

Minimal Sum Labeling of Graphs

Matěj Konečný, Stanislav Kučera, Jana Novotná, Jakub Pekárek,
Štěpán Šimsa, and Martin Töpfer*

Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic,
matejkon@gmail.com, stanislav.kucera@outlook.com, janka.novot@seznam.cz,
edalegos@gmail.com, simsa.st@gmail.com, mtopfer@gmail.com

Abstract. A graph G is called a *sum graph* if there is a so-called *sum labeling* of G , i.e. an injective function $\ell : V(G) \rightarrow \mathbb{N}$ such that for every $u, v \in V(G)$ it holds that $uv \in E(G)$ if and only if there exists a vertex $w \in V(G)$ such that $\ell(u) + \ell(v) = \ell(w)$. We say that sum labeling ℓ is *minimal* if there is a vertex $u \in V(G)$ such that $\ell(u) = 1$. In this paper, we show that if we relax the conditions (either allow non-injective labelings or consider graphs with loops) then there are sum graphs without a minimal labeling, which partially answers the question posed by Miller in [6] and [5].

1 Introduction

An undirected graph $G = (V, E)$ is a *sum graph* if there exists an injective function $\ell : V(G) \rightarrow \mathbb{N}$ such that every pair of vertices $u \neq v \in V(G)$ is connected via an edge of G if and only if there exists a vertex $w \in V(G)$ such that $\ell(w) = \ell(u) + \ell(v)$. We call the function ℓ a *sum labeling* or *labeling function*. The value $\ell(u) + \ell(v)$ is an *edge-number* of the edge uv and it is *guaranteed* by vertex w . The *sum number* $\sigma(G)$ of a graph is defined as the least integer, such that $G + \bar{K}_{\sigma(G)}$ (G with $\sigma(G)$ additional isolated vertices) is a sum graph.

The concept of sum graphs was introduced by Harary [4] in 1990. It was further developed by Gould and Rödl [3] and Miller [6], [7] who showed general upper and lower bounds on $\sigma(G)$ of order $\Omega(|E|)$ for a given general graph G and better bounds for specific classes of graphs.

For some graphs the exact sum numbers are known: $\sigma(T_n) = 1$ for trees (of order $n \geq 2$) [2], $\sigma(C_n) = 2$ for cycles ($n \geq 3$, $n \neq 4$) and $\sigma(C_4) = 3$ [4], $\sigma(K_n) = 2n - 3$ for complete graphs ($n \geq 4$) [1], $\sigma(H_{2,n}) = 4n - 5$ for cocktail party graphs ($n \geq 2$) [6] and for complete bipartite graphs [8].

In the work of Miller et al. [6] and [5], an open question was raised whether every sum graph has a labeling that uses number 1. Such labelings are called *minimal*. In [6], a minimal labeling of complete bipartite graphs is presented. In [5], an upper bound on $\sigma(G)$ for G being a disjoint union of graphs G_1, G_2, \dots, G_n is shown. If at least one of the disjoint graphs has minimal labeling then

$$\sigma(G) \leq \sum_{i=1}^n \sigma(G_i) - (n - 1).$$

* Supported by project CE-ITI P202/12/G061 of GA ČR and grant SVV-2017-260452.

In our work, we approach sum graphs from a different perspective. Instead of grounding our research on the properties of graphs, our basic objects are sets of integers. Given an integer set M , the rules of sum labeling uniquely define a graph such that it is a sum graph and its labeling consists exactly of all the integers from the given set. From our research we provide two negative answers to questions parallel to the one raised by Miller et al.

While it is natural to require graphs to have no loops, when we construct a sum graph from an integer set, it seems more natural to allow loops (i.e. for every not necessarily distinct vertices $u, v \in V(G)$ we have $uv \in E(G)$ if and only if there exists $w \in V(G)$ such that $\ell(u) + \ell(v) = \ell(w)$). We call these graphs *sum graphs with loops*. When we say *sum graphs without loops* we refer to the previous definition, where the vertices are required to be distinct. We show the following:

Theorem 1. *There exists an infinite family of sum graphs with loops which admit no minimal labeling.*

Another relaxation of the original problem (without loops) is to allow the integer set M to be a multiset. This of course causes the labeling function to cease being injective, thus we call such graphs *non-injective sum graphs*. Nevertheless, in this approach we may consider graphs without loops and obtain the following similar result:

Theorem 2. *There exists an infinite family of non-injective sum graphs (without loops) which admit no minimal labeling.*

1.1 Preliminaries

Let G be a sum graph with some labeling ℓ . We call set $M = \{\ell(v) : v \in V(G)\}$ *label-set* of G .

For a finite multiset of natural numbers $S \subset \mathbb{N}$, let G_S be the graph with elements of S being its vertices and for every $u, v \in S$ let there be an edge $uv \in E(G_S)$ if and only if $u + v \in S$. Depending on context, we sometimes allow G_S to have loops. We say that set S *induces* a graph G_S . Let f denote the natural bijection of vertices of G_S and integers in S . For any integer $i \in S$ we denote $\psi(i)$ the subset of $V(G_S)$ such that a vertex $v \in \psi(i)$ if and only if $f(v) = i$. We say that an integer i *induces* a vertex v if $v \in \psi(i)$.

We say that two vertices u, v of a graph G without loops are *equivalent* if it holds that $N(u) \setminus \{v\} = N(v) \setminus \{u\}$.

Lemma 1. *Let G be a graph with labeling ℓ and let u, v be two of its vertices. If $\ell(u) = \ell(v)$, then u and v are equivalent.*

Proof. Suppose $\ell(u) = \ell(v)$. Consider any $w \in V(G)$ other than u and v . The edge-numbers of wv and wu are the same, so either both edges are present or none of them is. Since we do not consider loops, the only remaining edge to consider is uv . If $uv \notin E(G)$, then clearly $N(u) = N(v)$. If $uv \in E(G)$, then all neighbors of v are also neighbors of u except u itself and vice versa. In both cases, u and v are equivalent.

