# Polygon Queries for Convex Hulls of Points* 

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#### Abstract

We study the following range searching problem: Preprocess a set $P$ of $n$ points in the plane with respect to a set $\mathcal{O}$ of $k$ orientations in the plane so that given an $\mathcal{O}$-oriented convex polygon $Q$, the convex hull of $P \cap Q$ can be computed efficiently, where an $\mathcal{O}$-oriented polygon is a polygon whose edges have orientations in $\mathcal{O}$. We present a data structure with $O\left(n k^{3} \log ^{2} n\right)$ space and $O\left(n k^{3} \log ^{2} n\right)$ construction time, and an $O\left(h+s \log ^{2} n\right)$-time query algorithm for any query $\mathcal{O}$-oriented convex $s$-gon $Q$, where $h$ is the complexity of the convex hull. Also, we can compute the perimeter or area of the convex hull of $P \cap Q$ in $O\left(s \log ^{2} n\right)$ time using the data structure.


## 1 Introduction

Range searching is one of the most thoroughly studied problems in computational geometry for decades from 1970s. Range trees and $k d$-trees were proposed as data structures for orthogonal range searching, and their sizes and query times have been improved over the years. The most efficient data structures for orthogonal range searching for points in the plane 6] and in higher dimensions [7] are due to Chazelle.

There are variants of the range searching problem that allow other types of query ranges, such as circles or triangles. Many of them can be solved using partition trees or a combination of partition trees and cutting trees. The simplex range searching problem, which is a higher dimensional analogue of the triangular range searching, has gained much attention in computational geometry as many other problems with more general ranges can be reduced to it. As an application, it can be used to solve the hidden surface removal in computer graphics [4, 11].

The polygon range searching is a generalization of the simplex range searching in which the search domain is a convex polygon. Willard [18] gave a data structure, called the polygon tree, with $O(n)$ space and an $O\left(n^{0.77}\right)$-time algorithm for counting the number of points lying inside an arbitrary query polygon of constant complexity. The query time was improved later by Edelsbrunner and Welzl 13 to $O\left(n^{0.695}\right)$. By using the stabbing numbers of spanning trees, Chazelle and Welzl [8 gave a data structure of size $O(n \log n)$ with an $O(\sqrt{k n} \log n)$-time query algorithm for computing the number of points lying inside a query convex $k$-gon for arbitrary values of $k$ with $k \leq n$. When $k$ is fixed for all queries, the size of the data structure drops to $O(n)$. Quite a few heuristic techniques and frameworks have been proposed to process polygon range queries on large-scale spatial data in a parallel and distributed manner on top of MapReduce [12]. For overviews of results on range searching, see the survey by Agarwal and Erickson [2].

In this paper, we consider the following polygon range searching problem: Preprocess a set $P$ of $n$ points with respect to a set $\mathcal{O}$ of $k$ orientations in the plane so that given an $\mathcal{O}$-oriented

[^0]convex polygon $Q$, the convex hull of $P \cap Q$, and its perimeter and area, can be computed efficiently, where an $\mathcal{O}$-oriented polygon is a polygon whose edges have orientations in $\mathcal{O}$. Here, an orientation is defined as a unit vector in the plane. We say that an edge has an orientation o if the edge is parallel to $o$.

Whereas orthogonal and simplex range queries can be carried out efficiently, it is quite expensive for queries of arbitrary polygons in general. This is a phenomenon that occurs in many other geometry problems. In an effort to overcome such inefficiency and provide robust computation, there have been quite a few works on "finite orientation geometry", for instance, computing distances [17] in fixed orientations, finding the contour of the union of a collection of polygons with edges of restricted orientations [16], and constructing Voronoi diagrams [3, 9 , using a distance metric induced by a convex $k$-gon. In the line of this research, we suggests the polygon queries whose edges have orientations from a fixed set of orientations. Such a polygon query, as an approximation of an arbitrary polygon, can be used in appropriate areas of application, for instance, in VLSI-design, and possibly takes advantages of the restricted number of orientations and robustness in computation.

Previous Works. Brass et al. [5 gave a data structure on $P$ for a query range $Q$ and a few geometric extent measures, including the convex hull of $P \cap Q$ and its perimeter or area. They gave a data structure with $O\left(n \log ^{2} n\right)$ space and $O\left(n \log ^{3} n\right)$ construction time that given a query axis-parallel rectangle $Q$, reports the convex hull of $P \cap Q$ in $O\left(\log ^{5} n+h\right)$ time and its perimeter or area in $O\left(\log ^{5} n\right)$ time, where $h$ is the complexity of the convex hull.

Both the data structure and query algorithm for reporting the convex hull of $P \cap Q$ were improved by Modiu et al. [14]. They gave a data structure with $O(n \log n)$ space and $O(n \log n)$ construction time that given a query axis-parallel rectangle $Q$, reports the convex hull of $P \cap Q$ in $O\left(\log ^{2} n+h\right)$ time.

For computing the perimeter of the convex hull of $P \cap Q$, the running time of the query algorithm by Brass et al. was improved by Abrahamsen et al. [1]. For a query axis-parallel rectangle $Q$, their data structure supports $O\left(\log ^{3} n\right)$ query time. Also, they presented a data structure of size $O\left(n \log ^{3} n\right)$ for supporting $O\left(\log ^{4} n\right)$ query time for a 5 -gon whose edges have three predetermined orientations.

Our Result. Let $P$ be a set of $n$ points and let $\mathcal{O}$ be a set of $k$ orientations in the plane.

- We present a data structure on $P$ that allows us to compute the perimeter or area of the convex hull of points of $P$ contained in any query $\mathcal{O}$-oriented convex $s$-gon in $O\left(s \log ^{2} n\right)$ time. We can construct the data structure with $O\left(n k^{3} \log ^{2} n\right)$ space in $O\left(n k^{3} \log ^{2} n\right)$ time. Note that $s$ is at most $2 k$ because $Q$ is convex. When the query polygon has a constant complexity, as for the case of $\mathcal{O}$-oriented triangle queries, the query time is only $O\left(\log ^{2} n\right)$.
- For queries of reporting the convex hull of the points contained in a query $\mathcal{O}$-oriented convex $s$-gon, the query algorithm takes $O(h)$ time in addition to the query times for the perimeter or area case, without increasing the size and construction time for the data structure, where $h$ is the complexity of the convex hull.
- For $k=2$, we can construct the data structure with $O(n \log n)$ space in $O(n \log n)$ time whose query time is $O\left(\log ^{2} n\right)$ for computing the perimeter or area of the convex hull of $P \cap Q$ and $O\left(\log ^{2} n+h\right)$ for reporting the convex hull.
- Our data structure can be used to improve the $O\left(n \log ^{4} n\right)$-time algorithm by Abrahamsen et al. [1 for computing the minimum perimeter-sum bipartition of $P$. Their data structure

(a)

(b)

Figure 1: (a) Canonical cells in a standard range tree. (b) Canonical cells in a grid-like range tree.
requires $O\left(n \log ^{3} n\right)$ space and allows to compute the perimeter of the convex hull of points of $P$ contained in a 5 -gon whose edges have three predetermined orientations. If we replace their data structure with ours, we can obtain an $O\left(n \log ^{2} n\right)$-time algorithm for their problem using $O\left(n \log ^{2} n\right)$ space.

