively prime integers and let $h<p+q-3$. If $\frac{a}{b}=\operatorname{Left}_{h+3}\left(\frac{p}{q}\right)$ and $\frac{c}{d}=\operatorname{Right}_{h+3}\left(\frac{p}{q}\right)$, then, assuming $G(-1, p, q)=0$ :

$$
L^{d}(h, p, q)= \begin{cases}\widetilde{G}(a+b-2, p, q)+\widetilde{G}(c+d-2, p, q) & \text { if } a+b+c+d=h+4 \\ \widetilde{G}(h+2, p, q) & \text { otherwise }\end{cases}
$$

Proof. Let us start with a special case of $\frac{a}{b}=\frac{0}{1}$. Then $\frac{c}{d}=\frac{1}{h+2}$, so $q>(h+2) p$ and $\widetilde{G}(k, p, q)=(k+1) p$ for $k \leq h+1$. Consequently, by Lemma 3.3,

$$
L^{d}(h, p, q)=\min _{k=0}^{h}(\widetilde{G}(k, p, q)+\widetilde{G}(h-k, p, q))=(h+2) p .
$$

Due to $a+b+c+d=0+1+1+h+2=h+4$, this is equal to the claimed value of $\widetilde{G}(-1, p, q)+\widetilde{G}(h+1, p, q)=$ $0+(h+2) p$. Symmetrically, the lemma holds if $\frac{c}{d}=\frac{1}{0}$. Thus, below we assume $\frac{1}{h+2}<\frac{p}{q}<\frac{h+2}{1}$.

By Fact 5.4, $\frac{a+c}{b+d}$ is a best (left or right) approximation of $\frac{p}{q}$, so $\max (a+b, c+d) \leq h+3<a+b+c+d$. Moreover,

$$
G(a q, p, q)=\left\lfloor\frac{a q}{p}\right\rfloor+\left\lfloor\frac{a q}{q}\right\rfloor=b-1+a \quad \text { and } \quad G(d p, p, q)=\left\lfloor\frac{d p}{p}\right\rfloor+\left\lfloor\frac{d p}{q}\right\rfloor=d+c-1
$$

so $\widetilde{G}(a+b-2, p, q)+\widetilde{G}(c+d-2, p, q)=a q+d p$
First, suppose that $a+b+c+d<h+4$. Assume without loss of generality that $\widetilde{G}(h+2, p, q)=\alpha p$ is a multiple of $p$. Note that $d<\alpha<b+d$ due to

$$
G((b+d) p, p, q)=b+d+\left\lfloor\frac{(b+d) p}{q}\right\rfloor \geq a+b+c+d-1 \geq h+4 .
$$

Consequently, Fact 5.4 yields $\alpha p \bmod q<d p \bmod q$. Hence

$$
G((\alpha-d) p, p, q)=(\alpha-d)+\left\lfloor\frac{\alpha p-d p}{q}\right\rfloor=(\alpha-d)+\left\lfloor\frac{\alpha p}{q}\right\rfloor+\left\lfloor\frac{-d p}{q}\right\rfloor=h+3-c-d
$$

and therefore

$$
L^{d}(h, p, q) \geq \widetilde{G}(h+2-c-d, p, q)+\widetilde{G}(c+d-2, p, q)=(\alpha-d) p+d p=\widetilde{G}(h+2, p, q)
$$

On the other hand, Lemma 4.5 yields $H^{d}(\alpha p, p, q) \geq G(\alpha p, p, q)-2=h+1$, so $L^{d}(h, p, q) \leq \widetilde{G}(h+2, p, q)$.
Finally, suppose that $a+b+c+d=h+4$. Lemma 3.3 immediately yields $L^{d}(h, p, q) \geq \widetilde{G}(a+b-$ $2, p, q)+\widetilde{G}(c+d-2, p, q)=a q+d p$. For the proof of the inverse inequality, let us take $k$ such that $L^{d}(h, p, q)=\widetilde{G}(k, p, q)+\widetilde{G}(h-k, p, q)$, and define $x=\widetilde{G}(k, p, q)$ and $y=\widetilde{G}(h-k, p, q)$. Consequently,