To get an analogous definition for a graph with loops one would not exclude the vertices v and u from the neighborhoods. This rather subtle difference actually makes dealing with sum graphs with loops much easier.

We define an operation that removes loops from sum graphs. Consider a sum graph G with loops. Let us take one by one each vertex v with a loop and choose any integer $k \geq 2$ (independently for each vertex). We replace v with a clique K_k and connect all neighbors of v to all vertices from K_k . We denote set of all possible results of this operation by $\mathcal{C}(G)$, and denote $\mathcal{C}^k(G)$ the unique result where we fix all values k from the construction to a given fixed value. Let G be induced by a multiset M , we may equivalently define $\mathcal{C}(G)$ as all graphs induced by all possible multisets obtained from the set M via raising the multiplicity of membership of any $i \in M$ such that $2i \in M$.

2 Sum Graphs with Loops

This section serves as an introduction to the topic of this paper. We show that there is a sum graph with loops that admits no minimal sum labeling. Though this is weaker result than Theorem 1, we provide a direct proof without usage of complex tools. The full proof of Theorem 1 is given later as a consequence of Theorem 2.

For a proof, we use the graph induced by the set $\{2, 3, 4, 6, 7\}$. This specific graph was chosen based on a result of a computer experiment as the smallest graph induced by an arithmetic sequence with difference 1 starting from 2 with one element missing such that no minimal labeling was found. The proof goes through several cases and is given in the Appendix.

Theorem 3. *There exists a sum graph with loops that admits no minimal labeling.*

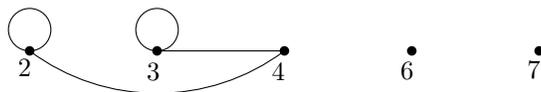


Fig. 1. The sum graph induced by the set $\{2, 3, 4, 6, 7\}$.

Although Theorem 3 does not require the labeling function to be injective, the theorem holds also under the constraint of injectivity. The labeling used to induce the graph is certainly injective, and the theorem shows that there exists no minimal labeling; thus, in particular no injective minimal labeling.

It is easy to observe that sets $\{1, 2, \dots, k\}$ induce graphs with the maximum number of edges out of all sum graphs with loops on the same number of vertices. In fact, it can be shown that any graph with the same number of vertices and edges is necessarily isomorphic to this graph. Based on this observation, it seems

reasonable to assume that the set $\{2, 3, \dots, k-1, k\}$ might induce a graph with specific structure and possibly exclude all labelings with 1 as 1 was removed from the inducing set. This idea does not hold, as the same graph is induced for example by the set $\{1, 2, \dots, k, 3k\}$.

However, the situation seems to change dramatically once we remove one more value. Let us call *gap-graphs* of size k all graphs induced by the set obtained from $\{2, \dots, k\}$ by removing one element we call a *gap*. Hence, a gap-graph of size k has $k-2$ vertices. While for small values of k some gap-graphs have a minimal labeling, we conjecture that for $k \geq 10$ none of the gap-graphs of size k with gap i such that $3 < i < k$ has a minimal labeling. While we do not prove this conjecture, it serves as a basic inspiration for our main result and as a consequence of our main result, we prove Theorem 1 by providing a partial proof of the conjecture for sufficiently large n and a specific choice of gap.

3 Non-injective Sum Graphs without Loops

In this section, we construct graphs with (non-injective) sum labelings such that they do not admit minimal labelings.

Based on the result from the previous section and the conjecture about gap-graphs, it is a natural question whether we can modify the gap-graph idea to remove all loops and keep the desired properties that may prevent existence of minimal labelings. Let us consider the gap-graph G induced by the set $\{2, 3, 4, 6, 7\}$ from Theorem 1, and its loopless modification $H = \mathcal{C}^2(G)$. The sum graph H is induced by a multiset $\{2, 2, 3, 3, 4, 6, 7\}$ by definition. Unfortunately a graph isomorphic to H is induced by a multiset $\{1, 5, 2, 2, 4, 6, 9\}$ (with the same ordering of vertices). This numbering can be naturally extended to a (non-injective) labeling of any graph from $\mathcal{C}(G)$. While this gives us a negative result, we show that for large enough gap-graphs and at least some choice of gap, the construction \mathcal{C} does in fact guarantee all produced graphs to admit no minimal labeling.

We first develop some tools applicable (up to minor adjustments) to all flavors of sum graphs (injective, non-injective, with or without loops). Namely, that labels of each sum labeling of a sum graph can be described as a set of arithmetic sequences. We show how several simultaneous description of this form limit each other and in doing so reflect some structural aspects of underlying sum graphs directly into their label-sets.

The graphs we work with are based on arithmetic sequence of integers. We define graph A_n as the unique sum graph with loops induced by the set $\{2, 3, \dots, n-1, n, n+2\}$, in other words by a set of all integers from 2 up to $n+2$ without the second-last value $n+1$. Note that graph A_n has exactly n vertices.

3.1 Sequence Description

Let us denote an arithmetic sequence $(a, a+d, a+2d, \dots, a+jd)$ with difference d as $(id+a)_i$ where we always consider i going from 0 up to some unspecified integer. We also generally refer to sequences with difference d as *d-sequences*.

Let us fix a vertex v and call it *generator*. The set of *terminals* associated with this generator is defined as $V(G) \setminus N(v)$. Note that v has exactly $n - \deg(v)$ terminals. It is an important observation is that unless v has a loop, it is its own terminal. We say that a terminal w is a *proper terminal* of v if $w \neq v$, and is *improper terminal* otherwise.