## 2 Axis-Parallel Rectangle Queries for Convex Hulls

We first consider axis-parallel rectangle queries. Given a set $P$ of $n$ points in the plane, Modiu et al. [14] gave a data structure on $P$ with $O(n \log n)$ space that reports the convex hull of $P \cap Q$ in $O\left(\log ^{2} n+h\right)$ time for any query axis-parallel rectangle $Q$, where $h$ is the complexity of the convex hull. We show that their data structure with a modification allows us to compute the perimeter of the convex hull of $P \cap Q$ in $O\left(\log ^{2} n\right)$ time.

### 2.1 Data Structure

We first briefly introduce the data structure given by Modiu et al., which is called a two-layer grid-like range tree. To obtain a data structure for computing the parameter of the convex hull of $P \cap Q$ for a query axis-parallel rectangle $Q$, we store information in each node of the two-layer grid-like range tree.

Two-layer Grid-like Range Tree. The two-layer grid-like range tree is a variant of the two-layer standard range tree on $P$. The two-layer standard range tree on $P$ is a two-level balanced binary search tree [10]. The level-1 tree is a balanced binary search tree $T_{x}$ on the points of $P$ with respect to their $x$-coordinates. Each node $\alpha$ in $T_{x}$ corresponds to a vertical slab $I(\alpha)$. The node $\alpha$ has a balanced binary search tree on the points of $P \cap I(\alpha)$ with respect to their $y$-coordinates as its level-2 tree. In this way, each node $v$ in a level- 2 tree corresponds to an axis-parallel rectangle $B(v)$.

For any query axis-parallel rectangle $Q$, there is a set $\mathcal{V}$ of $O\left(\log ^{2} n\right)$ nodes of the level-2 trees such that the rectangles $B(v)$ of $v \in \mathcal{V}$ are pairwise interior disjoint, $Q \cap B(v) \neq \emptyset$ for every $v \in \mathcal{V}$, and $\bigcup_{v \in \mathcal{V}}(P \cap B(v))=P \cap Q$. For $v \in \mathcal{V}$, we call $B(v)$ a canonical cell for $Q$. One drawback of this structure is that the canonical cells for $Q$ are not aligned with respect to their horizontal sides in general. See Figure 1(a).

To overcome this drawback, Modiu et al. [14] gave the two-layer grid-like range tree so that the canonical cells for any query axis-parallel rectangle $Q$ are aligned across all nodes $\alpha$ in the level-1 tree with $I(\alpha) \cap Q \neq \emptyset$. The two-layer grid-like range tree is also a two-level tree whose level-1 tree is a balanced binary search tree $T_{x}$ on the points of $P$ with respect to their $x$-coordinates. Each node $\alpha$ of $T_{x}$ is associated with the level- 2 tree $T_{y}(\alpha)$ which is a binary


Figure 2: (a) A node $v$ of the level-2 tree has two children $u_{1}$ and $u_{2}$ such that $B(v)$ is partitioned into $B\left(u_{1}\right) B\left(u_{2}\right)$. The dashed line segments are bridges stored in this node. The left one is the $c w$-bridge, and the right one is the $c c w$-bridge. (b) The part of $\mathrm{CH}(v)$ from $x$ to $y$ in clockwise order is decomposed into two polygonal curves $\gamma_{1}$ and $\gamma_{2}$ with respect to the bridges of $\mathrm{CH}(v)$.
search tree on the points of $P \cap I(\alpha)$. But, unlike the standard range tree, $T_{y}(\alpha)$ is obtained from $T_{y}$ by removing the subtrees rooted at all nodes whose corresponding rectangles have no point in $P \cap I(\alpha)$ and by contracting all nodes which have only one child, where $T_{y}$ is a balanced binary search tree on the points of $P$ with respect to their $y$-coordinates. Therefore, $T_{y}(\alpha)$ is not balanced but a full binary tree of height $O(\log n)$, and it is called a contracted tree on $P \cap I(\alpha)$. By construction, the canonical cells for any axis-parallel rectangle $Q$ are aligned.

Lemma 1 ([14). The two-layer grid-like range tree on a set of $n$ points in the plane can be computed in $O(n \log n)$ time. Moreover, its size is $O(n \log n)$.

Information Stored in a Node of a Level-2 Tree. To compute the perimeter of the convex hull of $P \cap Q$ for a query axis-parallel rectangle $Q$ efficiently, we store additional information for each node $v$ of the level- 2 trees as follows. The node $v$ has two children in the level- 2 tree that $v$ belongs to. Let $u_{1}$ and $u_{2}$ be the two children of $v$ such that $B\left(u_{1}\right)$ lies above $B\left(u_{2}\right)$. By construction, $B(v)$ is partitioned into $B\left(u_{1}\right)$ and $B\left(u_{2}\right)$.

Consider the convex hull $\mathrm{CH}(v)$ of $B(v) \cap P$ and the convex hull $\mathrm{CH}\left(u_{i}\right)$ of $B\left(u_{i}\right) \cap P$ for $i=1,2$. There are at most two edges of $\mathrm{CH}(v)$ that appear on neither $\mathrm{CH}\left(u_{1}\right)$ nor $\mathrm{CH}\left(u_{2}\right)$. We call such an edge a bridge of $\mathrm{CH}(v)$ with respect to $\mathrm{CH}\left(u_{1}\right)$ and $\mathrm{CH}\left(u_{2}\right)$, or simply a bridge of $\mathrm{CH}(v)$. Note that a bridge of $\mathrm{CH}(v)$ has one endpoint on $\mathrm{CH}\left(u_{1}\right)$ and the other endpoint on $\mathrm{CH}\left(u_{2}\right)$. We call the bridge of $\mathrm{CH}(v)$ whose clockwise endpoint lies on $\mathrm{CH}\left(u_{1}\right)$ and counterclockwise endpoint lies on $\mathrm{CH}\left(u_{2}\right)$ along the boundary of $\mathrm{CH}(v)$ the $c w$-bridge of $\mathrm{CH}(v)$. We call the other bridge of $\mathrm{CH}(v)$ the $c c w$-bridge of $\mathrm{CH}(v)$. See Figure 2 (a).

