

The Crossing Number of Seq-Shellable Drawings of Complete Graphs

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Abstract. The Harary-Hill conjecture states that for every $n > 0$ the complete graph on n vertices K_n , the minimum number of crossings over all its possible drawings equals

$$H(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

So far, the lower bound of the conjecture could only be verified for arbitrary drawings of K_n with $n \leq 12$. In recent years, progress has been made in verifying the conjecture for certain classes of drawings, for example 2-page-book, x -monotone, x -bounded, shellable and bishellable drawings. Up to now, the class of bishellable drawings was the broadest class for which the Harary-Hill conjecture has been verified, as it contains all beforehand mentioned classes. In this work, we introduce the class of *seq-shellable* drawings and verify the Harary-Hill conjecture for this new class. We show that bishellability implies seq-shellability and exhibit a non-bishellable but seq-shellable drawing of K_{11} , therefore the class of seq-shellable drawings strictly contains the class of bishellable drawings.

1 Introduction

Let $G = (V, E)$ be an undirected graph and K_n the complete graph on $n > 0$ vertices. The crossing number $cr(G)$ of G is the smallest number of edge crossings over all possible drawings of G . In a drawing D every vertex $v \in V$ is represented by a point and every edge $uv \in E$ with $u, v \in V$ is represented by a simple curve connecting the corresponding points of u and v . The Harary-Hill conjecture states the following.

Conjecture 1 (Harary-Hill [8]). Let K_n be the complete graph with n vertices, then

$$cr(K_n) = H(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

There are construction methods for drawings of K_n that lead to exactly $H(n)$ crossings, for example the class of *cylindrical* drawings first described by Harary and Hill [9]. For a cylindrical drawing, we put $\lfloor \frac{n}{2} \rfloor$ vertices on the top rim and the remaining $\lceil \frac{n}{2} \rceil$ vertices on the bottom rim of a cylinder. Edges between

vertices on the same rim (lid or bottom) are connected with straight lines on the lid or bottom. Two vertices on opposite rims are connected with an edge along the geodesic between the two vertices. The drawing of K_6 in figure 1 (a) is homeomorphic to a planarized cylindrical drawing of K_6 .

However, there is no proof for the lower bound of the conjecture for arbitrary drawings of K_n with $n > 12$. The cases for $n \leq 10$ are shown by Guy [8] and for $n = 11$ by Pan and Richter [12]. Guy [8] argues that $cr(K_{2n+1}) \geq H(2n+1)$ implies $cr(K_{2(n+1)}) \geq H(2(n+1))$, hence $cr(K_{12}) \geq H(12)$. McQuillan et al. showed that $cr(K_{13}) \geq 219$ [11]. Ábrego et al. [4] improved the result to $cr(K_{13}) \in \{223, 225\}$.

Beside these results for arbitrary drawings, there has been success in proving the Harary-Hill conjecture for different classes of drawings. So far, the conjecture has been verified for 2-page-book [1], x -monotone [2,6,15], x -bounded [2], shellable [2] and bishellable drawings [5]. The class of bishellable drawings comprises all beforehand mentioned classes, and until now it was the largest class of drawings for which the Harary-Hill conjecture has been verified. Ábrego et al. [5] showed that the Harary-Hill conjecture holds for bishellable drawings using cumulated k -edges.

Our contribution. In this work, we introduce the new class of *seq-shellable* drawings and verify the Harary-Hill conjecture for this new class. We show that bishellability implies seq-shellability and exhibit a drawing of K_{11} which is seq-shellable but not bishellable. Therefore, we establish that the class of seq-shellable drawings is strictly larger than the class of bishellable drawings.

The outline of this paper is as follows. In section 2 we present the preliminaries, and in particular the background on k -edges, cumulated k -edges and their usage for verifying the Harary-Hill conjecture. In section 3 we define *simple sequences* and their usage for proving lower bounds on the number of invariant edges. We present the definition of seq-shellability, verify the Harary-Hill conjecture for the new class and show its superiority towards the class of bishellable drawings. Finally, in section 4 we draw our conclusion and close with open questions.