Lemma 2. *Let G be a sum graph without loops, let ℓ be a fixed labeling of G , let v be a fixed generator and let w be a proper terminal of v . Then there is no vertex in G labeled $\ell(v) + \ell(w)$.*

Proof. For contradiction, let a vertex $u \in V(G)$ be labeled $\ell(v) + \ell(w)$. Then the edge-number of the edge vw is guaranteed by u , thus $vw \in E(G)$. This however makes w a neighbor of v and we reach a contradiction.

The previous lemma does not hold for improper terminal v , as the edge used in the proof would be a loop and thus would not be an edge of G even though it is technically guaranteed. In case of sum graphs with loops this issue does not arise.

Lemma 3. *Let G be a sum graph with fixed labeling ℓ , let M denote the label-set of G , let v be a fixed generator and let u be a non-terminal of v . Then there exists a sequence $S = (i\ell(v) + \ell(u))_i \subseteq M$ such that its last element is a label of a proper terminal of v .*

Proof. Since u is a non-terminal of v , there exists an edge uv with edge-number $\ell(v) + \ell(u)$. Clearly, this edge is guaranteed by some vertex u_1 labeled $\ell(v) + \ell(u)$. The vertex u_1 cannot be v as $\ell(u_1) > \ell(v)$. If u_1 is a proper terminal, then $S = (\ell(u), \ell(u) + \ell(v))$ and we are done. Otherwise, u_1 is a non-terminal and we iterate the previous argument, building a sequence S' from u_1 and setting $S = (\ell(u)).S'$ where the dot operation denotes sequence concatenation.

Lemma 4. *Let G be a sum graph, and let v be a fixed generator with k terminals and label $\ell(v) = g$. Then for every labeling ℓ and associated label-set M , the following holds:*

1. *The label-set M can be described as a union of at most k distinct arithmetic sequences with difference g .*
2. *The last element of each of the sequences is a label of a terminal.*
3. *Each label of a proper terminal is the last element of one of the sequences.*
4. *If v has j non-equivalent terminals and the description of M has j sequences, then one of the sequences is a singleton sequence (g) .*

Proof. Let u be a non-terminal of v such that it has the lowest label of all non-neighbors. From Lemma 3 we have a sequence S_1 such that it covers some elements of M and ends in a terminal. If $M \setminus S_1$ contains non-terminals, we iterate by using the Lemma 3 with the lowest remaining non-terminal label. If $M \setminus S_1$ contains no further non-terminals and is non-empty, then we create a singleton sequence $(\ell(w))$ for each proper terminal w . Clearly the only element of M

that can remain not-covered is $\ell(v)$, as v is improper terminal. In such case we create one more singleton sequence $(\ell(v))$. Suppose $S_1 = (ig + a)_i$ and $S_2 = (ig + b)_i$, for some integers a, b , are two constructed non-singleton sequences (S_1 was constructed first). Since both S_1 and S_2 have the same difference, $a \notin S_2$ and $a < b$ by the choice of a and b , both sequences are distinct. Clearly, no singleton sequence shares an element with any other sequence. Since there are exactly k terminals, each sequence ends with a terminal label and all sequences are distinct, we get that there are at most k sequences in total. Naturally, M is in the union of all of the constructed sequences and the sequences contain no extra elements. This proves points 1 and 2.

From Lemma 2, we have that if a sequence contains the label of a proper terminal, the hypothetical next element of the sequence is not a label of any vertex. Thus labels of proper terminals can only be the last elements in sequences, which proves the point 3.

Let v have j non-equivalent terminals. From Lemma 1 we have that there are j distinct labels of terminals in M . If there are j sequences in the description of M via generator v , we deduce from the previous points that each of the distinct labels is the last element of a distinct sequence. In particular, the label g must be the last in a sequence as v is its own (improper) terminal. Since all labels are positive and the difference of each sequence is g , the label g must form a singleton sequence (g) .

Note that some labels generated by a sequence may be labels of several vertices, if these are equivalent.

Let v be a vertex of a fixed graph G . Let v -cover denote the set of sequences covering the label-set M of G as described in Lemma 4. Each such set is associated with one difference value. If α is this difference value (i.e. $\alpha = \ell(v)$), then we also reference to such a cover as α -cover.

3.2 Cover Merging

While the results of the previous section do not give particularly strong results when all the degrees are low (in respect to the number of vertices), it does give strong limits on potential labelings once we have one or more vertices with almost full degree. In this section, we expand our tools to impose additional constraints when applying the previous results to multiple vertices simultaneously.

Let G be a graph with fixed proper labeling and let M be its label-set. Then each vertex g_i induces a cover C_i of M , as described in Lemma 4. Each such cover C_i is associated with a difference d_i ($d_i = \ell(g_i)$) and a number of terminals of g_i (including g_i) denoted t_i .

We say that two covers C_i and C_j are *mergeable* if the number of non-equivalent vertices in G (and thus also the size of any label-set of G) is at least $2 \cdot (t_i - 1) \cdot (t_j - 1) + 3$. A proof of the following lemma is given in the Appendix.

Lemma 5. *Let C_i and C_j be mergeable covers, then there exists a pair of sequences S_i, S_j from covers C_i resp. C_j such that they share at least three elements.*

Lemma 6. *For any fixed mergeable covers C_1, C_2 there exist positive integers j, k , such that $GCD(j, k) = 1$, $j \leq t_1 - 1$, $k \leq t_2 - 1$ and the equation $k \cdot d_1 = j \cdot d_2$ holds.*

Proof. Let us fix two mergeable covers C_1 and C_2 . Let S_1, S_2 denote sequences such that $S_1 \in C_1$, $S_2 \in C_2$ and $|S_1 \cap S_2| \geq 3$. Let x_0, x_1, x_2 be the three smallest elements of $S_1 \cap S_2$ so that $x_0 < x_1 < x_2$. Since all of them come from both sequences with no holes, we may denote the distance between elements $m = x_1 - x_0 = x_2 - x_1$.