For each node $v$ of the level- 2 trees, we store the two bridges of $\mathrm{CH}(v)$ and the length of each polygonal chain of $\mathrm{CH}(v)$ lying between the two bridges. In addition, we store the length of each polygonal chain connecting an endpoint $e$ of a bridge of $\mathrm{CH}(v)$ and an endpoint $e^{\prime}$ of a bridge of $\mathrm{CH}(p(v))$ for the parent node $p(v)$ of $v$ along the boundary of $\mathrm{CH}(v)$ if $e$ and $e^{\prime}$ are contained in $B(v)$. We do this for every pair of the endpoints of the bridges of $\mathrm{CH}(v)$ and $\mathrm{CH}(p(v))$ that are contained in $B(v)$. Since only a constant number of bridges are involved, the information stored for $v$ is also of constant size. Each bridge can be computed in time linear in the number of vertices of $\mathrm{CH}(u)$ which do not appear on $\mathrm{CH}(v)$ for a child $u$ of $v$. The length of each polygonal chain we store for $v$ can also be computed in this time. Notice that a vertex of $\mathrm{CH}(u)$ which does not appear on $\mathrm{CH}(v)$ does not appear on $\mathrm{CH}\left(v^{\prime}\right)$ for any ancestor $v^{\prime}$ of $v$. Therefore, the total running time for computing the bridges is linear in the total number of points corresponding to
the leaf nodes of the level- 2 trees, which is $O\left(n \log ^{2} n\right)$.
We will use the following lemma for our query algorithm.
Lemma 2. Given a node $v$ of a level-2 tree and two vertices $x, y$ of $\mathrm{CH}(v)$, we can compute the length of the part of the boundary of $\mathrm{CH}(v)$ from $x$ to $y$ in clockwise order along the boundary of $\mathrm{CH}(v)$ in $O(\log n)$ time.
Proof. Let $\gamma$ be the part of the boundary of $\mathrm{CH}(v)$ from $x$ to $y$ in clockwise order along the boundary of $\mathrm{CH}(v)$. Let $u_{1}$ and $u_{2}$ be the children of $v$ such that $B\left(u_{1}\right)$ lies above $B\left(u_{2}\right)$. We consider the case that only one bridge lies on $\gamma$. We assume further that $x$ is contained in $B\left(u_{2}\right)$, and $y$ is contained in $B\left(u_{1}\right)$. The other cases can be handled analogously. See Figure 2(b). Then $\gamma$, excluding the bridge $b$ of $\mathrm{CH}(v)$ lying on $\gamma$, consists of two polygonal curves, $\gamma_{1}$ and $\gamma_{2}$, with $\gamma_{1}$ lying before $b$ and $\gamma_{2}$ after $b$ along $\gamma$ from $x$.

We show how to compute the length of $\gamma_{1}$ only. The same method works for $\gamma_{2}$. To do this, we traverse the level-2 tree along the path from the root to the leaf corresponding to $x$ and process nodes as follows. For each node $v^{\prime}$ on the path, our task is computing the length of a polygonal chain of $\mathrm{CH}\left(v^{\prime}\right)$ connecting $x$ and an endpoint of a bridge of $\mathrm{CH}\left(v^{\prime}\right)$.

We first consider the case that $\gamma_{1}$ contains a bridge of $\mathrm{CH}\left(u_{2}\right)$. The chain $\gamma_{1}$, excluding the bridges of $\mathrm{CH}\left(u_{2}\right)$, consists of at most three pieces because there are at most two bridges of $\mathrm{CH}\left(u_{2}\right)$. One of the pieces has one endpoint on $x$ and the other on an endpoint of a bridge of $\mathrm{CH}\left(u_{2}\right)$, and each of the other pieces has one endpoint on an endpoint of a bridge of $\mathrm{CH}\left(u_{2}\right)$ and the other on an endpoint of a bridge of $\mathrm{CH}(v)$. Therefore, the lengths of the pieces of $\gamma_{1}$ which are not incident to $x$ are stored in $u_{2}$. Thus, it suffices to compute the piece of $\gamma_{1}$ with endpoints on $x$ and an endpoint of a bridge of $\mathrm{CH}\left(u_{2}\right)$. To do this, we visit the child $w$ of $u_{2}$ such that $B(w)$ contains $x$ and compute the length of the piece of $\gamma_{1}$ recursively.

Consider the case that $\gamma_{1}$ contains no bridge of $\mathrm{CH}\left(u_{2}\right)$. In this case, we find the first endpoint $e$ of a bridge of $\mathrm{CH}\left(u_{2}\right)$ that appears first along its boundary from $x$ in clockwise order. The length of $\gamma_{1}$ is equal to the length of the part $\gamma_{x e}$ of the boundary of $\mathrm{CH}\left(u_{2}\right)$ from $x$ to $e$ in clockwise order minus the length of the part $\gamma_{z e}$ of the boundary from the endpoint $z$ of $\gamma_{1}$ other than $x$ to $e$ in clockwise order. The length of $\gamma_{z e}$ is stored in $u_{2}$. Thus it suffices to compute the length of $\gamma_{x e}$. Since $\gamma_{x e}$ connects $x$ and an endpoint, $e$, of a bridge of $\mathrm{CH}\left(u_{2}\right)$, we visit the child $w$ of $u_{2}$ such that $B(w)$ contains $x$ and compute the length of $\gamma_{x e}$ recursively.

In this way, we traverse the tree along the path from $v$ to a leaf node in $O(\log n)$ time. Finally, we obtain the length of $\gamma_{1}$.

### 2.2 Query Algorithm

Let $Q$ be an axis-parallel rectangle. We present an algorithm for computing the perimeter of the convex hull of $P \cap Q$ in $O\left(\log ^{2} n\right)$ time. We call the part of the convex hull from its topmost vertex to its rightmost vertex in clockwise order along its boundary the urc-hull of $P \cap Q$. In the following, we compute the length of the urc-hull $\gamma$ of $P \cap Q$ in $O\left(\log ^{2} n\right)$ time. The lengths of the other parts of the convex hull of $P \cap Q$ can be computed analogously.

We use the algorithm by Overmars and van Leeuwen [15] for computing the outer tangents between any two convex polygons.

Lemma 3 ([15]). Given any two convex polygons stored in two binary search trees of height $O(\log n)$, we can compute the outer tangents between them in $O(\log n)$ time, where $n$ is the total complexity of the convex hulls.

We compute the set $\mathcal{V}$ of the canonical cells for $Q$ in $O\left(\log ^{2} n\right)$ time. Recall that the size of $\mathcal{V}$ is $O\left(\log ^{2} n\right)$. We consider the cells of $\mathcal{V}$ as grid cells of a grid with $O(\log n)$ rows and $O(\log n)$
columns. We use $C(i, j)$ to denote the grid cell of the $i$ th row and $j$ th column such that the leftmost cell in the topmost row is $C(1,1)$. See Figure 1(b). Notice that a grid cell $C(i, j)$ might not be contained in $\mathcal{V}$.

Recall that we want to compute the urc-hull of $P \cap Q$. To do this, we compute the point $p_{x}$ with largest $x$-coordinate and the point $p_{y}$ with largest $y$-coordinate from $P \cap Q$ in $O(\log n)$ time using the range tree 10. Then we find the cells of $\mathcal{V}$ containing each of them in the same time. Let $C\left(i_{1}, j_{1}\right)$ and $C\left(i_{2}, j_{2}\right)$ be the cells of $\mathcal{V}$ containing $p_{y}$ and $p_{x}$, respectively.