2 Preliminaries

Formally, a *drawing* D of a graph G on the plane is an injection ϕ from the vertex set V into the plane, and a mapping of the edge set E into the set of simple curves, such that the curve corresponding to the edge $e = uv$ has endpoints $\phi(u)$ and $\phi(v)$, and contains no other vertices [14]. We call an intersection point of the interior of two edges a crossing and a shared endpoint of two adjacent edges is not considered a crossing. The crossing number $cr(D)$ of a drawing D equals the number of crossings in D and the crossing number $cr(G)$ of a graph G is the minimum crossing number over all its possible drawings. We restrict our discussions to *good* drawings of K_n , and call a drawing *good* if (1) any two of the curves have finitely many points in common, (2) no two curves have a

point in common in a tangential way, (3) no three curves cross each other in the same point, (4) any two edges cross at most once and (5) no two adjacent edges cross. It is known that every drawing with a minimum number of crossings is good [13]. In the discussion of a drawing D , we call the points also vertices, the curves edges and V denotes the set of vertices (i.e. points), and E denotes the edges (i.e. curves) of D . If we subtract the drawing D from the plane, a set of open discs remain. We call $\mathcal{F}(D) := \mathbb{R}^2 \setminus D$ the set of *faces* of the drawing D . If we remove a vertex v and all its incident edges from D , we get the subdrawing $D - v$. Moreover, we might consider the drawing to be on the surface of the sphere S^2 , which is equivalent to the drawing on the plane due to the homeomorphism between the plane and the sphere minus one point.

In [5] Ábrego et al. introduce bishellable drawings.

Definition 1 (Bishellability [5]). *For a non-negative integer s , a drawing D of K_n is s -bishellable if there exist sequences a_0, a_1, \dots, a_s and $b_s, b_{s-1}, \dots, b_1, b_0$, each sequence consisting of distinct vertices of K_n , so that with respect to a reference face F :*

- (i) *For each $i \in \{0, \dots, s\}$, the vertex a_i is incident to the face of $D - \{a_0, a_1, \dots, a_{i-1}\}$ that contains F ,*
- (ii) *for each $i \in \{0, \dots, s\}$, the vertex b_i is incident to the face of $D - \{b_0, b_1, \dots, b_{i-1}\}$ that contains F , and*
- (iii) *for each $i \in \{0, \dots, s\}$, the set $\{a_0, a_1, \dots, a_i\} \cap \{b_{s-i}, b_{s-i-1}, \dots, b_0\}$ is empty.*

The class of bishellable drawings contains all drawings that are $(\lfloor \frac{n}{2} \rfloor - 2)$ -bishellable. In order to show that if a drawing D is $(\lfloor \frac{n}{2} \rfloor - 2)$ -bishellable, the Harary-Hill conjecture holds for D , Ábrego et al. use the notion of k -edges. The origins of k -edges lie in computational geometry and problems over n -point set, especially problems on halving lines and k -set [3]. An early definition in the geometric setting goes back to Erdős et al [7]. Given a set P of n points in general position in the plane, the authors add a directed edge $e = (p_i, p_j)$ between the two distinct points p_i and p_j , and consider the continuation as line that separates the plane into the left and right half plane. There is a (possibly empty) point set $P_L \subseteq P$ on the left side of e , i.e. left half plane. Erdős et al. assign $k := \min(|P_L|, |P \setminus P_L|)$ to e . Later, the name k -edge emerged and Lovász et al. [10] used k -edges for determining a lower bound on the crossing number of rectilinear graph drawings. Finally, Ábrego et al. [1] extended the concept of k -edges from rectilinear to topological graph drawings.

Every edge in a good drawing D of K_n is a k -edge with $k \in \{0, \dots, \lfloor \frac{n}{2} \rfloor - 1\}$. Let D be on the surface of the sphere S^2 , and $e = uv$ be an edge in D and $F \in \mathcal{F}(D)$ be an arbitrary but fixed face; we call F the *reference face*. Together with any vertex $w \in V \setminus \{u, v\}$, the edge e forms a triangle uvw and hence a closed curve that separates the surface of the sphere into two parts. For an arbitrary but fixed orientation of e one can distinguish between the left part and the right part of the separated surface. If F lies in the left part of the surface, we say the triangle has orientation $+$ else it has orientation $-$. For e there are $n - 2$

possible triangles in total, of which $0 \leq i \leq n-2$ triangles have orientation $+$ (or $-$) and $n-2-i$ triangles have orientation $-$ (or $+$ respectively). We define $k := \min(i, n-2-i)$ and say e is an k -edge with respect to the reference face F and its k -value equals k with respect to F . Ábrego et al. [1] show that the crossing number of a drawing is expressible in terms of the number of k -edges for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ with respect to the reference face. The following definition of the *cumulated* number of k -edges is helpful in determining the lower bound of the crossing number.

Definition 2 (Cumulated k -edges [1]). Let D be good drawing and $E_k(D)$ be the number of k -edges in D with respect to a reference face $F \in \mathcal{F}(D)$ and for $k \in \{0, \dots, \lfloor \frac{n}{2} \rfloor - 1\}$. We call

$$E_{\leq k}(D) := \sum_{i=0}^k (k+1-i)E_i(D)$$

the *cumulated number of k -edges with respect to F* .