$$
\begin{array}{r}
a+b+c+d=h+4=\left\lfloor\frac{x-1}{p}\right\rfloor+\left\lfloor\frac{x-1}{q}\right\rfloor+\left\lfloor\frac{y-1}{p}\right\rfloor+\left\lfloor\frac{y-1}{q}\right\rfloor+4=\left\lceil\frac{x}{p}\right\rceil+\left\lceil\frac{y}{p}\right\rceil+\left\lceil\frac{x}{q}\right\rceil+\left\lceil\frac{y}{q}\right\rceil \geq \\
\left\lceil\frac{x+y}{p}\right\rceil+\left\lceil\frac{x+y}{q}\right\rceil \geq\left\lceil\frac{a q+d p}{p}\right\rceil+\left\lceil\frac{a q+d p}{q}\right\rceil=d+\left\lceil\frac{a q}{p}\right\rceil+a+\left\lceil\frac{d p}{q}\right\rceil=d+b+a+c .
\end{array}
$$

Each intermediate inequality must therefore be an equality, so we conclude that

$$
\left\lceil\frac{x}{p}\right\rceil+\left\lceil\frac{y}{p}\right\rceil=\left\lceil\frac{x+y}{p}\right\rceil=\left\lceil\frac{a q+d p}{p}\right\rceil=b+d \quad \text { and } \quad\left\lceil\frac{x}{q}\right\rceil+\left\lceil\frac{y}{q}\right\rceil=\left\lceil\frac{x+y}{q}\right\rceil=\left\lceil\frac{a q+d p}{q}\right\rceil=a+c .
$$

If $p \mid x$ and $p \mid y$, then $\frac{x+y}{p}=b+d$, so $\left\lceil\frac{(b+d) p}{q}\right\rceil=a+c$. Hence $\frac{a+c}{b+d} \geq \frac{p}{q}$, and consequently $\frac{a+c}{b+d}$ is either a right semiconvergent of $\frac{p}{q}$ or is equal to $\frac{p}{q}$. In both cases, Fact 5.4 implies $(-(b+d) p) \bmod q<$ $\min ((-x) \bmod q,(-y) \bmod q)$. This lets us derive a contradiction:

$$
\left\lceil\frac{x}{q}\right\rceil+\left\lceil\frac{y}{q}\right\rceil=\frac{x+y+(-x) \bmod q+(-y) \bmod q}{q}>\frac{(b+d) p+2((-(b+d) p) \bmod q)}{q} \geq\left\lceil\frac{(b+d) p}{q}\right\rceil .
$$

Symmetrically, $q \mid x$ and $q \mid y$ yields an analogous contradiction.
Thus, without loss of generality we may assume $p \mid x$ and $q \mid y$. However, the conditions $x+y \geq a q+d p$ and $\left\lceil\frac{x+y}{p}\right\rceil=\left\lceil\frac{a q+d p}{p}\right\rceil$ yield $(-y) \bmod p=(-(x+y)) \bmod p \leq(-(a q+d p)) \bmod p=(-d p) \bmod p$. By Fact 5.4, this implies $y=d p$. Symmetrically, $x=a q$. Thus, $L^{d}(h, p, q)=a q+d p$, as claimed.

Lemma 5.7 applies to $h<p+q-3$; the following fact lets us deal with $h \geq p+q-3$. It appeared in [5], but we provide an alternative proof for completeness.

Fact 5.8 ([5, Theorem 4]). Let $p, q$ be relatively prime positive integers. For each $h \geq 0$, we have

$$
L^{d}(h, p, q)=L^{d}(h \bmod (p+q-2), p, q)+\left\lfloor\frac{h}{p+q-2}\right\rfloor \cdot p q .
$$

Moreover, $L^{d}(p+q-3, p, q)=p q$.
Proof. First, note that $\widetilde{G}(k, p, q)+\widetilde{G}(p+q-3-k, p, q)=p q$ holds for $0 \leq k \leq p+q-3$. Hence, $L^{d}(p+q-3, p, q)=p q$ holds as claimed due to Lemma 3.3.