We traverse the cells of $\mathcal{V}$ starting from $C\left(i_{1}, j_{1}\right)$ until we reach $C\left(i_{2}, j_{2}\right)$ as follows. We find every cell $C(i, j) \in \mathcal{V}$ with $i_{1} \leq i \leq i_{2}$ and $j_{1} \leq j \leq j_{2}$ such that no cell $C\left(i^{\prime}, j^{\prime}\right)$ with $i<i^{\prime}$ is in $\mathcal{V}$ or no cell $C\left(i^{\prime}, j^{\prime}\right)$ with $j>j^{\prime}$ is in $\mathcal{V}$. There are $O(\log n)$ such cells, and we call them extreme cells. We can compute all extreme cells in $O\left(\log ^{2} n\right)$ time. Note that the urc-hull of $P \cap Q$ is the urc-hull of points contained in the extreme cells. To compute the urc-hull of $P \cap Q$, we traverse the extreme cells in the lexicographical order with respect to the first index and then the second index.

During the traversal, we maintain the urc-hull of the points contained in the cells we visited so far using a binary search tree of height $O(\log n)$. Imagine that we have just visited a cell $C \in \mathcal{V}$ in the traversal. Let $\delta_{1}$ denote the urc-hulls of the points contained in the cells we visited before the visit to $C$. Let $\delta_{2}$ denote the urc-hulls of the points contained in the cells we visited so far, including $C$. Due to the data structure we maintained, we have a binary search tree of height $O(\log n)$ for the convex hull CH of the points contained in $C$. Moreover, we have a binary search tree of height $O(\log n)$ for $\delta_{1}$ from the traversal to the cells we visited so far. Therefore, we compute the outer tangents (bridges) between them in $O(\log n)$ time by Lemma 3. The urc-hull $\delta_{2}$ is the concatenation of three polygonal curves: a part of CH , the bridge, and a part of $\delta_{1}$. Thus we can represent $\delta_{2}$ using a binary search tree of height one plus the maximum of the heights of the binary search trees for CH and $\delta_{1}$.

Since we traverse $O(\log n)$ cells in total, we obtain a binary search tree of height $O(\log n)$ representing the urc-hull of $P \cap Q$ after the traversal. The traversal takes $O\left(\log ^{2} n\right)$ time. Notice that the urc-hull consists of $O(\log n)$ polygonal curves that are parts from the convex hulls stored in cells of $\mathcal{V}$ and $O(\log n)$ bridges connecting them. We can compute the length of the polygonal curve in $O\left(\log ^{2} n\right)$ time in total by Lemma 2 .

Theorem 4. Given a set $P$ of $n$ points in the plane, we can construct a data structure with $O(n \log n)$ space in $O(n \log n)$-time preprocessing that allows us to compute the perimeter of the convex hull of $P \cap Q$ in $O\left(\log ^{2} n\right)$ time for any query axis-parallel rectangle $Q$.

Since the data structure with its construction and the query algorithm can be used for any pair of orientations which are not necessarily orthogonal through an affine transformation, they work for any pair of orientations with the same space and time complexities.

Corollary 5. Given a set $P$ of $n$ points and a set $\mathcal{O}$ of two orientations in the plane, we can construct a data structure with $O(n \log n)$ space in $O(n \log n)$-time preprocessing that allows us to compute the perimeter of the convex hull of $P \cap Q$ in $O\left(\log ^{2} n\right)$ time for any query $\mathcal{O}$-oriented rectangle $Q$.

## $3 \mathcal{O}$-oriented Triangle Queries for Convex Hulls

In this section, we are given a set $P$ of $n$ points and a set $\mathcal{O}$ of $k$ distinct orientations in the plane. We preprocess the two sets so that we can compute the perimeter of $P \cap Q$ for any query $\mathcal{O}$-oriented triangle $Q$ in the plane efficiently. We construct a three-layer grid-like range tree on $P$ with respect to every 3 -tuple ( $o_{1}, o_{2}, o_{3}$ ) of the orientations in $\mathcal{O}$, which is a generalization of


Figure 3: (a) A node of $T_{1}$ corresponds to a slab of orientation $o_{1}$. (b) A node of a level-2 tree corresponds to a parallelogram having two sides of orientation $o_{1}$ and two sides of orientation $o_{2}$. (c) A node $v$ of a level- 3 tree corresponds to an $\left\{o_{1}, o_{2}, o_{3}\right\}$-polygon $B(v)$.
the two-layer grid-like range tree described in Section 2.1. A straightforward query algorithm takes $O\left(\log ^{3} n\right)$ time since there are $O\left(\log ^{2} n\right)$ canonical cells for a query $\left\{o_{1}, o_{2}, o_{3}\right\}$-oriented triangle $Q$. However, it is unclear how to obtain a faster query algorithm as the query algorithm described in Section 2 does not generalize to this problem directly. A main reason is that a canonical cell for any query $\left\{o_{1}, o_{2}, o_{3}\right\}$-oriented triangle is a $\left\{o_{1}, o_{2}, o_{3}\right\}$-oriented polygon, not a parallelogram. This makes it unclear how to apply the approach in Section 2 to this case.

In this section, we present an $O\left(\log ^{2} n\right)$-time query algorithm for this problem. Our algorithm improves this straightforward algorithm by a factor of $\log n$. To do this, we classify canonical cells for $Q$ into two types. We can handle the cells of the first type as we do in Section 2 and compute the convex hull of the points of $P$ contained in them. Then we handle the cells of the second type by defining a specific ordering to these cells so that we can compute the convex hull of the points of $P$ contained in them efficiently. Then we merge the two convex hulls to obtain the convex hull of $P \cap Q$.

### 3.1 Data Structure

We construct a three-layer grid-like range tree on $P$ with respect to every 3 -tuple of the orientations in $\mathcal{O}$. Let $\left(o_{1}, o_{2}, o_{3}\right)$ be a 3 -tuple of the orientations in $\mathcal{O}$. For an index $i=1,2,3$, we call the projection of a point in the plane onto a line orthogonal to $o_{i}$ the $o_{i}$-projection of the point. Let $T_{i}$ be a balanced binary search tree on the $o_{i}$-projections of the points of $P$ for $i=1,2,3$.

Three-layer Grid-like Range Tree. The level- 1 tree of the grid-like range tree is $T_{1}$. Each node of $T_{1}$ corresponds to a slab of orientation $o_{1}$. For each node of the level- 1 tree, we construct a contracted tree of the $o_{2}$-projections of the points contained in the slab. A node of a level- 2 tree corresponds to an $\left\{o_{1}, o_{2}\right\}$-oriented parallelogram. For each node of a level- 2 tree, we construct a contracted tree of the $o_{3}$-projections of the points contained in the $\left\{o_{1}, o_{2}\right\}$-oriented parallelogram. A node $v$ of a level-3 tree corresponds to an $\left\{o_{1}, o_{2}, o_{3}\right\}$-oriented polygon $B(v)$ with at most six vertices. See Figure 3 for an illustration.