We also write *cumulated k -edges* or *cumulated k -value* instead of cumulated number of k -edges. Lower bounds on $E_{\leq k}(D)$ for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 2$ translate directly into a lower bound for $cr(D)$.

Lemma 1. [1] Let D be a good drawing of K_n and $F \in \mathcal{F}(D)$. If $E_{\leq k}(D) \geq 3\binom{k+3}{3}$ for all $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 2$ with respect to F , then $cr(D) \geq H(n)$. \square

If a vertex v is incident to the reference face, the edges incident to v have a predetermined distribution of k -values.

Lemma 2. [1] Let D be a good drawing of K_n , $F \in \mathcal{F}(D)$ and $v \in V$ be a vertex incident to F . With respect to F , vertex v is incident to two i -edges for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2$. Furthermore, if we label the edges incident to v counter clockwise with e_0, \dots, e_{n-2} such that e_0 and e_{n-2} are incident to the face F , then e_i is a k -edge with $k = \min(i, n-2-i)$ for $0 \leq i \leq n-2$. \square

Examples for lemma 2 are the vertices incident to F in figure 1. We denote the cumulated k -values for edges incident to a vertex v in a drawing D with $E_{\leq k}(D, v)$. Due to lemma 2 it follows that $E_{\leq k}(D, v) = \sum_{i=0}^k (k+1-i) \cdot 2 = 2\binom{k+2}{2}$.

Next, we introduce *invariant k -edges*. Consider removing a vertex $v \in V$ from a good drawing D of K_n , resulting in the subdrawing $D-v$. By deleting v and its incident edges every remaining edge loses one triangle, i.e. for an edge $uw \in E$ there are only $(n-3)$ triangles $uw x$ with $x \in V \setminus \{u, v\}$ (instead of the $(n-2)$ triangles in drawing D). The k -value of any edge $e \in E$ is defined as the minimum count of $+$ or $-$ oriented triangles that contain e . If the lost triangle had the same orientation as the minority of triangles, the k -value of e is reduced by one else it stays the same. Therefore, every k -edge in D with respect to $F \in \mathcal{F}(D)$ is either a k -edge or a $(k-1)$ -edge in the subdrawing $D-v$ with respect to $F' \in \mathcal{F}(D-v)$ and $F \subseteq F'$. We call an edge e *invariant* if e has the same k -value with respect to F in D as for F' in D' . We denote the number

of cumulated invariant k -edges between D and D' (with respect to F and F' respectively) with $I_{\leq k}(D, D')$, i.e. $I_{\leq k}(D, D')$ equals the sum of the number of invariant i -edges for $0 \leq i \leq k$.

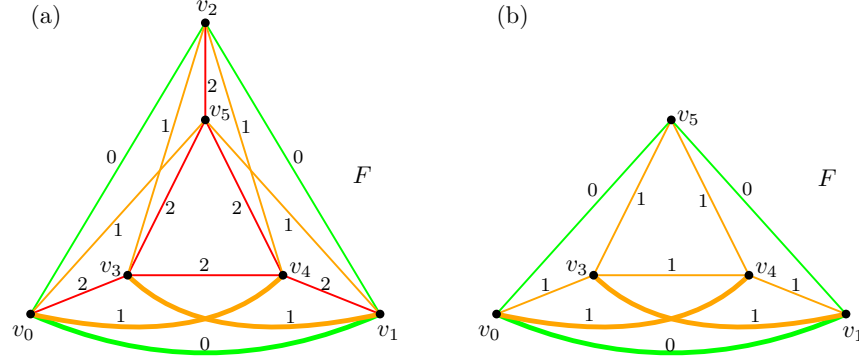


Fig. 1. Example: (a) shows a crossing optimal drawing D of K_6 with the k -values at the edges. (b) shows the subdrawing $D - v_2$ and its k -values. The fat highlighted edges v_0v_1 , v_0v_4 and v_1v_3 are invariant and keep their k -values. The reference face is the outer face F .

For a good drawing D of K_n , we are able to express the value of cumulated k -edges with respect to a reference face $F \in \mathcal{F}(D)$ recursively by adding up the cumulated $(k - 1)$ -value of a subdrawing $D - v$, the contribution of the edges incident to v and the number of *invariant edges* between D and $D - v$.

Lemma 3. [5] *Let D be a good drawing of K_n , $v \in V$ and $F \in \mathcal{F}(D)$. With respect to the reference face F , we have*

$$E_{\leq k}(D) = E_{\leq k-1}(D - v) + E_{\leq k}(D, v) + I_{\leq k}(D, D - v).$$

□

Ábrego et al. [5] use an inductive proof over k to show that for a bishellable drawing D of K_n $E_{\leq k}(D) \geq 3 \binom{k+3}{3}$ for all $k \in \{0, \dots, \lfloor \frac{n}{2} \rfloor - 2\}$. Together with lemma 1 follows $cr(D) \geq H(n)$.