For the first part of the statement, it suffices to prove that $H^{d}(n+p q, p, q)=H^{d}(n, p, q)+p+q-2$ for each $n \geq q$. The function $G$ satisfies an analogous equality, so Lemma 3.3 immediately yields $H^{d}(n+p q, p, q) \leq p+q+2+H^{d}(n, p, q)$. The other inequality also follows from Lemma 3.3 unless each optimum value $l$ for $n+p q$ satisfies $n \leq l<p q$. However, for such $l$ (and $q<n<p q$ ), we have

$$
\begin{aligned}
& G(l, p, q)+G(n+p q-l-1, p, q)=\left\lfloor\frac{l}{p}\right\rfloor+\left\lfloor\frac{l}{q}\right\rfloor+\left\lfloor\frac{n+p q-l-1}{p}\right\rfloor+\left\lfloor\frac{n+p q-l-1}{p}\right\rfloor \geq \\
&\left\lfloor\frac{n+p q}{p}\right\rfloor-1+\left\lfloor\frac{n+p q}{q}\right\rfloor-1=G(n+p q, p, q)+G(0, p, q)
\end{aligned}
$$

a contradiction. This concludes the proof.
Theorem 5.9. Given integers $p, q \geq 1$ such that $\operatorname{gcd}(p, q) \notin\{p, q\}$ and an integer $h \geq 0$, the value $L(h, p, q)$ can be computed in $\mathcal{O}(\log p+\log q)$ time.
Proof. We proceed as in the proof of Corollary 4.9, except that we apply Fact 5.8 and Lemma 5.7 to compute $L^{d}(h, p, q)$. Fact 5.8 reduces the problem to determining $L^{d}\left(h^{\prime}, p, q\right)$, where $h^{\prime}=h \bmod (p+q-$ 2). We use Corollary 5.6 to compute $\operatorname{Left}_{h^{\prime}+3}\left(\frac{p}{q}\right)$ and $\operatorname{Right}_{h^{\prime}+3}\left(\frac{p}{q}\right)$ in $\mathcal{O}\left(\log h^{\prime}\right)$ time. The values $\widetilde{G}(r, p, q)$ can be determined in $\mathcal{O}(\log r)$ time using binary search (restricted to multiples of $p$ or $q$ ). The overall running time for $L^{d}(h, p, q)$ is $\mathcal{O}\left(\log h^{\prime}\right)=\mathcal{O}(\log p+\log q)$, so for $L(h, p, q)$ it is also $\mathcal{O}(\log p+\log q)$.

## 6 Closed-Form Formula for $L(h, \cdot, \cdot)$

In this section we show how to compute a compact representation of the function $L(h, \cdot, \cdot)$ in $\mathcal{O}(h \log h)$ time. We start with such representations for $\widetilde{G}$ and $L^{d}$.

Assume that $h<p+q-3$. For $0<i \leq h+4$, let us define fractions

$$
l_{i}=\frac{i-1}{h+4-i}, \quad m_{i}=\frac{i}{h+4-i}
$$

called the $h$-special points and the $h$-middle points, respectively. Now, The function $\widetilde{G}$ can be expressed as follows (see Fig. 3):
Lemma 6.1. If $\operatorname{gcd}(p, q)=1$ and $h<p+q-3$, then

$$
\widetilde{G}(h+2, p, q)= \begin{cases}(h+4-i) \cdot p & \text { if } l_{i} \leq \frac{p}{q} \leq m_{i} \\ i \cdot q & \text { if } m_{i} \leq \frac{p}{q} \leq l_{i+1}\end{cases}
$$

Proof. Note that $\widetilde{G}(h+2, p, q)=n$ is equivalent to $G(n-1, p, q) \leq h+2<G(n, p, q)$. Additionally, observe that $\widetilde{G}(h+2, p, q)$ is a multiple of $p$ or $q$. We have two cases.

