NONBIPARTITE DULMAGE-MENDELSOHN DECOMPOSITION FOR BERGE DUALITY

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Abstract. The *Dulmage-Mendelsohn decomposition* is a classical *canonical decomposition* in matching theory applicable for bipartite graphs, and is famous not only for its application in the field of matrix computation, but also for providing a prototypal structure in matroidal optimization theory. The Dulmage-Mendelsohn decomposition is stated and proved using the two color classes, and therefore generalizing this decomposition for nonbipartite graphs has been a difficult task. In this paper, we obtain a new canonical decomposition that is a generalization of the Dulmage-Mendelsohn decomposition for arbitrary graphs, using a recently introduced tool in matching theory, the *basilica decomposition*. Our result enables us to understand all known canonical decompositions in a unified way. Furthermore, we apply our result to derive a new theorem regarding *barriers*. The duality theorem for the maximum matching problem is the celebrated *Berge formula*, in which dual optimizers are known as barriers. Several results regarding maximal barriers have been derived by known canonical decompositions, however no characterization has been known for general graphs. In this paper, we provide a characterization of the family of *maximal barriers* in general graphs, in which the known results are developed and unified.

1. Introduction

We establish the Dulmage-Mendelsohn decomposition for general graphs. The *Dulmage-Mendelsohn decomposition* [2–4], or the *DM decomposition* in short, is a classical canonical decomposition in matching theory [15] applicable for bipartite graphs. This decompositions is famous for its application for combinatorial matrix theory, especially, for providing an efficient solution for a system of linear equations [1, 4], and is also important in matroidal optimization theory.

Canonical decompositions of a graph are fundamental tools in matching theory [15]. A canonical decomposition partitions a given graph in a way uniquely determined for the graph, and describes the structure of maximum matchings using this partition. The classical canonical decompositions are the *Gallai-Edmonds* [5,6] and *Kotzig-Lovász* decompositions [11–13], in addition to the DM decomposition. The DM and Kotzig-Lovász decompositions are applicable for bipartite graphs and *factor-connected graphs*, respectively. The Gallai-Edmonds decomposition partitions an arbitrary graph into three parts, that is, the so-called $D(G)$, $A(G)$, and $C(G)$ parts. Comparably recently, a new canonical decomposition was proposed: the *basilica decomposition* [7, 8]. This decomposition is applicable for arbitrary graphs and contains a generalization of the Kotzig-Lovász decomposition and a refinement the Gallai-Edmonds decomposition. (The $C(G)$ part can be decomposed nontrivially.)

In this paper, using the basilica decomposition, we establish an analogue of the DM decomposition for general graphs. Our result accordingly provides a paradigm that enables us to handle any graph and understand the known canonical decompositions in a unified way. In the theory of original DM decomposition, the concept

Date: August 3, 2017.

of the *DM components* of a bipartite graph is first defined, and then it is proved that these components form a poset with respect to a certain binary relation.

This theory is heavily depend on the two color classes of a bipartite graph and cannot be easily generalized for nonbipartite graphs. In our generalization, we first define a generalization of the DM components using the basilica decomposition. To capture the structure formed by these components in nonbipartite graphs, we introduce a little more complexed concept: *posets with a transitive forbidden relation*. We then prove that the generalized DM components form a poset with a transitive forbidden relation for certain binary relations.

Using this generalized DM decomposition, we derive a characterization of the family of *maximal barriers* in general graphs. The *Berge formula* is a combinatorial min-max theorem, in which maximum matchings are the optimizers of one hand, and the optimizers of the other hand are known as *barriers* [15]. That is, barriers are the dual optimizers of the maximum matchings problem. Barriers are heavily employed as a tool for studying matchings. However, not so much is known about barriers themselves [15]. Aside from several observations that are derived rather easily from the Berge formula, several substantial results are known about (inclusion-wise) maximal barriers, which are provided by canonical decompositions.