Here, we also use lemma 3 and show that for a seq-shellable drawing D of K_n the lower bounds on $E_{\leq k}(D)$ hold for all $k \in \{0, \dots, \lfloor \frac{n}{2} \rfloor - 2\}$. But in contrast to [5], we use a more general and at the same time easy to follow approach to guarantee lower bounds on the number of invariant edges $I_{\leq k}(D, D - v)$ for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 2$.

3 Seq-Shellability

Before we proceed with the definition of seq-shellability, we introduce simple sequences.

3.1 Simple sequences

We use simple sequences to guarantee a lower bound of the number of invariant edges in the recursive formulation of the cumulated k -value.

Definition 3 (Simple sequence). Let D be a good drawing of K_n , $F \in \mathcal{F}(D)$ and $v \in V$ with v incident to F . Furthermore, let $S_v = (u_0, \dots, u_k)$ with $u_i \in V \setminus \{v\}$ be a sequence of distinct vertices. If u_0 is incident to F and vertex u_i is incident to a face containing F in subdrawing $D - \{u_0, \dots, u_{i-1}\}$ for all $1 \leq i \leq k$, then we call S_v simple sequence of v .

Before we continue with a result for lower bounds on the number of invariant edges using simple sequences, we need the following lemma.

Lemma 4. Let D be a good drawing of K_n , $F \in \mathcal{F}(D)$ and $u, v \in V$ with u and v incident to F . The edge uv touches F either over its full length or not at all (except its endpoints).

Proof. Assume that D is a good drawing of K_n in which the edge uv touches F only partly. We can exclude the case that an edge cuts a part out of uv by crossing it more than once due to the goodness of the drawing (see figure 2 (a)). The case that an edge crosses the whole face F and separates it into two faces is also impossible, because this would contradict that both u and v are incident to F . Therefore, a vertex x has to be on the same side of uv as F and a vertex y on the other side such that the edge xy crosses uv . But the edge xu cannot cross any edge uz with $z \in V \setminus \{u\}$ as this would contradict the goodness of D and xu cannot leave the superface of x without separating v from F (see figure 2 (b) and (c)). We have the symmetric case for v . Consequently, uv cannot touch F beside its endpoints u and v (see figure 2 (d)), a contradiction to the assumption. \square

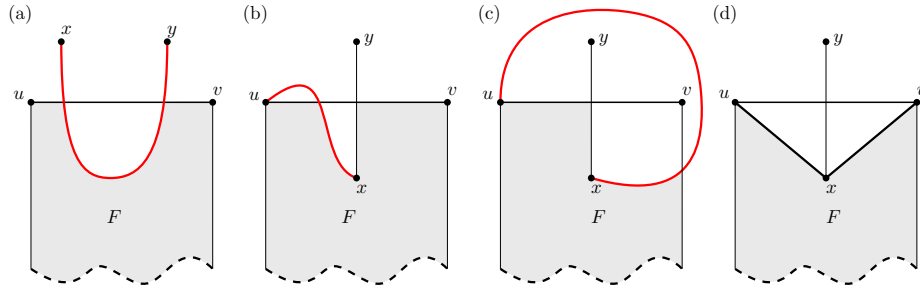


Fig. 2. (a) Due to the goodness of D an edge cannot *cut* a part out of the edge uv . (b) The edges uv and ux cross, both have vertex u as endpoint thus the drawing is not good. (c) The drawing is good but vertex v is not incident to the face F . (d) The edge uv is crossed, the drawing is good and both vertices u and v are incident to F , however uv is not incident to F .

Corollary 1. *Let D be a good drawing of K_n , $F \in \mathcal{F}(D)$ and $u, v \in V$ with both u and v incident to F . If and only if uv is a j -edge, there are exactly j or $n - 2 - j$ vertices on the same side of uv as the reference face F . \square*

The following lemma provides a lower bound for the number of invariant edges in the case that F is incident to at least two vertices and we remove one of them.