$$
\begin{aligned}
& \frac{\mathbf{0}}{\mathbf{1 0}} \stackrel{10 p}{\leftarrow} \frac{1}{10} \xrightarrow{q} \frac{\mathbf{1}}{\mathbf{9}} \stackrel{9 p}{\leftarrow} \frac{\mathbf{2}}{9} \xrightarrow{2 q} \frac{\mathbf{2}}{\mathbf{8}} \stackrel{8 p}{\leftarrow} \frac{3}{8} \xrightarrow{3 q} \frac{\mathbf{3}}{\mathbf{7}} \stackrel{7 p}{\leftarrow} \stackrel{4}{\rightarrow} \stackrel{4 q}{\mathbf{6}} \stackrel{6 p}{\leftarrow} \frac{5}{6} \xrightarrow{5 q} \frac{\mathbf{5}}{\mathbf{5}} \\
& \frac{\mathbf{0}}{\mathbf{1 3}} \stackrel{13 p}{\leftarrow} \frac{1}{\mathbf{1}} \stackrel{q}{\rightarrow} \frac{\mathbf{1}}{\mathbf{1 2}} \stackrel{12 p}{\leftarrow} \frac{2}{12} \xrightarrow{2 q} \frac{\mathbf{2}}{\mathbf{1 1}} \stackrel{11 p}{11} \xrightarrow{3} \rightarrow \frac{\mathbf{3}}{\mathbf{1 0}} \stackrel{10 p}{10} \xrightarrow{4} \stackrel{4 q}{\mathbf{4}} \stackrel{9 p}{\leftarrow} \frac{5}{9} \xrightarrow{5 q} \frac{\mathbf{5}}{\mathbf{8}} \stackrel{8 p}{\leftarrow} \frac{6}{8} \stackrel{6 q}{\rightarrow} \frac{\mathbf{6}}{\mathbf{7}} \stackrel{7 p}{\leftarrow} \frac{\mathbf{7}}{\mathbf{7}}
\end{aligned}
$$

Figure 3: Graphical representations of the closed-form formulae for $\widetilde{G}(9, p, q)$ (above) and $\widetilde{G}(12, p, q)$ (below) for $p<q$ : partitions of $[0,1]$ into intervals w.r.t. $p / q$ and linear functions of $p$ and $q$ for each interval. The respective special points are shown in bold.

Case 1: The condition $\widetilde{G}(h+2, p, q)=j \cdot q$ for $j \in \mathbb{Z}_{>0}$ is equivalent to:

$$
\left\lfloor\frac{j q}{p}\right\rfloor+j \geq h+3 \quad \text { and } \quad\left\lfloor\frac{j q-1}{p}\right\rfloor+j-1 \leq h+2
$$

i.e.,

$$
\left\lfloor\frac{j q}{p}\right\rfloor \geq h+3-j \quad \text { and } \quad\left\lceil\frac{j q}{p}\right\rceil=\left\lfloor\frac{j q-1}{p}\right\rfloor+1 \leq h+4-j .
$$

In other words, we have $h+3-j \leq \frac{j q}{p} \leq h+4-j$, i.e.,

$$
m_{j}=\frac{j}{h+4-j} \leq \frac{p}{q} \leq \frac{j}{h+3-j}=l_{j+1}
$$

Case 2: The condition $\widetilde{G}(h+2, p, q)=j \cdot p$ for $j \in \mathbb{Z}_{>0}$ is equivalent to:

$$
\left\lfloor\frac{j p}{q}\right\rfloor+j \geq h+3 \quad \text { and } \quad\left\lfloor\frac{j p-1}{q}\right\rfloor+j-1 \leq h+2
$$

i.e.,

$$
\left\lfloor\frac{j p}{q}\right\rfloor \geq h+3-j \quad \text { and } \quad\left\lceil\frac{j q}{p}\right\rceil=\left\lfloor\frac{j p-1}{q}\right\rfloor+1 \leq h+4-j .
$$

In other words, we have $h+3-j \leq \frac{p}{q} \leq h+4-j$, i.e.,

$$
l_{h+4-j}=\frac{h+3-j}{j} \leq \frac{p}{q} \leq \frac{h+4-j}{j}=m_{h+4-j}
$$

The family of intervals $\left[m_{i}, l_{i+1}\right]$ and $\left[l_{i}, m_{i}\right]$ has the property that any two distinct intervals in this family have disjoint interiors. Hence, the values of $\widetilde{G}(h, p, q)$ are as claimed.