As both x_1, x_2 belong to both sequences, it must hold that $m = k \cdot d_1 = j \cdot d_2$ for some positive integers j, k such that x_2 is k -th element following x_1 in S_1 and also j -th element following x_1 in S_2 .

Suppose $d_1 = d_2$, then $k = j = 1 \leq t_1 - 1, t_2 - 1$ and the lemma holds trivially. We may assume $d_1 \neq d_2$. From definition of a single sequence, S_1 contains all k possible elements from interval $(x_1, x_2]$. Similarly, S_2 contains all j elements from $(x_1, x_2]$. If $GCD(k, j) > 1$ then there is some $m_0 < m$ such that $x_1 + m_0$ is an element of both sequences which would contradict the minimality of x_1, x_2 . More generally, if any two elements x_3, x_4 of S_1 such that $x_1 < x_3 < x_4 \leq x_2$ belonged to the same sequence $S_0 \in C_2$, then $x_1 + (x_4 - x_3) \in S_1 \cap S_2$ which would contradict the minimality of x_2 . Analogously, we would reach a contradiction if any such x_3, x_4 belonged to any $S'_0 \in C_1$. Thus the k elements of S_1 from the interval $(x_1, x_2]$ fall into distinct sequences from C_2 and we have $k \leq t_2$ as t_2 limits the number of sequences in C_2 . Analogously we get $j \leq t_1$.

From Lemma 4, we know that if the k elements of S_1 fall into t_1 distinct sequences from C_2 , then one of them has to be the singleton sequence (d_2) . Recall that we chose x_1, x_2 from $S_1 \cap S_2$ so that they are preceded by some element x_0 from $S_1 \cap S_2$. This means that $x_1 > d_2$ and thus the singleton sequences from C_2 cannot, in fact, play any role. Thus, the limit on the number of sequences involved can be further reduced to $k \leq t_2 - 1$. Symmetrically, we obtain that $j \leq t_1 - 1$.

Lemma 7. *For any fixed mergeable covers C_1, C_2 of a graph G without loops, let $d_1 = 1$. Then $d_2 \leq t_2 - 1$.*

Proof. Consider the equation from Lemma 6. If $d_1 = 1$ then $k = j \cdot d_2$. From the same lemma we also know that $GCD(j, k) = 1$, so necessarily $j = 1$ and $k = d_2$. Finally, we also have inequality $k \leq t_2 - 1$ which together give $d_2 \leq t_2 - 1$.

3.3 Vertices $\psi(2)$ and $\psi(3)$ in $\mathcal{C}(A_n)$

In this section, we explore the exact structure of terminals of graphs from $\mathcal{C}(A_n)$. Based on a number of simple properties we show that given large enough n , no graph from $\mathcal{C}(A_n)$ admits label 1 on any vertex from $\psi(2)$ and $\psi(3)$.

For a vertex v of a fixed labeling, let $\tau(v)$ denote the set of proper terminals of v . Lemma 8 summarizes some basic observations. For its proof see the Appendix.

Lemma 8. For any graph $G \in \mathcal{C}(A_n)$ such that $n \geq 39$, let us fix arbitrary vertices v_i such that $v_i \in \psi(i)$ for values of i from 2 to 6. Then for any integers j, k such that $2 \leq j, 2 \leq k \leq 6$ all of the following holds:

1. Vertex v_k has exactly k proper terminals (and $k + 1$ terminals in total), all of which have distinct labels.
2. If $j \leq 3$, then the vertices v_j and v_k are mergeable.
3. If labeling is minimal and either $\ell(v_2)$ or $\ell(v_3)$ equals to 1, then $\ell(v_k) \leq k$.
4. There is exactly one proper terminal in the intersection of all $\tau(v_k)$, and this terminal has the highest label in the graph.
5. If j is not the highest label, then either $j \notin \tau(v_2)$ or $j \notin \tau(v_3)$.
6. $\tau(v_k) \subset \tau(v_j)$ whenever $j \geq k + 2$.
7. None of the chosen vertices is a proper terminal of any of the other chosen vertices and thus $\ell(v_j) + \ell(v_k) \in M$ for any $j \neq k$.

For convenience, we extend the meaning of *terminal*. We characterize possible labelings of graphs in terms of presence or absence of values in respect to the label-set of the graph. For a vertex v we say an integer value k is a (proper) *terminal* for v as a shortcut for the fact that there exists a vertex w , which is a (proper) terminal for v and $\ell(w) = k$. We only deal with terminals of vertices from $\psi(2), \dots, \psi(6)$ whose all proper terminals are non-equivalent and thus have distinct labels. Hence the integer terminals and the vertex terminals are in one-to-one correspondence for these vertices.

Let G be a sum graph with a minimal labeling ℓ . Let M be the label-set of G in respect to ℓ and let $v \in V(G)$ be such that $\ell(v) = 1$. According to Lemma 4, the vertex v describes M as a union of several distinct integer intervals separated by some values that are not elements of M .

We say that an interval is *long* if its first six labels and its last six labels do not intersect. In particular, its last six labels are strictly bigger than 6. The property of the long intervals we want to use is that for any v such that $v \in \psi(i)$ for $i \leq 6$, any terminal among the last six elements of a long interval is always a proper terminal of v . Proofs of Lemma 9 and Lemma 10 are very similar, thus the proof of Lemma 9 is given in the Appendix.

Lemma 9. For any graph $G \in \mathcal{C}(A_n)$ such that $n \geq 39$, there is no labeling such that $\ell(v) = 1$ for any $v \in \psi(2)$.

Lemma 10. For any graph $G \in \mathcal{C}(A_n)$ such that $n \geq 39$, there is no labeling such that $\ell(v) = 1$ for any $v \in \psi(3)$.