Lemma 5. *Let D be a good drawing of K_n , $F \in \mathcal{F}(D)$ and $v, w \in V$ with v and w incident to F . If we remove v from D , then w is incident to at least $\lfloor \frac{n}{2} \rfloor - 1$ invariant edges.*

Proof. We label the edges incident to w counter clockwise with e_0, \dots, e_{n-2} such that e_0 and e_{n-2} are incident to the face F , and we label the vertex at the other end of e_i with u_i . Furthermore, we orient all edges incident to w as outgoing edges. Due to lemma 2 we know that w has two i -edges for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2$. Edge e_i obtains its i -value from the minimum of say $+$ oriented triangles and edge e_{n-2-i} obtains its i -value from the minimum $-$ oriented triangles (or vice versa). Assume that vw is incident to F , i.e. vw is a 0-edge and all triangles vwu for $u \in V \setminus \{v, w\}$ have the same orientation. Consequently, all e_i or all e_{n-2-i} for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2$ are invariant. In the case that vw is not incident to F and is a j -edge, there are j triangles vwu_h with $u_h \in V \setminus \{v, w\}$, $0 \leq h \leq j - 1$ or $n - 1 - j \leq h \leq n - 2$ and u_h is on the same side of vw as F (corollary 1). This means, each triangle wu_hv is part of the majority of orientations for the k -value of edge wu_h , therefore removing v does not change the k -value and there are j additional invariant edges incident to w if we remove v . \square

The following lemma provides a lower bound for the number of cumulated invariant k -edges if we remove a vertex that has a simple sequence.

Lemma 6. *Let D be a good drawing of K_n , $F \in \mathcal{F}(D)$ and $v \in V$ with v incident to F . If v has a simple sequence $S_v = (u_0, \dots, u_k)$, then*

$$I_{\leq k}(D, D - v) \geq \binom{k+2}{2}$$

with respect to F and for all $k \in \{0, \dots, \lfloor \frac{n}{2} \rfloor - 2\}$.

Proof. Let $k \in \{0, \dots, \lfloor \frac{n}{2} \rfloor - 2\}$. We know that u_0 has at least $k + 1 \leq \lfloor \frac{n}{2} \rfloor - 1$ invariant edges with respect to F and removing v . After removing vertex u_0 from drawing D , vertices v and u_1 are incident to F . Since $k \leq \lfloor \frac{n}{2} \rfloor - 2 \leq \lfloor \frac{n-1}{2} \rfloor - 1$ and u_0 has an edge to u_1 in drawing D , vertex u_1 has at least k invariant edges with respect to F and removing v in drawing D . In general, after removing vertices u_0, \dots, u_{i-1} from drawing D , vertices v and u_i are incident to F . For $u \in \{u_0, \dots, u_{i-1}\}$ the edge uu_i in drawing D may be invariant or non-invariant, and we have $k + 1 - i \leq \lfloor \frac{n}{2} \rfloor - 1 - i \leq \lfloor \frac{n-i}{2} \rfloor - 1$. Therefore, u_i has at least $k - i + 1$ invariant edges in drawing D with respect to F and removing v . Summing up

leads to

$$I_{\leq k}(D, D - v) \geq \sum_{i=0}^k (k+1-i) = \binom{k+2}{2}.$$

□

3.2 Seq-shellable drawings

With help of simple sequences we define k -seq-shellability. For a sequence of distinct vertices a_0, \dots, a_k we assign to each vertex a_i with $0 \leq i \leq k \leq n-2$ a simple sequence S_i , under the condition that S_i does not contain any of the vertices a_0, \dots, a_{i-1} .

Definition 4 (Seq-Shellability). *Let D be a good drawing of K_n . We call D k -seq-shellable for $k \geq 0$ if there exists a face $F \in \mathcal{F}(D)$ and a sequence of distinct vertices a_0, \dots, a_k such that a_0 is incident to F and*

1. *for each $i \in \{1, \dots, k\}$, vertex a_i is incident to the face containing F in drawing $D - \{a_0, \dots, a_{i-1}\}$ and*
2. *for each $i \in \{0, \dots, k\}$, vertex a_i has a simple sequence $S_i = (u_0, \dots, u_{k-i})$ with $u_j \in V \setminus \{a_0, \dots, a_i\}$ for $0 \leq j \leq k-i$ in drawing $D - \{a_0, \dots, a_{i-1}\}$.*

Notice that if D is k -seq-shellable for $k > 0$, then the subdrawing $D - a_0$ is $(k-1)$ -seq-shellable. Moreover, if D is k -seq-shellable, then D is also j -seq-shellable for $0 \leq j \leq k$.