Information Stored in a Node of a Level-3 Tree. Without loss of generality, we assume $o_{3}$ is parallel to the $x$-axis. To compute the perimeter of the convex hull of $P \cap Q$ for a query $\mathcal{O}$-oriented triangle $Q$, we store additional information for each node $v$ of a level-3 tree as follows. The node $v$ has two children $u_{1}$ and $u_{2}$ in the level- 3 tree that $v$ belongs to such that $B\left(u_{1}\right)$ lies


Figure 4: A node $v$ of a level- 3 tree has two children $u_{1}$ and $u_{2}$ such that $B(v)$ is partitioned into $B\left(u_{1}\right)$ and $B\left(u_{2}\right)$ with $B\left(u_{1}\right)$ lying above $B\left(u_{2}\right)$. There are at most two bridges of $\mathrm{CH}(v)$, each has one endpoint on $\mathrm{CH}\left(u_{1}\right)$ and the other on $\mathrm{CH}\left(u_{2}\right)$.
above $B\left(u_{2}\right)$. By construction, $B(v)$ is partitioned into $B\left(u_{1}\right)$ and $B\left(u_{2}\right)$. See Figure 4 for an illustration.

Consider the convex hull $\mathrm{CH}(v)$ of $P \cap B(v)$ and the convex hull $\mathrm{CH}\left(u_{i}\right)$ of $P \cap B\left(u_{i}\right)$ for $i=1,2$. There are at most two edges of $\mathrm{CH}(v)$ that appear on neither $\mathrm{CH}\left(u_{1}\right)$ nor $\mathrm{CH}\left(u_{2}\right)$. We call such an edge a bridge of $\mathrm{CH}(v)$ with respect to $\mathrm{CH}\left(u_{1}\right)$ and $\mathrm{CH}\left(u_{2}\right)$, or simply a bridge of $\mathrm{CH}(v)$. Note that a bridge of $\mathrm{CH}(v)$ has one endpoint on $\mathrm{CH}\left(u_{1}\right)$ and the other endpoint on $\mathrm{CH}\left(u_{2}\right)$. As we do in Section 2, for each node $v$ of the level-3 trees, we store two bridges of $\mathrm{CH}(v)$ and the length of each polygonal chain of $\mathrm{CH}(v)$ lying between the two bridges. Also, we store the length of each polygonal chain connecting an endpoint of a bridge of $\mathrm{CH}(v)$ and an endpoint of a bridge of $\mathrm{CH}(p(v))$ for the parent $p(v)$ of $v$ along the boundary of $\mathrm{CH}(v)$ if the two endpoints appear on $\mathrm{CH}(v)$. The following lemma can be proven in a way similar to Lemma 2 .

Lemma 6. Given a node $v$ of a level-3 tree and two vertices $x, y$ of $\mathrm{CH}(v)$, we can compute the length of the part of the boundary of $\mathrm{CH}(v)$ from $x$ to $y$ in clockwise order along the boundary of $\mathrm{CH}(v)$ in $O(\log n)$ time.

### 3.2 Query Algorithm

In this subsection, we present an $O\left(\log ^{2} n\right)$-time query algorithm for computing the perimeter of the convex hull of $P \cap Q$ for a query $\left\{o_{1}, o_{2}, o_{3}\right\}$-oriented triangle $Q$. Let $T$ be the three-layer grid-like range tree constructed with respect to $\left(o_{1}, o_{2}, o_{3}\right)$.

### 3.2.1 Computing Canonical Cells

We obtain $O\left(\log ^{2} n\right)$ cells of $T$, called canonical cells of $Q$, such that the union of $P \cap C$ coincides with $P \cap Q$ for all the canonical cells $C$ as follows. We first search the level-1 tree of $T$ along the endpoints of the $o_{1}$-projection of $Q$. Then we obtain $O(\log n)$ nodes such that the union of the slabs corresponding to the nodes contains $Q$. Then we search the level- 2 tree associated with each such node along the endpoints of the $o_{2}$-projection of $Q$. Then we obtain $O\left(\log ^{2} n\right)$ nodes in total such that the union of the $\left\{o_{1}, o_{2}\right\}$-parallelograms corresponding to the nodes contains $Q$. We discard all $\left\{o_{1}, o_{2}\right\}$-parallelograms not intersecting $Q$. Some of the remaining $\left\{o_{1}, o_{2}\right\}$-parallelograms are contained in $Q$, but the others intersect the boundary of $Q$ in their interiors. For the nodes corresponding to the $\left\{o_{1}, o_{2}\right\}$-parallelograms intersecting the boundary of $Q$, we search their level-3 trees along the $o_{3}$-projection of $Q$.

As a result, we obtain $\left\{o_{1}, o_{2}\right\}$-parallelograms from the level-2 trees and $\left\{o_{1}, o_{2}, o_{3}\right\}$-polygons from the level-3 trees of size $O\left(\log ^{2} n\right)$ in total. See Figure 5 . We call them the canonical cells of $Q$ and denote the set of them by $\mathcal{V}$. Also, we use $\mathcal{V}_{p}$ and $\mathcal{V}_{h}$ to denote the subsets of $\mathcal{V}$ consisting


Figure 5: Canonical cells for a triangle. Four $\left\{o_{1}, o_{2}\right\}$-oriented parallelogram cells from level-2 trees and $26\left\{o_{1}, o_{2}, o_{3}\right\}$-oriented polygon cells from level- 3 trees.
of $\left\{o_{1}, o_{2}\right\}$-parallelograms from the level- 2 trees and $\left\{o_{1}, o_{2}, o_{3}\right\}$-polygons from the level- 3 trees, respectively. We can compute them in $O\left(\log ^{2} n\right)$ time.

### 3.2.2 Computing Convex Hulls for Each Subset

We first compute the convex hull $\mathrm{CH}_{p}$ of the points contained in the cells of $\mathcal{V}_{p}$ and the convex hull $\mathrm{CH}_{h}$ of the points contained in the cells of $\mathcal{V}_{h}$. Then we merge them into the convex hull of $P \cap Q$ in Section 3.2.3. We can compute $\mathrm{CH}_{p}$ in $O\left(\log ^{2} n\right)$ time due to Corollary 5. This is because the cells are aligned with respect to two axes which are parallel to $o_{1}$ and $o_{2}$ each. Then we obtain a binary search tree of height $O(\log n)$ representing $\mathrm{CH}_{p}$. Thus in the following, we focus on compute $\mathrm{CH}_{h}$.