Proof. Let us fix arbitrary vertices v_i such that $v_i \in \psi(i)$ for values of i from 2 to 6 and let us denote their labels as $\alpha := \ell(v_2), \beta := \ell(v_3), \gamma := \ell(v_4), \delta := \ell(v_5), \epsilon := \ell(v_6)$. For contradiction let $\beta = 1$. We use the observations from the previous Lemma 8 and Lemma 7 to reach contradiction.

As v_3 has four terminals, M composes of at most three non-trivial intervals with possible trivial interval $\{1\}$, let X, Y, Z denote the intervals other than $\{1\}$ so that $X < Y < Z$. Let k be the last label of X or Y .

As v_2 has only two proper terminals, $\alpha \leq 2$ and thus $\alpha = 2$. Since k is a terminal for v_3 , it is not a terminal for v_2 . Together, we get that $k + 2$ is a label and so both gaps between the intervals X, Y, Z have size exactly 1.

Let us focus on the interval X and let k from now on denote its last label. From k being a terminal for v_3 we have that k is also a terminal for v_5 and v_6 . Thus there are at least three distinct non-labels strictly between k and $k + 7$, and so the third closest non-label following k is at most $k + 6$. As there are at most two of them in the gaps separating the three intervals, we have that the sum of lengths of Y and Z is at most 3 (together with 3 non-labels summing up to 6). From this we get that X is long.

Since $\alpha = 2$, there is one terminal for v_2 in X . If Y or Z has length two, then the first element is also a terminal for v_2 . Together with the highest label in M we would reach a contradiction with v_2 having only two proper terminals.

We have that $Y = \{k + 2\}$ and $Z = \{k + 4\}$. The label k is a terminal for v_3 and label $k - 1$ is a terminal for v_2 . Both are also terminals for v_5 . Therefore, we need to set δ so that both $k + \delta$ and $k - 1 + \delta$ fall into $\{k + 1, k + 3, k + 5\}$. But there is no such value and we reach a contradiction.

3.4 Smallest Labels of $\mathcal{C}(A_n)$

In this section we give limitations on ordering of labels in general case. We apply this together with our previously acquired knowledge to further limit the position of label 1 in graphs from $\mathcal{C}(A_n)$.

Lemma 11. *Let G be a sum graph, let v_1, v_2 be equivalent vertices of G with k terminals. Let ℓ be any labeling of G and M the label-set associated with ℓ . Then $\ell(v_1)$ is one of the k smallest labels in M . Furthermore, if $\ell(v_1) = \ell(v_2)$ then $\ell(v_1)$ is one of the $k - 1$ smallest labels in M .*

Proof. Consider v_1 and let us count the number of labels from M not appearing on any neighbor of v_1 . The vertex v_1 has exactly k terminals including itself. Therefore there are at least $|M| - k$ labels appearing on the neighbors of v_1 and so the edges incident with v_1 carry at least $|M| - k$ distinct edge-numbers expressed as a sum of $\ell(v_1)$ and a positive label of one of the neighbors. Together we get that M contains at least $|M| - k$ values strictly greater than $\ell(v_1)$.

Assume that $\ell(v_1) = \ell(v_2)$. We can improve the previous argument by the fact that the label of v_1 is present on a neighbor of v_1 (namely v_2). This improves the bound on labels in M strictly greater than $\ell(v_1)$ to at least $|M| - (k - 1)$.

Consider graph A_n for $n \geq 39$ with some labeling ℓ , let α denote a label of some vertex from $\psi(2)$ and let β denote a label of some vertex from $\psi(3)$. Recall Lemma 8 giving explicit amounts of terminals of vertices from $\psi(2)$ and $\psi(3)$. As a corollary of the previous Lemma 11 we have that α is among the 3 smallest labels from M and β is among the 4 smallest labels from M . And as we already know from Lemma 9 and Lemma 10, $\alpha, \beta \neq 1$.

Recall Lemma 6, which gives us limitations on mutual relations in between labels of mergeable vertices. As we know that $\psi(2)$ are mergeable with $\psi(3)$, we know that one of the following must hold: $\alpha = 2\beta$, $2\alpha = \beta$, $3\alpha = \beta$ or $3\alpha = 2\beta$.

We use these facts to show that Lemma 11 can be applied to vertices from $\psi(2)$ and $\psi(3)$ in its stronger form, thus fully determining the smallest three labels in minimal labelings of graphs from $\mathcal{C}(A_n)$.

Lemma 12. *In any minimal labeling ℓ of a graph $G \in \mathcal{C}(A_n)$ such that $n \geq 39$, $\ell(v_1) = \ell(v_2)$ for any $v_1, v_2 \in \psi(2)$.*

Proof. For contradiction, let $v_1, v_2 \in \psi(2)$ have distinct labels α_1, α_2 . Without loss of generality, $\alpha_1 < \alpha_2$. We use the observations from Lemma 8. As v_1 is mergeable with v_2 and each has three terminals, Lemma 6 implies that necessarily $2\alpha_1 = \alpha_2$.

Let β denote a label of any vertex from $\psi(3)$. From mergeability, both values α_1, α_2 relate to β , from Lemma 6. The only values for any of the two alphas are $2\beta, \frac{1}{2}\beta, \frac{1}{3}\beta, \frac{2}{3}\beta$. Since $2\alpha_1 = \alpha_2$, the only two suitable values are $\alpha_1 = \frac{1}{3}\beta$ and thus $\alpha_2 = \frac{2}{3}\beta$.

Consider the α_2 -cover of the label-set M . Since both α_1 and α_2 are among the three smallest elements of M , according to Lemma 11, we know that the two proper α_2 -sequences start with elements 1 and α_1 , as these are smaller than α_2 . The value α_2 is not an element of either of the two sequences, thus $\{\alpha_2\}$ forms an improper sequence and consequently $2\alpha_2$ is not a label. From the last point of Lemma 8, the value $\alpha_1 + \beta = 4\alpha_1 = 2\alpha_2$ is a label and we reach a contradiction.