Lemma 7. *If D is a good drawing of K_n and D is k -seq-shellable with $k \in \{0, \dots, \lfloor \frac{n}{2} \rfloor - 2\}$, then $E_{\leq k}(D) \geq 3 \binom{k+3}{3}$.*

Proof. We proceed with induction over k . For $k = 0$ the reference face is incident to at least three 0-edges and it follows that

$$E_{\leq 0}(D) \geq 3 = 3 \binom{0+3}{3}.$$

For the induction step, let D be k -seq-shellable with a_0, \dots, a_k and the sequences S_0, \dots, S_k . Consider the drawing $D - a_0$ which is $(k-1)$ -seq-shellable for a_1, \dots, a_k and S_1, \dots, S_k . Since $k-1 \leq (\lfloor \frac{n}{2} \rfloor - 2) - 1 \leq (\lfloor \frac{n-1}{2} \rfloor - 2)$, we assume

$$E_{\leq k-1}(D - a_0) \geq 3 \binom{k+2}{3}.$$

We use the recursive formulation introduced in lemma 3, i.e.

$$E_{\leq k}(D) = E_{\leq k-1}(D - a_0) + E_{\leq k}(D, a_0) + I_{\leq k}(D, D - a_0).$$

Because a_0 is incident to F , we have $E_{\leq k}(D, a_0) = 2\binom{k+2}{2}$, and with the simple sequence S_0 of a_0 follows $I_{\leq k}(D, D - a_0) \geq \binom{k+2}{2}$ (see lemma 6). Together with the induction hypothesis, we have

$$E_{\leq k}(D) \geq 3\binom{k+2}{3} + 2\binom{k+2}{2} + \binom{k+2}{2} = 3\binom{k+3}{3}.$$

□

Using lemmas 1 and 7, we are able to verify the Harary-Hill conjecture for seq-shellable drawings.

Theorem 1. *If D is a good drawing of K_n and D is $(\lfloor \frac{n}{2} \rfloor - 2)$ -seq-shellable, then $cr(D) \geq H(n)$.*

Proof. Let D be a good drawing of K_n and $(\lfloor \frac{n}{2} \rfloor - 2)$ -seq-shellable. Since D is $(\lfloor \frac{n}{2} \rfloor - 2)$ -seq-shellable, it is also k -seq-shellable for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 2$. We apply lemma 7 and have $E_{\leq k}(D) \geq 3\binom{k+3}{3}$ for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor - 2$ and the result follows with lemma 1. □

If a drawing D of K_n is $(\lfloor \frac{n}{2} \rfloor - 2)$ -seq-shellable, we omit the $(\lfloor \frac{n}{2} \rfloor - 2)$ part and say D is seq-shellable. The class of seq-shellable drawings contains all drawings that are $(\lfloor \frac{n}{2} \rfloor - 2)$ -seq-shellable.

Theorem 2. *The class of seq-shellable drawings strictly contains the class of bishellable drawings.*

Proof. First, we show that k -bishellability implies k -seq-shellability. Let D be a k -bishellable drawing of K_n with the associated sequences a_0, \dots, a_k and b_0, \dots, b_k . In order to show that D is k -seq-shellable, we choose a_0, \dots, a_k as vertex sequence and k simple sequences S_i for $0 \leq i \leq k$ such that $S_i = (b_0, \dots, b_{k-i})$. We assign simple sequence S_i to vertex a_i for each $0 \leq i \leq k$ and see that D is indeed seq-shellable. Furthermore, drawing H of K_{11} in figure 3 is not bishellable but seq-shellable. It is impossible to find sequences a_0, \dots, a_3 and b_0, \dots, b_3 in H that fulfill the definition of bishellability. However, H is seq-shellable for face F , vertex sequence (v_0, v_2, v_3, v_4) and the simple sequences $S_0 = (v_1, v_2, v_7, v_4)$, $S_1 = (v_1, v_8, v_6)$, $S_2 = (v_1, v_8)$ and $S_3 = (v_1)$. □

The distinctive difference between seq-shellability and bishellability is that the latter demands a symmetric structure in the sense that we can mutually exchange the sequences a_0, \dots, a_k and b_0, \dots, b_k . Thus, the sequence b_0, \dots, b_{k-i} has to be the simple sequence of a_i in the subdrawing $D - \{a_0, \dots, a_{i-1}\}$ for all $0 \leq i \leq k$ and vice versa, i.e. the sequence a_0, \dots, a_{k-i} has to be the simple sequence of b_i in the subdrawing $D - \{b_0, \dots, b_{i-1}\}$ for all $0 \leq i \leq k$. With seq-shellability we do not have this requirement. Here we have the vertex sequence a_0, \dots, a_k and each vertex a_i with $0 \leq i \leq k$ has its own (independent) simple sequence S_i .