Without loss of generality, assume that $Q$ lies above the $x$-axis. Let $\ell$ be the side of $Q$ of orientation $o_{3}$. We assign a pair of indices to each cell of $\mathcal{V}_{h}$, which consists of a row index and a column index as follows. The cells of $\mathcal{V}_{h}$ come from $O(\log n)$ level-3 trees of the range tree. This means that each cell of $\mathcal{V}_{h}$ is contained in the cell corresponding to the root of one of such level-3 trees. These root cells are pairwise interior disjoint and intersect $\ell$. For each cell $v$ of $\mathcal{V}_{h}$ contained in the $i$ th leftmost root cell along $\ell$, we assign $i$ to it as the row index of $v$. The bottom side of a cell of $\mathcal{V}_{h}$ is parallel to the $x$-axis. Consider the $y$-coordinates of all bottom sides of the cells of $\mathcal{V}_{h}$. By construction, there are $O(\log n)$ distinct $y$-coordinates although the size of $\mathcal{V}_{h}$ is $O\left(\log ^{2} n\right)$. We assign an index $j$ to the cells of $\mathcal{V}_{h}$ whose bottom side has the $j$ th largest $y$-coordinates as their column indices. Then each cell of $\mathcal{V}_{h}$ has an index $(i, j)$, where $i$ is its row index and $j$ is its column index. Any two distinct cells of $\mathcal{V}_{h}$ have distinct indices. We let $C(i, j)$ be the cell of $\mathcal{V}_{h}$ with index $(i, j)$.

Due to the indices we assigned, we can apply a procedure similar to Graham's scan algorithm for computing $\mathrm{CH}_{h}$. We show how to compute the urc-hull of $\mathrm{CH}_{h}$ only. The other parts of the boundary of $\mathrm{CH}_{h}$ can be computed analogously. To do this, we choose $O(\log n)$ cells as follows. Note that a cell of $\mathcal{V}_{h}$ is a polygon with at most 6 vertices. A trapezoid cell $C(i, j)$ of $\mathcal{V}_{h}$ is called an extreme cell if there is no cell $C\left(i^{\prime}, j^{\prime}\right) \in \mathcal{V}_{h}$ such that $i<i^{\prime}$ and $j>j^{\prime}$, or $i<i^{\prime}$ and $j<j^{\prime}$. Here, we need the disjunction. Otherwise, we cannot find some trapezoidal cell containing a vertex of the urc-hull. See Figure 6. There are $O(\log n)$ extreme cells of $\mathcal{V}_{h}$. In addition to these extreme cells, we choose every cell of $\mathcal{V}_{h}$ which are not trapezoids, that is, convex $t$-gons with $t=3,5,6$. Note that there are $O(\log n)$ such cells because such cells are incident to the corners of the cells of $\mathcal{V}_{p}$. In this way, we choose $O(\log n)$ cells of $\mathcal{V}_{h}$ in total.

Lemma 7. A cell of $\mathcal{V}_{h}$ containing a vertex of the urc-hull of $\mathrm{CH}_{h}$ is an extreme cell of $\mathcal{V}_{h}$ if it is a trapezoid.

Proof. Let $v$ be a vertex of the urc-hull of $\mathrm{CH}_{h}$ and $C=C(i, j)$ be the trapezoid cell of $\mathcal{V}_{h}$


Figure 6: (a) We choose $C(i, j)$ if and only if at least one of the two gray regions contains no cell of $\mathcal{V}_{h}$. (b) It is not sufficient to choose the only cells such that no cell of $\mathcal{V}_{h}$ is contained in their lower gray regions since the urc-hull might have its vertices in a cell whose lower gray regions contains a cell of $\mathcal{V}_{h}$. (c) Similarly, it is not sufficient to choose the only cells such that no cell of $\mathcal{V}_{h}$ is contained in their upper gray regions.
containing $v$. Consider the region $H$ lying to the right of the line containing the right side of $C$. The lines containing the top and bottom sides of $C$ subdivide $H$ into three subregions. Since $v$ is a vertex of the urc-hull, the topmost or bottommost subregion contains no point of $P \cap Q$, that is, there is no cell $C\left(i^{\prime}, j^{\prime}\right) \in \mathcal{V}_{h}$ such that $i<i^{\prime}$ and $j>j^{\prime}$, or $i<i^{\prime}$ and $j<j^{\prime}$. Therefore, $C$ is an extreme cell.

By Lemma 7, the convex hull $\mathrm{CH}_{h}$ coincides with the convex hull of the convex hulls of points in the cells chosen by the previous procedure. For each column $j$, we consider the cells with column index $j$ chosen by the previous procedure one by one in increasing order with respect to their row indices, and compute the convex hull of points contained in those cells. Then we consider the column indices one by one in increasing order, and compute the convex hull of the convex hulls for column indices. This takes $O\left(\log ^{2} n\right)$ time in total as we do in Section 2.2 .

In this way, we can obtain a binary search tree of height $O(\log n)$ representing the urc-hull of $\mathrm{CH}_{h}$. The urc-hull consists of $O(\log n)$ polygonal curves that are parts of the boundaries of the convex hulls stored in cells of $\mathcal{V}_{h}$ and $O(\log n)$ bridges connecting them. Therefore, we can compute the lengths of the polygonal curves in $O\left(\log ^{2} n\right)$ time in total.

### 3.2.3 Merging the Two Convex Hulls

The convex hull CH of $P \cap Q$ coincides with the convex hull of $\mathrm{CH}_{p}$ and $\mathrm{CH}_{h}$. To compute it, we need the following lemma.

Lemma 8. The boundary of $\mathrm{CH}_{p}$ intersects the boundary of $\mathrm{CH}_{h}$ at most $O(\log n)$ times. We can compute the intersection points in $O\left(\log ^{2} n\right)$ time in total.
Proof. Consider two edges, one from $\mathrm{CH}_{p}$ and one from $\mathrm{CH}_{h}$, intersecting each other. One of them is a bridge with endpoints lying on two distinct cells of $\mathcal{V}$. This is because the cells of $\mathcal{V}$ are pairwise interior disjoint. Moreover, each bridge in $\mathrm{CH}_{p}\left(\right.$ or $\left.\mathrm{CH}_{h}\right)$ intersects the boundary of $\mathrm{CH}_{h}$ ( or $\mathrm{CH}_{p}$ ) at most twice since $\mathrm{CH}_{h}$ and $\mathrm{CH}_{p}$ are convex. Since there are $O(\log n)$ bridges in $\mathrm{CH}_{p}$ and $\mathrm{CH}_{h}$, there are $O(\log n)$ intersection points between the boundary of $\mathrm{CH}_{p}$ and the boundary of $\mathrm{CH}_{h}$.

To compute the intersection points, we compute the intersection points between each bridge of a convex hull and the boundary of the other convex hull. For each bridge, we can compute the two intersection points in $O(\log n)$ time since we have a binary search tree for each convex hull of height $O(\log n)$ [15]. Therefore, we can compute all intersection points in $O\left(\log ^{2} n\right)$ time.

We first compute the intersection points of the boundaries of $\mathrm{CH}_{p}$ and $\mathrm{CH}_{h}$ in $O\left(\log ^{2} n\right)$ time by Lemma 8, and then sort them along the boundary of their convex hull in clockwise order
in $O(\log n \log \log n)$ time. Note that this order is the same as the clockwise order along the boundary of $\mathrm{CH}_{p}$ (and $\mathrm{CH}_{h}$ ). Then we locate each intersection point on the boundary of each convex hull with respect to the bridges in $O(\log n)$ time in total.