Corollary 1. *For any minimal labeling ℓ of A_n , $\ell(v) < \ell(w)$ for any $v \in \psi(2)$ and $w \in \psi(3)$.*

Proof. The previous Lemma 12 guarantees the additional condition of Lemma 11 on labels of vertices from $\psi(2)$, thus $\ell(v)$ is at most second smallest label in A_n . As 1 is also a label, $\ell(v)$ is exactly the second smallest and thus $1 \neq \ell(w) > \ell(v)$.

Lemma 13. *In any minimal labeling ℓ of a graph $G \in \mathcal{C}(A_n)$ such that $n \geq 39$, $\ell(v_1) = \ell(v_2)$ for any $v_1, v_2 \in \psi(3)$.*

The proof of Lemma 13 is analogous to the proof of Lemma 12 and is given in the Appendix. We are now ready to prove the last limitation on the placement of label 1 in minimal labelings of graphs from $\mathcal{C}(A_n)$ in order to exclude all possible minimal labelings.

Lemma 14. *In any labeling ℓ of a graph $G \in \mathcal{C}(A_n)$ such that $n \geq 39$, if $v \in \psi(i)$ and $\ell(v) = 1$, then $i \leq 3$.*

Proof. Let α be a label of a vertex $v \in \psi(2)$ and let β be a label of a vertex $w \in \psi(3)$. From Lemma 12 and Lemma 13, we know that these values are uniquely determined by ℓ . For contradiction, let $\alpha, \beta > 1$. Let M denote the label-set of G . From the previous Lemma 12 and Lemma 13, we have the additional conditions to apply Lemma 11 to v and w in its stronger form. Together we have

that $\alpha < \beta$ (corollary of Lemma 12) and labels $1, \alpha, \beta$ are the three smallest labels in M .

Let x be a vertex such that $\ell(x) = 1$. Suppose x is not a terminal for v . Then $\alpha + 1$ is a label. Since β is the first label greater than α , we have $\beta = \alpha + 1$. From the fact that $\alpha > 1$ and β is not a multiple of α and vice versa, the mergeability of v and w leaves only one possible relation, $3\alpha = 2\beta$. We conclude that $\alpha = 2$ and $\beta = 3$. Consider any β -sequence, any of its consecutive elements fall into distinct α -sequences. As there exists at least one β -sequence with at least 8 elements, the two proper α -sequences must overlap to satisfy the last 4 elements, none of which can fall into the improper α -sequence. Let k be a label of a proper terminal of v . As x is induced by a number of size at least 4, any proper terminal label of v is also a terminal label of x . Thus, $k + 1$ is not a label. This means that the other α -sequence not terminating in k cannot extend over $k + 1$ and thus, either ends before k or begins after $k + 1$. Applying the same argument to the other sequence, we get that the two α -sequences must not overlap as none can extend over the last element of the other. This is a contradiction and x must be a proper terminal for v .

Since 1 is a proper terminal of v , the α -cover of M has only one non-trivial sequence with the first element k (for some yet unknown integer k). Values β and $\beta + \alpha$ are elements of M with difference α , and thus are consecutive elements of the only proper α -sequence. Since the only terminal shared between v and w has the biggest label in M , x is not a terminal of w and thus $\beta + 1$ is a label and must belong to the proper α -sequence (as $\beta > \alpha$). Together, we have that the proper α -sequence contains elements β and $\beta + 1$; thus, the difference α must equal to 1, which is a contradiction.

3.5 Results

Proof (of Theorem 2). Let G be any graph from $\mathcal{C}(A_n)$ where $n \geq 39$ and let ℓ be any labeling of G . For contradiction, let $\ell(v) = 1$ for some vertex v . Clearly $v \in \psi(i)$ for some integer i , $1 < i \leq n + 2$. As shown by Lemma 9 and Lemma 10, $i > 3$. The Lemma 14, based on the previously mentioned lemmas, shows the complementary fact that $i \leq 3$. This is of course a contradiction.

The constant 39 is an artifact of used methods and may in fact be much smaller. While minimal labeling exists for graphs from $\mathcal{C}(A_6)$, based on a computer search we conjecture that there is in fact no minimal labeling for any graph from $\mathcal{C}(A_n)$ for any $n \geq 7$.

Proof (of Theorem 1). Let G be a graph A_n where $n \geq 39$. For contradiction let G admit a minimal labeling. We replace each vertex of G with loop by a clique, obtaining a graph $H \in \mathcal{C}(G)$, and assign all the new vertices from each new clique the label of the original vertex this clique replaces. We have a graph H with sum labeling using the same labels as the labeling of G . Hence if G admits a minimal sum labeling then H must also admit a minimal sum labeling, which is a contradiction with Theorem 2.

4 Conclusion

We have shown that the set $\{2, 3, \dots, n-1, n, n+2\}$ for $n \geq 39$ induces a family of sum graphs with loops which admit no minimal labeling. Furthermore the loops can be replaced by a cliques of sizes at least two and we obtain an infinite family of non-injective sum graphs (without loops) which also admit no minimal labeling.

The constant 39 is an artefact of used methods and it might be much smaller. While minimal labeling exists for graphs from $\mathcal{C}(A_6)$, based on a computer search, we put forward the following conjecture:

Conjecture 1. Let $G \in \mathcal{C}(A_n)$ such that $n \geq 7$, then G allows no minimal labeling.

Further computer experiments indicate that it is not necessary to restrict to the omission of the second-last element from the sequence $\{2, 3, \dots, n-1, n, n+1, n+2\}$. Thus, we put forward the following conjecture regarding sum graphs with loops:

Conjecture 2. Let G be a gap-graph of size k , where $k \geq 10$, with gap i such that $3 < i < k$, then G admits no minimal labeling.