Figure 5 shows a gadget that visualizes the difference between bishellability and seq-shellability: (a) shows a substructure with nine vertices that may occur

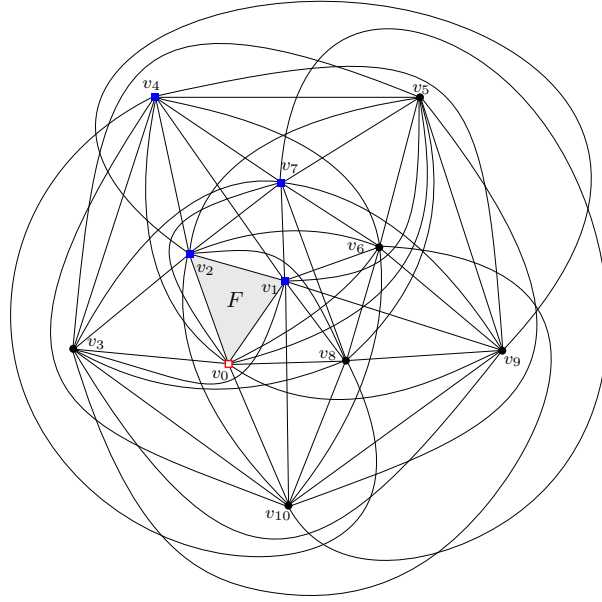


Fig. 3. Drawing H of K_{11} which is not bishellable for any face, however it is seq-shellable for face F , vertex sequence (v_0, v_2, v_3, v_4) and the simple sequences $S_0 = (v_1, v_2, v_7, v_4)$, $S_1 = (v_1, v_8, v_6)$, $S_2 = (v_1, v_8)$ and $S_3 = (v_1)$. Vertex v_0 and the vertices of S_0 are highlighted as unfilled and filled squares.

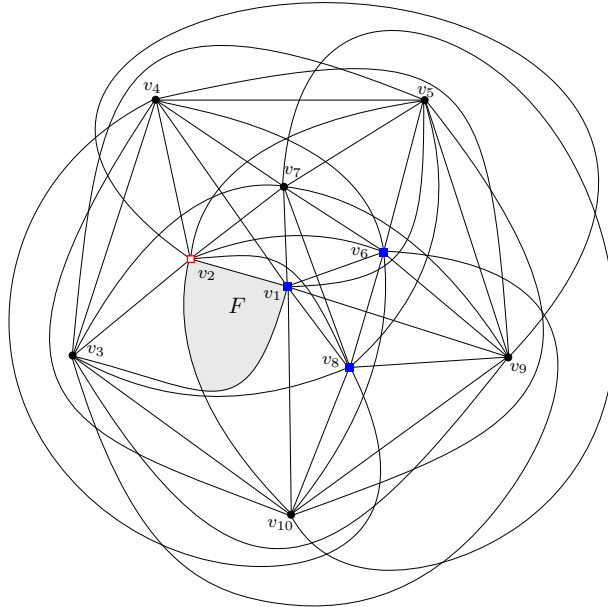


Fig. 4. Subdrawing $H - v_0$ after removing vertex v_0 and its incident edges. The second vertex of the vertex sequence v_2 is incident to the face containing F and has simple sequence S_1 . Vertex v_2 and the vertices of S_1 are highlighted as unfilled and filled squares.

in a drawing. We have the simple sequence v_1, v_2, v_4 for vertex v_3 in (b) and (c). Therefore, we can remove vertex v_3 and are able to guarantee the number of invariant edges. After removing vertex v_3 in (d), there are simple sequences for vertex v_1 and v_2 , thus the substructure is seq-shellable. However, it is impossible to apply the definition of bishellability. We may use, for example, sequence v_1, v_2, v_4 as a_0, \dots, a_k sequence and we need a second sequence (the b sequence) that satisfies the exclusion condition of the bishellability, i.e. for each $i \in \{0, \dots, k\}$, the set $\{a_0, a_1, \dots, a_i\} \cap \{b_{k-i}, b_{k-i-1}, \dots, b_0\}$ has to be empty (see definition 1). The first vertex of our second sequence (i.e. b_0) has to be v_3 , because b_0 has to be incident to F . Now, for the second vertex we have to satisfy $\{a_0, a_1\} \cap \{b_1, b_0\} = \emptyset$, thus the second vertex has to be different from the first two vertices of the sequence v_1, v_2, v_4 . Because we only can choose between vertices v_1 and v_2 , we cannot select a second vertex for our b sequence. Thus, the structure is not bishellable. We can argue the same way for the other possible sequences in the gadget.

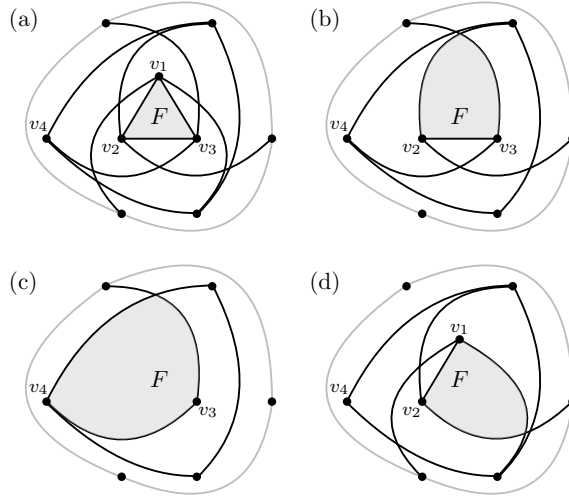


Fig. 5. The gadget does not allow for a bishellability sequence, because only one of the two sequences a_0, \dots, a_k or b_0, \dots, b_k can be chosen due to condition three of the definition of bishellability. However, the gadget is seq-shellable.