There are $O(\log n)$ edges of the convex hull CH of $\mathrm{CH}_{p}$ and $\mathrm{CH}_{h}$ that do not appear on the boundaries of $\mathrm{CH}_{p}$ and $\mathrm{CH}_{h}$. To distinguish them with the bridges on the boundaries of $\mathrm{CH}_{p}$ and $\mathrm{CH}_{h}$, we call the edges on the boundary of CH appearing neither $\mathrm{CH}_{p}$ nor $\mathrm{CH}_{h}$ the hull-bridges. Also we call the bridges on $\mathrm{CH}_{p}$ and $\mathrm{CH}_{h}$ with endpoints in two distinct cells of $\mathcal{V}$ the node-bridges.

The boundary of the convex hull of $\mathrm{CH}_{p}$ and $\mathrm{CH}_{h}$ consists of $O(\log n)$ hull-bridges and $O(\log n)$ polygonal curves each of which connects two hull-bridges along $\mathrm{CH}_{p}$ or $\mathrm{CH}_{h}$. We compute all hull-bridges in $O\left(\log ^{2} n\right)$ time.

Lemma 9. All hull-bridges can be computed in $O\left(\log ^{2} n\right)$ time in total.
Proof. Let $\left\langle p_{1}, \ldots, p_{m}\right\rangle$ be the sequence of the intersection points of the boundaries of $\mathrm{CH}_{p}$ and $\mathrm{CH}_{h}$ sorted along the boundary of the convex hull of the intersection points with $m=O(\log n)$. For an index $i$ with $1 \leq i<m$, We use $\mathrm{CH}_{p}[i]$ and $\mathrm{CH}_{h}[i]$ to denote the parts of the boundaries of $\mathrm{CH}_{p}$ and $\mathrm{CH}_{h}$ from $p_{i}$ to $p_{i+1}$, respectively, in clockwise order along their boundaries.

Every hull-bridge is an outer tangent of $\mathrm{CH}_{p}[i]$ and $\mathrm{CH}_{h}[i+1]$ or an outer tangent of $\mathrm{CH}_{h}[i]$ and $\mathrm{CH}_{p}[i+1]$ for an index $1 \leq i<m$. Therefore, it suffices to compute all outer tangents of $\mathrm{CH}_{p}[i]$ and $\mathrm{CH}_{h}[i+1]$ (and $\mathrm{CH}_{h}[i]$ and $\left.\mathrm{CH}_{p}[i]\right)$. Note that some of the outer tangents are not hull-bridges, but we can determine whether an outer tangent is a hull-bridge or not in constant time by considering the edges of $\mathrm{CH}_{p}$ and $\mathrm{CH}_{h}$ incident to the endpoints of the outer tangent.

We can compute the outer tangents of $\mathrm{CH}_{p}[i]$ and $\mathrm{CH}_{h}[i+1]$ in $O(\log n)$ time using the algorithm in [15] since we have a binary search tree of height $O(\log n)$ representing $\mathrm{CH}_{p}$ (and $\left.\mathrm{CH}_{h}\right)$. Therefore, we can compute all hull-bridges in $O\left(\log ^{2} n\right)$ time.

As a result, we obtain a binary search tree of height $O(\log n)$ representing the convex hull CH of $P \cap Q$. We can compute the length of each polygonal curve connecting two hull-bridges in $O(\log n)$ time by Lemma 2 and the fact that there are $O(\log n)$ node-bridges lying on сн. Therefore, we have the following theorem.

Theorem 10. Given a set $P$ of $n$ points and a set $\mathcal{O}$ of $k$ orientations in the plane, we can construct a data structure with $O\left(n k^{3} \log ^{2} n\right)$ space in $O\left(n k^{3} \log ^{2} n\right)$ time that allows us to compute the perimeter of the convex hull of $P \cap Q$ in $O\left(\log ^{2} n\right)$ time for any query $\mathcal{O}$-oriented triangle $Q$.

## $4 \mathcal{O}$-oriented Polygon Queries for Convex Hulls

The data structure in Section 3 can be used for more general queries. We are given a set $P$ of $n$ points in the plane and a set $\mathcal{O}$ of $k$ orientations. Let $Q$ be a query $\mathcal{O}$-oriented convex $s$-gon. Since $Q$ is convex, $s$ is at most $2 k$. Assume that we are given the three-layer grid-like range tree on $P$ with respect to the set $\mathcal{O}$ including the axis-parallel orientations. We want to compute the perimeter of the convex hull of $P \cap Q$ in $O\left(s \log ^{2} n\right)$ time.

We draw vertical line segments through the vertices of $Q$ to subdivide $Q$ into at most $2 k$ trapezoids. We subdivide each trapezoid further using the horizontal lines passing through its vertices into at most two triangles and one parallelogram. The edges of a triangle and a parallelogram, say $\triangle$, have orientations in the set $\mathcal{O}$ including the axis-parallel orientations. Thus, we can compute the convex hull of $\triangle \cap P$ in $O\left(\log ^{2} n\right)$ time and represent it using a binary search tree of height $O(\log n)$. By Lemma 3, we can compute the convex hull of the points contained in each trapezoid in $O\left(s \log ^{2} n\right)$ time in total and represent them using balanced binary search trees of height $O(\log n)$.

Let $A_{1}, \ldots, A_{t}$ be the trapezoids from the leftmost one to the rightmost one for $t \leq k$. We consider the trapezoids one by one from $A_{1}$ to $A_{t}$. Assume that we have just handled the trapezoid $A_{i}$ and we want to handle $A_{i+1}$. Assume further that we already have the convex hull $\mathrm{CH}_{i}$ of the points contained in $A_{j}$ for all $j \leq i$. Since the convex hull of the points in $A_{i+1}$ is disjoint from $\mathrm{CH}_{i}$, we can compute $\mathrm{CH}_{i+1}$ in $O(\log n)$ time using Lemma 3. In this way, we can compute the convex hull of $P \cap Q$ in $O\left(s \log ^{2} n\right)$ time in total. Moreover, we can compute its perimeter in the same time as we did before. If $s$ is a constant as for the case of $\mathcal{O}$-oriented triangle queries, it takes only $O\left(\log ^{2} n\right)$ time.

Theorem 11. Given a set $P$ of $n$ points in the plane and a set $\mathcal{O}$ of $k$ orientations, we can construct a data structure with $O\left(n k^{3} \log ^{2} n\right)$ space in $O\left(n k^{3} \log ^{2} n\right)$ time that allows us to compute the perimeter of the convex hull of $P \cap Q$ in $O\left(s \log ^{2} n\right)$ time for any $\mathcal{O}$-oriented convex s-gon.