Combined with Lemma 5.7, Lemma 6.1 yields a closed-form formula for $L^{d}$. Note that for each $i$, we have $l_{i} \leq \operatorname{Left}_{h+3}\left(m_{i}\right) \leq m_{i} \leq \operatorname{Right}_{h+3}\left(m_{i}\right) \leq l_{i+1}$, but none of the inequalities is strict in general. In particular, $\operatorname{Left}_{h+3}\left(m_{i}\right)=m_{i}=\operatorname{Right}_{h+3}\left(m_{i}\right)$ if $\operatorname{gcd}(i, h+4-i)>1$.
Corollary 6.2. Let $p, q$ be relatively prime positive integers and let $h \leq p+q-3$ be a non-negative integer. Suppose that $l_{i} \leq \frac{p}{q} \leq l_{i+1}$ and define reduced fractions $\frac{a_{i}}{b_{i}}=\operatorname{Left}_{h+3}\left(m_{i}\right)$ and $\frac{c_{i}}{d_{i}}=\operatorname{Right}_{h+3}\left(m_{i}\right)$. Then:

$$
L^{d}(h, p, q)= \begin{cases}(h+4-i) \cdot p & \text { if } l_{i} \leq \frac{p}{q} \leq \frac{a_{i}}{b_{i}} \\ a_{i} q+d_{i} p & \text { if } \frac{a_{i}}{b_{i}}<\frac{p}{q}<\frac{c_{i}}{d_{i}} \\ i \cdot q & \text { if } \frac{c_{i}}{d_{i}} \leq \frac{p}{q} \leq l_{i+1}\end{cases}
$$

Proof. First, observe that for $h=p+q-3$, we have $\frac{p}{q}=l_{p+1}$ and $m_{p}<\frac{p}{q}<m_{p+1}$, so $\frac{c_{p}}{d_{p}} \leq \frac{p}{q} \leq \frac{a_{p+1}}{b_{p+1}}$. As claimed, $L^{d}(h, p, q)=(h+4-(p+1)) \cdot p=p \cdot q$.

Below, we assume $h<p+q-3$. Let $\frac{a}{b}=\operatorname{Left}_{h+3}\left(\frac{p}{q}\right)$ and $\frac{c}{d}=\operatorname{Right}_{h+3}\left(\frac{p}{q}\right)$. We shall prove that $a+b+c+d=h+4$ if and only if $\frac{a_{i}}{b_{i}}<\frac{p}{q}<\frac{c_{i}}{d_{i}}$ for some $i$.

First, suppose that $a+b+c+d=h+4$. This means that $\frac{a+c}{b+d} \in \mathcal{F}_{h+4} \backslash \mathcal{F}_{h+3}$, so $\frac{a+c}{b+d}=m_{i}$ for some $i$, and therefore $\frac{a}{b}=\frac{a_{i}}{b_{i}}$ and $\frac{c}{d}=\frac{c_{i}}{d_{i}}$. Consequently, $\frac{a_{i}}{b_{i}}<\frac{p}{q}<\frac{c_{i}}{d_{i}}$. In the other direction, $\frac{a_{i}}{b_{i}}<\frac{p}{q}<\frac{c_{i}}{d_{i}}$ implies $\frac{a}{b}=\frac{a_{i}}{b_{i}}$ and $\frac{c}{d}=\frac{c_{i}}{d_{i}}$, so $\frac{a}{b}<m_{i}<\frac{c}{d}$. By Fact 5.4, this yields $a+b+c+d \leq h+4$. Moreover, $\frac{a+c}{b+d} \notin \mathcal{F}_{h+3}$, so $a+b+c+d=h+4$.

Since $G\left(a_{i} q, p, q\right)=a_{i}+b_{i}-1$ and $G\left(d_{i} p, p, q\right)=c_{i}+d_{i}-1$ by Fact 5.4, we have $a_{i} q+d_{i} p=$ $\widetilde{G}\left(a_{i}+b_{i}-2, p, q\right)+\widetilde{G}\left(c_{i}+d_{i}-2, p, q\right)$. Now, Lemmas 5.7 and 6.1 yield the final formula.