Our result for maximal barriers shows a reasonable consistency regarding our generalization of the DM decomposition, considering the relationship between each known canonical decomposition and maximal barriers. Each known canonical decomposition can be used to state the structure of maximal barriers. The original DM decomposition provides a characterization of the family of maximal barriers in bipartite graphs in terms of ideals in the poset; minimum vertex covers in bipartite graphs are equivalent to maximal barriers. The Gallai-Edmonds decomposition derives a characterization of the intersection of all maximal barriers (that is, the $A(G)$ part) [15]; this characterization is known as the *Gallai-Edmonds description*. The Kotzig-Lovász decomposition is used for characterizing the family of maximal barriers in factor-connected graphs [15]; this result is known as *Lovász's canonical partition theorem* [14, 15]. The basilica decomposition provides the structure of a given maximal barrier in general graphs, which contains a common generalization of the Gallai-Edmonds description and Lovász's canonical partition theorem. Hence, a generalization of the DM decomposition would be reasonable if it can characterize the family of maximal barriers, and our generalization attains this in a way analogical to the classical DM decomposition, that is, in terms of ideals in the poset with a transitive forbidden relation.

Our results may imply a new possibility in matroidal optimization theory. In submodular function theory, the bipartite maximum matching problem is an important exemplary problem, and the DM decomposition therefore has a special role in this theory. Our nonbipartite DM decomposition may be a clue to a new phase of submodular function theory that can be brought in by capturing these concepts.

The remainder of this paper is organized as follows: In Section 2, we explain the basic definitions. In Section 3, we present the preliminary results from the basilica decomposition theory. In Section 4, we introduce the new concept of posets with a transitive forbidden relation. In Section 5, we provide our main result, the nonbipartie DM decomposition. In Section 6, we present preliminary definitions and results regarding barriers. We then prove in Section 7 that our generalization of the DM decomposition can be used to characterize the family of maximal barriers. In Section 8, we show how our result contains the original DM decomposition for bipartite graphs. In Section 9, we remark that our nonbipartite DM decomposition can be computed in polynomial time.

2. Notation

2.1. **General Definitions.** For basic notation for sets, graphs, and algorithms, we mostly follow Schrijver [16]. In this section, we explain exceptions or nonstandard definitions. In Section 2, unless otherwise stated, let G be a graph. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. We treat paths and circuits as graphs. For a path P and vertices x and y from P, xPy denotes the subpath of P between x and y. The singleton set $\{x\}$ is often denoted by just x. We often treat a graph as the set of its vertices.

2.2. **Graph Operations.** In the remainder of this section, let $X \subseteq V(G)$. The subgraph of G induced by X is denoted by $G[X]$. The graph $G[V(G)\setminus X]$ is denoted by $G - X$. The contraction of G by X is denoted by G/X. Let $F \subseteq E(G)$. The graph obtained by deleting F from G without removing vertices is denoted by $G-F$. Let H be a subgraph of G. The graph obtained by adding F to H is denoted by $H + F$. Regarding these operations, we identify vertices, edges, subgraphs of the newly created graph with the naturally corresponding items of old graphs.

2.3. **Functions on Graphs.** A *neighbor* of X is a vertex from $V(G) \setminus X$ that is adjacent to some vertex from X. The neighbor set of X is denoted by $N_G(X)$. Let $Y \subseteq V(G)$. The set of edges joining X and Y is denoted by $E_G[X, Y]$. The set $E_G[X, V(G) \setminus X]$ is denoted by $\delta_G(X)$.

2.4. **Matchings.** A set $M \subseteq E(G)$ is a *matching* if $|\delta_G(v) \cap M| \leq 1$ holds for each $v \in V(G)$. For a matching M, we say that M *covers* a vertex v if $|\delta_G(v) \cap M| = 1$; otherwise, we say that M *exposes* v. A matching is *maximum* if it consists of the maximum number of edges. A graph can possess an exponentially large number of matchings. A matching is *perfect* if it covers every vertex. A graph is *factorizable* if it has a perfect matchings. A graph is *factor-critical* if, for each vertex v, $G - v$ is factorizable. A graph with only one vertex is defined to be factor-critical. The number of edges in a maximum matching is denoted by $\nu(G)$. The number of vertices exposed by a maximum matching is denoted by $\mathrm{def}(G)$; that is, $\mathrm{def}(G) := |V(G)| - 2\nu(G)$.

2.5. **Alternating Paths and Circuits.** Let $M \subseteq E(G)$. A circuit or path is said to be *M-alternating* if edges in M and not in M appear alternately. The precise definition is the following: A circuit C of G is M-alternating if $E(C) \cap M$ is a perfect matching of C. We define the three types of M-alternating paths. Let P be a path with ends s and t. We say that P is *M-forwarding* from s to t if $M \cap E(P)$ is a matching of P that covers every vertex except for t. We say that P is *M-saturated* between s and t if $M \cap E(P)$ is a perfect matching of P. We say that P is *M-exposed* between s and t if $M \cap E(P)$ is a matching of P that covers every vertex except for s and t. Any path with exactly one vertex x is defined to be an M-forwarding path from x to x, and is never treated as an M-exposed path. Any M-forwarding path has an even number of edges, which can be zero, whereas any M-saturated or -exposed path has an odd number of edges.