5 Acknowledgements

This paper is the output of the 2016 Problem Seminar. We would like to thank Jan Kratochvíl and Jiří Fiala for their guidance, help and tea.

References

1. Bergstrand, D., Harary, F., Hodges, K., Jennings, G., Kuklinski, L., Wiener, J.: The sum number of a complete graph, *Bull. Malaysian Math. Soc.* 12 (1989) 25-28.
2. Ellingham, M.N.: Sum graphs from trees, *Ars Combin.* 35 (1993) 335-349.
3. Gould, R.J., Rödl, V.: Bounds on the number of isolated vertices in sum graphs, *Graph Theory. Graph Theory, Combinatorics and Applications* 1 (1991): 553-562.
4. Harary, F.: Sum graphs and difference graphs. *Congr. Numer* 72 (1990): 101-108.
5. Miller, M., Ryan J.F., Smyth, W.F.: The sum number of a disjoint union of graphs. (2003).
6. Miller, M., Ryan, J.F., Smyth, W.F.: The sum number of the cocktail party graph. *Bulletin of the Institute of Combinatorics and its Applications* 22 (1998): 79-90.
7. Nagamochi, H., Miller, M.: Bounds on the number of isolates in sum graph labeling. *Discrete Mathematics* 240.1-3 (2001): 175-185.
8. Pyatkin, A.V.: New formula for the sum number for the complete bipartite graphs, *Discrete Math.* 239 (2001) 155-160

Appendix

Proof (of Theorem 3). Consider a sum graph induced by the set $M = \{2, 3, 4, 6, 7\}$. For contradiction, assume there is a labeling $\{a, b, c, d, e\}$ using label 1. Let v_i denote the unique vertex from $\psi(i)$ for every $i \in M$. As vertices v_2, v_3, v_4 are not equivalent, the labels a, b, c need to be distinct. If we take the greatest of the three labels, it belongs to a vertex incident with two edges with distinct edge-numbers that need to be guaranteed by labels d, e . We conclude that $a, b, c < d, e$. Therefore, the only values that can be equal to 1 are a, b and c . Without loss of generality we can also assume $a < b$ and $d < e$. Let us go through the remaining cases:

1. $a = 1$: since there is a loop incident with the vertex v_2 , there needs to be a vertex labeled 2 and from $b, c < d, e$ we see it has to be either b or c .
 - (a) $b = 2$: as there is no edge v_2v_3 , no label can be equal to 3. Since v_2 has a loop with edge-number 4, we have $c = 4$. Because of edges v_2v_4 and v_3v_4 we have $d = 5$ and $e = 6$. This is a contradiction because there is no edge v_2v_6 .
 - (b) $c = 2$: since v_2v_4 is an edge and $b < d, e$, it holds $b = 3$. From the edge v_3v_4 and the loop on v_3 , we conclude $d = 5$ and $e = 6$, which is a contradiction because there is no edge v_2v_6 .
2. $c = 1$: from the presence of the edge v_2v_4 and $b < d, e$ we have $b = a + 1$. From the edge v_3v_4 and loops on v_2 and v_3 we have three edge-numbers $2a, 2a + 2$ and $a + 2$ that are all distinct from labels a, b and c because $a > 1$. Two of these edge-numbers need to be equal, as there are only two vertices to guarantee all three. The only option is $2a = a + 2$, which gives us $a = 2$. As there is the edge v_2v_4 and $b < d, e$, it follows $b = 3$ and consequently $d = 4$ and $e = 6$. This is a contradiction because there is no edge between v_2v_6 .

Proof (of Lemma 5). Suppose that the number of non-equivalent vertices of G is at least $2t_i t_j + 1$. As there are at most t_i sequences in C_i and at most t_j sequences in C_j , there are at most $t_i \cdot t_j$ possible intersections. Since all labels are covered by both C_i and C_j , each label falls to one of the intersections. The claim holds via Pigeonhole principle.

Notice that if C_i contains the maximum number of distinct sequences, from Lemma 4 we know that one of the sequences is a singleton sequence (d_i) . Clearly this sequence can have at most one non-zero intersection with any sequence from C_j , and it can be of size at most one. From a symmetrical argument for C_j we get that there are actually at most $(t_i - 1) \cdot (t_j - 1)$ possible intersections that can contain more than one element and two extra intersections containing at most one element each. Thus, the lemma holds via Pigeonhole principle.

Proof (of Lemma 8). The first point is an observation based on the structure of A_n resp. $\mathcal{C}(A_n)$. The claim holds for A_n . As all the chosen vertices are induced by small values, the involved proper terminals are loopless pair-wise non-equivalent

vertices (induced by high values) in A_n , so the number of proper terminals is the same in G and all the proper terminals are non-equivalent in $\mathcal{C}(A_n)$.

To prove the second point, we first note that all vertices in A_n are non-equivalent. Thus, the same holds for any two vertices of G induced by distinct integer values. From this we have that there are at least n non-equivalent vertices in G . From the definition of mergeability, it suffices to show that v_3 and v_6 are mergeable, as the condition for these vertices is the most restrictive. Applying the first point, we get that the condition explicitly states $n \geq 2 \cdot 3 \cdot 6 + 3 = 39$, which is satisfied.

The third point is a direct consequence of the first two points and Lemma 7.

The fact that the vertex with the highest label is a terminal for every vertex follows as the highest label is necessarily the last element of a sequence in any cover of the label-set. On the other hand, there can be at most one universally shared element, as the proper terminals of v_2 are induced by values $n - 1$ and $n + 2$, while the proper terminals of v_3 are induced by the values $n - 2$, n and $n + 2$. None of these values induces multiple vertices in A_n , which together with the previous observation proves points 4 and 5.