Theorem 6.3. Let $2<p<q$ be relatively prime and let $4<h<p+q-2$. Suppose that $l_{i} \leq \frac{p}{q} \leq l_{i+1}$ and define reduced fractions $\frac{a_{i}}{b_{i}}=\operatorname{Left}_{h+3}\left(m_{i}\right)$ and $\frac{c_{i}}{d_{i}}=\operatorname{Right}_{h+3}\left(m_{i}\right)$. Then:

$$
L(h, p, q)= \begin{cases}\left\lceil\frac{h+1}{2}\right\rceil p+q-(h+1) \bmod 2 & \text { if } 0<\frac{p}{q}<1 /\left\lceil\frac{h}{2}\right\rceil \text { else } \\ (h+4-i) \cdot p & \text { if } l_{i} \leq \frac{p}{q} \leq \frac{a_{i}}{b_{i}}, \\ a_{i} q+d_{i} p & \text { if } \frac{a_{i}}{b_{i}}<\frac{p}{q}<\frac{c_{i}}{d_{i}}, \\ i \cdot q & \text { if } \frac{c_{i}}{d_{i}} \leq \frac{p}{q} \leq l_{i+1} .\end{cases}
$$

This compact representation of $L(h, p, q)$ (see Fig. 4 for an example) for a given $h$ has size $\mathcal{O}(h)$ and can be computed in time $\mathcal{O}(h \log h)$.

$$
\begin{aligned}
& \underset{\substack{\mathbf{b} \\
\text { left subinterval }}}{\mathbf{a}} \stackrel{b \cdot p}{\stackrel{c}{d}} \quad \underset{\text { middle subinterval }}{b} \stackrel{a \cdot q+d \cdot p}{\rightleftarrows} \frac{c}{d} \quad \underset{\text { right subinterval }}{\stackrel{a}{b}} \xrightarrow{c \cdot q} \frac{\mathbf{c}}{\mathbf{d}}
\end{aligned}
$$

Figure 4: Graphical representations of the closed-form formulae for $L(7, p, q)$ (middle) and $L(10, p, q)$ (below). Compared to $\widetilde{G}(9, p, q)$ and $\widetilde{G}(12, p, q)$, respectively, an initial subinterval and several middle subintervals are added. A general pattern for the left, middle, and right subintervals, is presented above. However, the left subinterval $\left(\frac{1}{5}, \frac{1}{4}\right)$ within $L(10, p, q)$ is an exception because is has been trimmed by the initial interval.

Proof. The formula follows from the formulae for $L^{s}$ (Lemma 2.4) and $L^{d}$ (Corollary 6.2) combined using Theorem 4.6. To compute the table for $L$ efficiently, we determine $\frac{a_{i}}{b_{i}}=\operatorname{Left}_{h+3}\left(m_{i}\right)$ and $\frac{c_{i}}{d_{i}}=$ Right $_{h+3}\left(m_{i}\right)$ using Corollary 5.6.

## 7 Relation to Standard Sturmian Words

For a finite directive sequence $\gamma=\left(\gamma_{1}, \ldots, \gamma_{m}\right)$ of positive integers, a Sturmian word $\operatorname{St}(\gamma)$ is recursively defined as $X_{m}$, where $X_{-1}=\mathrm{q}, X_{0}=\mathrm{p}$, and $X_{i}=X_{i-1}^{\gamma_{i}} X_{i-2}$ for $1 \leq i \leq m$; see [16, Chapter 2]. We classify directive sequences $\gamma$ (and the Sturmian words $\operatorname{St}(\gamma)$ ) into even and odd based on the parity of $m$.

Observation 7.1. Odd Sturmian words of length at least 2 end with pq, while even Sturmian words of length at least 2 end with qp .