As mentioned in Introduction, our data structure can be used to improve the algorithm and space requirement by Abrahamsen et al. [1. They considered the following problem: Given a set $P$ of $n$ points in the plane, partition $P$ into two subsets $P_{1}$ and $P_{2}$ such that the sum of the perimeters of $\mathrm{CH}\left(P_{1}\right)$ and $\mathrm{CH}\left(P_{2}\right)$ is minimized, where $\mathrm{CH}(A)$ is the convex hull of a point set $A$. They gave an $O\left(n \log ^{4} n\right)$-time algorithm for this problem using $O\left(n \log ^{3} n\right)$ space. Using our data structure, we can improve their running time to $O\left(n \log ^{2} n\right)$ and their space complexity to $O\left(n \log ^{2} n\right)$.

Corollary 12. Given a set $P$ of $n$ points in the plane, we can compute a minimum perimeter-sum bipartition of $P$ in $O\left(n \log ^{2} n\right)$ time using $O\left(n \log ^{2} n\right)$ space.

## 4.1 $\mathcal{O}$-oriented Polygon Queries for the Areas of Convex Hulls

We can modify our data structure to compute the area of the convex hull of $P \cap Q$ for any $\mathcal{O}$-oriented convex polygon query $Q$ without increasing the time and space complexities.

The modification of the data structure is on the information stored in each node of the grid-like range trees of $P$. Let $v$ be a node of a level-3 tree of a grid-like range tree $T$. Without loss of generality, we assume that the axis of the level-3 trees of $T$ is parallel to the $x$-axis. Let $u_{1}$ and $u_{2}$ be the two children of $v$ such that $B\left(u_{1}\right)$ lies above $B\left(u_{2}\right)$. We use $\mathrm{CH}(v)$ to denote the convex hull of the points contained in $B(v)$.

We store its two bridges and the area of the convex hull of each polygonal chain of $\mathrm{CH}(v)$ lying between two bridges. In addition, we store the area of the convex hull of each polygonal chain connecting an endpoint $e$ of a bridge of $\mathrm{CH}(v)$ and an endpoint $e^{\prime}$ of a bridge of $\mathrm{CH}(p(v))$ for the parent node $p(v)$ of $v$ with $e, e^{\prime} \in B(v)$ along the boundary of $\mathrm{CH}(v)$. We do this for every endpoint of the bridges of $\mathrm{CH}(v)$ and $\mathrm{CH}(p(v))$ that are contained in $B(v)$. Therefore, we have the following operation.

Lemma 13. Given a node $v$ of a level-3 tree and two vertices $x, y$ of $\mathrm{CH}(v)$, we can compute the area of the convex hull of the part of the boundary of $\mathrm{CH}(v)$ from $x$ to $y$ in clockwise order along the boundary of $\mathrm{CH}(v)$ in $O(\log n)$ time.

Proof. The proof is similar to the proof of Lemma 2. We obtain two paths such that the boundary of $\mathrm{CH}(v)$ is decomposed into $O(\log n)$ pieces each of which corresponds to a node in the two paths in $O(\log n)$ time.

In the proof of Lemma 2 , we simply add the lengths of all such pieces, each in $O(\log n)$ time, since the length of each piece is stored in a node of the paths. Instead, we add the areas of the convex hulls of all such pieces. Then we compute the area of the convex hull of the endpoints of all such pieces. Since all such convex hulls are pairwise interior disjoint, the total sum is the area
we want to compute. Therefore, we can compute the area of the convex hull of the part of the boundary of $\mathrm{CH}(v)$ from $x$ to $y$ in $O(\log n)$ time.

For an $\mathcal{O}$-oriented convex $s$-gon query $Q$, we can obtain $O(s \log n)$ pieces of the convex hull of $P \cap Q$ each of which is a straight line, or a polygonal curve lying on the boundary of $\mathrm{CH}(v)$ for some node $v$ of the grid-like range trees in $O\left(s \log ^{2} n\right)$ time due to Section 4 . These pieces are sorted along the boundary of CH . By construction, the convex hulls of all such pieces and the convex hull of all straight lines contain the convex hull of $P \cap Q$ and they are pairwise interior disjoint. The area of the convex hull of $P \cap Q$ is the sum of the areas of these convex hulls. Therefore, we can compute the area of the convex hull of $P \cap Q$ in $O\left(s \log ^{2} n\right)$ time.

Theorem 14. Given a set $P$ of $n$ points and a set $\mathcal{O}$ of $k$ orientations in the plane, we can construct a data structure with $O\left(n k^{3} \log ^{2} n\right)$ space in $O\left(n k^{3} \log ^{2} n\right)$ time that allows us to compute the area of the convex hull of $P \cap Q$ in $O\left(s \log ^{2} n\right)$ time for any $\mathcal{O}$-oriented convex $s$-gon.

### 4.2 O-oriented Polygon Queries for Reporting Convex Hulls

We can use our data structure to report the edges of the convex hull of $P \cap Q$ for any $\mathcal{O}$-oriented convex polygon query $Q$ without increasing the space and time complexities, except an additional $O(h)$ term in the query time for reporting the convex hull with $h$ edges, due to the following lemma.

Lemma 15. Given a node $v$ of a level-3 tree and two vertices $x, y$ of $\mathrm{CH}(v)$, we can report all edges of the convex hull of the part of the boundary of $\mathrm{CH}(v)$ from $x$ to $y$ in clockwise order along its boundary in $O(\log n+h(v))$ time, where $h(v)$ is the number of the edges reported.

Proof. The proof is similar to the proof of Lemma 2. We obtain two paths such that the boundary of $\mathrm{CH}(v)$ is decomposed into $O(\log n)$ pieces each of which corresponds to a node in the two paths in $O(\log n)$ time. For each such node $v$, we can report all edges of a part of the boundary of $\mathrm{CH}(v)$ in order once we have the endpoints of the part in time linear to the output size by traversing the subtree rooted at $v$ in a DFS order. Therefore, we can report all edges of $\mathrm{CH}(v)$ from $x$ to $y$ in $O(\log n+h(v))$ time in total.

For an $\mathcal{O}$-oriented convex $s$-gon query $Q$, we can decompose the boundary of $\mathrm{CH}(v)$ into $O(\log n)$ pieces each of which is a straight line, or a polygonal curve lying on the boundary of $\mathrm{CH}(v)$ for some node $v$ of the grid-like range trees in $O\left(\log ^{2} n\right)$ time due to Section 4. Using Lemma 15, we report the edges of the convex hull of $P \cap Q$ in $O\left(s \log ^{2} n+h\right)$ time, where $h$ is the number of the edges of the convex hull.

Theorem 16. Given a set $P$ of $n$ points and a set $\mathcal{O}$ of $k$ orientations in the plane, we can construct a data structure with $O\left(n k^{3} \log ^{2} n\right)$ space in $O\left(n k^{3} \log ^{2} n\right)$ time that allows us to report all edges of the convex hull of $P \cap Q$ in $O\left(s \log ^{2} n+h\right)$ time for any $\mathcal{O}$-oriented convex $s$-gon, where $h$ is the number of edges of the convex hull.

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