To prove the sixth point, let us consider $\tau(v_k)$ in A_n . Whenever vertex w induced by an integer i is a terminal for v_k then either $i + k = n + 1$ or $i + k > n + 2$. In both cases it also holds that $i + (k + 2) > n + 2$ and thus w is a terminal for v_{k+2} .

The final point holds as all vertices induced by values 2 to 6 are connected in A_n and thus also in G .

Proof (of Lemma 9). Let us fix arbitrary vertices v_i such that $v_i \in \psi(i)$ for values of i from 2 to 6 and let us denote their labels as $\alpha := \ell(v_2)$, $\beta := \ell(v_3)$, $\gamma := \ell(v_4)$, $\delta := \ell(v_5)$, $\epsilon := \ell(v_6)$. For contradiction let $\alpha = 1$. We use the observation from the previous Lemma 8 and Lemma 7 to reach contradiction.

As v_2 has three terminals, M composes of at most two non-trivial intervals with possible trivial interval $\{1\}$, let X, Y denote the intervals other than $\{1\}$ so that $X < Y$ and let k be the last label of X .

Label k is a terminal for v_2 , thus it is also a terminal for v_4, v_5 and v_6 . On the other hand, k is not a terminal for v_3 as its not the highest label in M .

Consider label of v_3 (denoted β). As k is a not terminal for v_3 , the element $k + \beta \leq k + 3$ is an element of Y , so there are at most two elements in the gap between the intervals X and Y . Furthermore, if the gap contains two elements, then β must be exactly 3.

Consider labels of v_4 and v_5 (denoted γ resp. δ). As k is a terminal for both of them (and v_2), we have that $k + \gamma \leq k + 4$, $k + \delta \leq k + 5$ and $k + \alpha = k + 1$ are three distinct non-labels. As there are at most two of them in between X and Y , we get that the sum of the lengths of the gap and the interval Y is at most 3. Thus Y contains at most two elements and X is long.

Consider the gap to have size 2. We have $\beta = 3$ and since the first interval is long, we have that v_3 has two terminals i, j such that $i + \beta, j + \beta$ fall into the gap. As v_3 has three proper terminals, Y has at most one element; as otherwise, it would hold two additional terminals for v_3 . We know that all terminals of v_2 and

v_3 are also terminals of v_5 . Since $\beta = 3$, we have that $k, k-1, k-2$ are terminals of v_5 . Thus, δ is a value such that none of the values $k + \delta, k-1 + \delta, k-2 + \delta$ are equal to k or $k+3$. As $\delta \leq 5$, there is no possible value of δ and we reach a contradiction. This shows that the size of the gap is 1.

Consider the interval Y to have size 1. The interval Y must consist of a single element $k+2$. As k is not a terminal for v_3 , necessarily $k+\beta$ is the only possible label $k+2$ and thus $\beta = 2$. Notice that the only proper terminals of v_3 are now $k-1$ and $k+1$. However v_3 has three proper terminals, which is a contradiction and we conclude that $|Y| = 2$.

Since k is a terminal for v_2 , it is also a terminal for v_4 , thus $k+\gamma$ is a non-label and $k+2 \leq k+\gamma \leq k+4$. There is only one such non-label, $k+4$ thus $\gamma = 4$. We have that $(k+1) - \gamma = k-3$ is a terminal of v_4 and thus also a terminal of v_6 . The label $k-3+\epsilon$ is a non-label. Since $\epsilon \neq \gamma$, the lowest possible non-label is $k+4$, meaning that ϵ must be at least 7, however $\epsilon \leq 6$ and we reach a contradiction with the existence of the assumed labeling ℓ .

Proof (of Lemma 13). We proceed analogously as in the proof of Lemma 12. For contradiction, let $v_1, v_2 \in \psi(3)$ have distinct labels β_1, β_2 . Without loss of generality, $\beta_1 < \beta_2$. As v_1 is mergeable with v_2 and each has four terminals, Lemma 6 implies that one of the equations $2\beta_1 = \beta_2; 3\beta_1 = \beta_2; 3\beta_1 = 2\beta_2$ holds.

Let α denote a label of any vertex from $\psi(2)$. From mergeability, both values β_1, β_2 relate to α , according to Lemma 6. Since from the Corollary 1 we have $\alpha < \beta_1, \beta_2$, the only possible values of β_1, β_2 are $2\alpha, 3\alpha, \frac{3}{2}\alpha$. The only two solutions are $\beta_1 = \frac{3}{2}\alpha; \beta_2 = 3\alpha$ and $\beta_1 = 2\alpha; \beta_2 = 3\alpha$.

Suppose the former solution holds. All values $1, \alpha, \beta_1$ are strictly smaller than β_2 and since β_2 belongs to the four smallest labels, there are no further elements smaller than β_2 . From the last point of Lemma 8, we have that $\alpha + \beta_1$ is a label. From the relations, we have that $\beta_1 < \alpha + \beta_1 < 3\alpha = \beta_2$. Thus, $\alpha + \beta_1$ is another distinct element strictly smaller than β_2 and we reach a contradiction.

Thus, the latter solution must apply. Consider β_2 -cover, it must consist of three proper sequences with first elements $1, \alpha$ and β_1 . As $\beta_1 = 2\alpha$, the only labels that are not multiples of α must be of form $1 + i\beta_2 = 1 + 3i\alpha$ for non-negative integer values i . Consider the α -cover, it must consist of two proper sequences with first elements 1 and α , as $2\alpha = \beta_1$ is a label. Thus, if there is more than one label that is not multiple of α , then there is label $1 + \alpha$. Together with the previous, $1 + \alpha$ is not a label and thus, there is at most one label that is not a multiple of α , label 1. This means that 1 is a terminal for vertex from $\psi(2)$ and simultaneously for a v_2 . This is a contradiction as $\psi(2)$ and $\psi(3)$ can only share the highest label.