For a directive sequence $\gamma=\left(\gamma_{1}, \ldots, \gamma_{m}\right)$, we define $\operatorname{fr}(\gamma)=\left[0 ; \gamma_{1}, \ldots, \gamma_{m}\right]$.
Fact $7.2\left(\left[16\right.\right.$, Proposition 2.2.24]). If $\operatorname{fr}(\gamma)=\frac{p}{q}$, then $\operatorname{St}(\gamma)$ contains $p$ characters q and $q$ characters p .
Example 7.3. We have $\frac{5}{7}=[0 ; 1,2,2]=[0 ; 1,2,1,1]$, so the Sturmian words with 5 q's and 7 p's are: $\operatorname{St}(1,2,2)=$ pqpqppqpqppq and $\operatorname{St}(1,2,1,1)=$ pqpqppqpqpqp.

For relatively prime integers $1<p<q$, we define $\operatorname{St}_{p, q}$ as a Sturmian word with $\operatorname{fr}(\gamma)=\frac{p}{q}$. Note that we always have two possibilities for $\mathrm{St}_{p, q}$ (one odd and one even), but they differ in the last two positions only. In fact, the first $p+q-2$ characters of $\mathrm{St}_{p, q}$ are closely related to the values $\widetilde{G}(i, p, q)$.

Fact 7.4 ([16, Proposition 2.2.15]). Let $1<p<q$ be relatively prime integers. If $i \leq p+q-3$, then

$$
\mathrm{St}_{p, q}[i]= \begin{cases}\mathrm{p} & \text { if } p \mid \widetilde{G}(i, p, q) \\ \mathrm{q} & \text { if } q \mid \widetilde{G}(i, p, q)\end{cases}
$$

As a result, the values $\widetilde{G}(i, p, q)$ can be derived from $\mathrm{St}_{p, q}$; see Table 3.
Fact 7.5 ([16, Exercise 2.2.9]). $\operatorname{St}\left(\gamma_{0}^{\prime}, \ldots, \gamma_{m^{\prime}}^{\prime}\right)$ is a prefix of $\operatorname{St}(\gamma)$ if and only if $\left[0 ; \gamma_{0}^{\prime}, \ldots, \gamma_{m^{\prime}}^{\prime}\right]$ is a semiconvergent of $\operatorname{fr}(\gamma)$.

Example 7.6. The semiconvergents of $[0 ; 1,2,2]=\frac{5}{7}=[0 ; 1,2,1,1]$ are $[0 ; 1,2,1]=\frac{3}{4},[0 ; 1,2]=\frac{2}{3}$, $[0 ; 1,1]=\frac{1}{2},[0 ; 1]=1,[0 ;]=\frac{0}{1}$ (and $\frac{1}{0}$ ). They correspond to the following Sturmian prefixes of $\operatorname{St}(1,2,2)=$ pqpqppqpqppq: $\operatorname{St}(1,2,1)=$ pqpqpppq, $\operatorname{St}(1,2)=$ pqpqp, $\operatorname{St}(1,1)=\operatorname{pqp}, \operatorname{St}(1)=$ pq, and St()$=\mathrm{p}$.

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{St}_{p, q}[i]$ | p | q | p | q | p | p | q | p | q | p | $\mathrm{p} / \mathrm{q}$ | $\mathrm{q} / \mathrm{p}$ |
| $\widetilde{G}(i, p, q)$ | $p$ | $q$ | $2 p$ | $2 q$ | $3 p$ | $4 p$ | $3 q$ | $5 p$ | $4 q$ | $6 p$ |  |  |
| $\widetilde{G}(i, p, q)$ | 5 | 7 | 10 | 14 | 15 | 20 | 21 | 25 | 28 | 30 |  |  |

Table 3: The Sturmian words $\operatorname{St}_{p, q}$ for $p=5$ and $q=7$ and the corresponding values of $\widetilde{G}(i, p, q)$ for $i<p+q-2$.

Corollary 7.7. Consider a proper prefix $P$ of Sturmian word $\operatorname{St}(\gamma)$. Moreover, let $\frac{a}{b}=\operatorname{Left}_{|P|}(\operatorname{fr}(\gamma))$ and $\frac{c}{d}=\operatorname{Right}_{|P|}(\operatorname{fr}(\gamma))$. The longest even Sturmian prefix of $P$ has length $a+b$, whereas the longest odd Sturmian prefix of $P$ has length $c+d$.