4 Conclusion

In this work, we introduced the new class of seq-shellable drawings and verified the Harary-Hill conjecture for this class. Seq-shellability is a generalization of bishellability, thus bishellability implies seq-shellability. In addition we exhibited a drawing of K_{11} which is seq-shellable but not bishellable, hence seq-shellability

is a proper extension of bishellability. So far, we are not aware of an optimal seq-shellable but non-bishellable drawing and we close with the following open questions:

1. Can we find a construction method to obtain optimal drawings of K_n that are seq-shellable but not bishellable?
2. Does there exist a non-bishellable but seq-shellable drawing of K_n with $10 \leq n < 14$, such that after removing the first vertex of the simple sequence the drawing $D - a_0$ is still non-bishellable. We found a drawing of K_{14} with this property.

References

1. Ábrego, B., Aichholzer, O., Fernández-Merchant, S., Ramos, P., Salazar, G.: The 2-page crossing number of K_n . In: Proceedings of the Twenty-eighth Annual Symposium on Computational Geometry. pp. 397–404. SoCG '12, ACM, New York, NY, USA (2012)
2. Ábrego, B.M., Aichholzer, O., Fernández-Merchant, S., Ramos, P., Salazar, G.: Shellable drawings and the cylindrical crossing number of K_n . *Discrete & Computational Geometry* 52(4), 743–753 (2014)
3. Ábrego, B.M., Cetina, M., Fernández-Merchant, S., Leaños, J., Salazar, G.: On $\leq k$ -edges, crossings, and halving lines of geometric drawings of K_n . *Discrete & Computational Geometry* 48(1), 192–215 (2012)
4. Ábrego, B., Aichholzer, O., Fernández-Merchant, S., Hackl, T., Pammer, J., Pilz, A., Ramos, P., Salazar, G., Vogtenhuber, B.: All good drawings of small complete graphs. In: Proc. 31st European Workshop on Computational Geometry (EuroCG). pp. 57–60 (2015)
5. Ábrego, B., Aichholzer, O., Fernández-Merchant, S., McQuillan, D., Mohar, B., Mutzel, P., Ramos, P., Richter, R., Vogtenhuber, B.: Bishellable drawings of K_n . In: Proc. XVII Encuentros de Geometría Computacional (EGC). pp. 17–20. Alicante, Spain (2017), <https://arxiv.org/pdf/1510.00549.pdf>
6. Balko, M., Fulek, R., Kyncl, J.: Crossing numbers and combinatorial characterization of monotone drawings of K_n . *Discrete & Computational Geometry* 53(1), 107–143 (2015)
7. Erdős, P., Lovász, L., Simmons, A., Straus, E.G.: Dissection graphs of planar point sets. In: *A Survey of Combinatorial Theory*, pp. 139–149. Elsevier (1973)
8. Guy, R.K.: A combinatorial problem. *Nabla, Bull. Malayan Math. Soc.* 7, 68–72 (1960)
9. Harary, F., Hill, A.: On the number of crossings in a complete graph. *Proceedings of the Edinburgh Mathematical Society* 13(4), 333–338 (1963)
10. Lovász, L., Vesztegombi, K., Wagner, U., Welzl, E.: Convex quadrilaterals and k -sets. *Contemporary Mathematics* 342, 139–148 (2004)
11. McQuillan, D., Pan, S., Richter, R.B.: On the crossing number of K_{13} . *J. Comb. Theory, Ser. B* 115, 224–235 (2015)
12. Pan, S., Richter, R.B.: The crossing number of K_{11} is 100. *Journal of Graph Theory* 56(2), 128–134 (2007)
13. Schaefer, M.: The graph crossing number and its variants: A survey. *The Electronic Journal of Combinatorics* (Dec. 22, 2017) 1000, DS21–May (2013)

14. Székely, L.A.: A successful concept for measuring non-planarity of graphs: The crossing number. *Electronic Notes in Discrete Mathematics* 5, 284–287 (2000)
15. Ábrego, B.M., Aichholzer, O., Fernández-Merchant, S., Ramos, P., Salazar, G.: More on the crossing number of K_n : Monotone drawings. *Electronic Notes in Discrete Mathematics* 44, 411 – 414 (2013)