Proof. By Fact 7.5, the longest even Sturmian prefix of $P$ is the longest Sturmian word $\operatorname{St}\left(\gamma^{\prime}\right)$ such that $\frac{a^{\prime}}{b^{\prime}}:=\operatorname{fr}\left(\gamma^{\prime}\right)$ is an even semiconvergent of $\operatorname{fr}(\gamma)$. Its length $a^{\prime}+b^{\prime} \leq|P|$ is largest possible, so by Fact 5.4 $\frac{a^{\prime}}{b^{\prime}}$ is the best left approximation of $\operatorname{fr}(\gamma)$ with $a^{\prime}+b^{\prime} \leq|P|$. This is precisely how $\frac{a}{b}=\operatorname{Left}_{|P|}(\operatorname{fr}(\gamma))$ is defined.

The proof for odd Sturmian prefixes is symmetric.
The following theorem can be seen as a restatement of Lemma 5.7 in terms of Sturmian words.
Theorem 7.8. Let $\mathrm{St}_{p, q}$ be a standard Sturmian word corresponding to $\frac{p}{q}$ and let $0 \leq h<p+q-3$. If $\mathrm{St}_{p, q}[0 . . h+3]$ is a Sturmian word, then $L^{d}(h, p, q)=\widetilde{G}(l-2, p, q)+\widetilde{G}(r-2, p, q)$, where $l$, $r$ are the lengths of the longest proper Sturmian prefixes of $\operatorname{St}_{p, q}[0 . . h+3]$ of different parities, and $\widetilde{G}(-1, p, q)=0$. Otherwise, $L^{d}(h, p, q)=\widetilde{G}(h+2, p, q)$.
Proof. To apply Lemma 5.7, we set $\frac{a}{b}=\operatorname{Left}_{h+3}\left(\frac{p}{q}\right.$ and $\frac{c}{d}=\operatorname{Right}_{h+3}\left(\frac{p}{q}\right)$. Observe that the mediant $\frac{a+c}{b+d}$ is a better approximation of $\frac{p}{q}$ than $\frac{a}{b}$ or $\frac{c}{d}$, and thus it is a semiconvergent of $\frac{p}{q}$. Thus, we always have $a+b+c+d \geq h+4$ and, by Fact 7.5, equality holds if and only if $\mathrm{St}_{p, q}$ has a Sturmian prefix of length $h+4$. In other words, the case distinction here coincides with the one in Lemma 5.7. If $a+b+c+d>h+4$, then we have $L^{d}(h, p, q)=\widetilde{G}(h+2, p, q)$. Otherwise, $L^{d}(h, p, q)=\widetilde{G}(a+b-2, p, q)+\widetilde{G}(c+d-2, p, q)$. However, due to Corollary 7.7, $\mathrm{St}_{p, q}[0 . . a+b-1]$ is an even Sturmian word corresponding to $(a, b)$, $\mathrm{St}_{p, q}[0 . . c+d-1]$ is an odd Sturmian word corresponding to $(c, d)$, and these are the longest Sturmian prefixes of $\mathrm{St}_{p, q}[0 . . h+2]$ of each parity.
Example 7.9. Consider a word $\mathrm{St}_{5,7}$ as in Table 3. The lengths of its proper even Sturmian prefixes are 2,7 , whereas the lengths of its proper odd Sturmian prefixes are $1,3,5$. Hence, $L^{d}(7,5,7)=\widetilde{G}(9,5,7)=$ 30 , since $\mathrm{St}_{5,7}[0 . .10]$ is not a Sturmian word. Moreover, $L^{d}(8,5,7)=\widetilde{G}(5,5,7)+\widetilde{G}(3,5,7)=20+14=34$, since $\mathrm{St}_{5,7}[0 . .11]=\mathrm{St}_{5,7}$ is a Sturmian word.

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[^0]:    ${ }^{1}$ An integer $p$ is a weak period of $X$ if $X[i] \approx X[i+p]$ for all $0 \leq i<n-p$.