A path P is an *ear* relative to X if the internal vertices of P are disjoint from X, whereas the ends are in X. A circuit C is an *ear* relative to X if exactly one vertex of C is in X; for simplicity, we call the vertex in $X \cap V(C)$ the *end* of the ear C. We call an ear P relative to X an *M-ear* if $P - X$ is empty or an M-saturated path, and $\delta_P(X) \cap M = \emptyset$.

2.6. **Berge Formula and Barriers.** We now explain the Berge Formula and the definition of barriers. An *odd component* (resp. *even component*) of a graph is a connected component with an odd (resp. even) number of vertices. The number of odd components of $G - X$ is denoted by $q_G(X)$. The set of vertices from odd components (resp. even components) of $G - X$ is denoted by D_X (resp. C_X).

Theorem 2.1 (Berge Formula [15])**.** For a graph G, $\mathrm{def}(G)$ is equal to the maximum value of $q_G(X) - |X|$, where X is taken over all subsets of $V(G)$.

The set of vertices that attains the maximum value in this relation is called a *barrier*. That is, a set of vertices X is a *barrier* if $\mathrm{def}(G) = q_G(X) - |X|$.

2.7. **Gallai-Edmonds Family and Structure Theorem.** The set of vertices that can be exposed by some maximum matchings is denoted by $D(G)$. The neighbor set of $D(G)$ is denoted by $A(G)$, and the set $V(G) \setminus D(G) \setminus A(G)$ is denoted by $C(G)$. The following statement about $D(G)$, $A(G)$, and $C(G)$ is the celebrated *Gallai-Edmonds structure theorem* [5, 6, 15].

Theorem 2.2 (Gallai-Edmonds Structure Theorem)**.** For any graph G,
- (i) $A(G)$ is a barrier for which $D_{A(G)} = D(G)$ and $C_{A(G)} = C(G)$;
- (ii) each odd component of $G - A(G)$ is factor-critical; and,
- (iii) every edge in $E_G[A(G), D(G)]$ is allowed, whereas no edge in $E_G[A(G), A(G) \cup C(G)]$ is allowed.

2.8. **Factor-Components.** An edge is *allowed* if it is contained in some maximum matching. Two vertices are *factor-connected* if they are connected by a path whose edges are allowed. A subgraph is *factor-connected* if any two vertices are factor-connected. A maximal factor-connected subgraph is called a *factor-connected component* or *factor-component*. A graph consists of its factor-components and edges joining them that are not allowed. The set of factor-components of G is denoted by $\mathcal{G}(G)$.

A factor-component C is *inconsistent* if $V(C) \cap D(G) \neq \emptyset$. Otherwise, C is said to be *consistent*. We denote the sets of consistent and inconsistent factor-components of G by $\mathcal{G}^+(G)$ and $\mathcal{G}^-(G)$, respectively. The next property is easily confirmed from the Gallai-Edmonds structure theorem.

Fact 2.3. A subgraph C of G is an inconsistent factor-component if and only if C is a connected component of $G[D(G) \cup A(G)]$. Any consistent factor-component has the vertex set contained in $C(G)$.

That is, the structure of inconsistent factor-components are rather trivial under the Gallai-Edmonds structure theorem.

3. Basilica Decomposition of Graphs

3.1. **Central Concepts.** In Section 3, we introduce the basilica decomposition of graphs [7, 8]. The theory of basilica decomposition is made up of the three central concepts:
- (i) a canonical partial order between factor-components (Theorem 3.2),
- (ii) the general Kotzig-Lovász decomposition (Theorem 3.4), and
- (iii) an interrelationship between the two (Theorem 3.5).

In Section 3.1, we explain these three concepts and give the definition of the basilica decomposition. Every statement in the following are from Kita [7, 8] [1] In Section 3, let G be a graph unless otherwise stated.

[1] The essential part of the structure described by the basilica decomposition lies in the factorizable graph $G[C(G)]$. Therefore, statements for factorizable graphs [7, 8] can be straightforwardly generalized for arbitrary graphs under the Gallai-Edmonds structure theorem.

Definition 3.1. We call a set $X \subseteq V(G)$ *separating* if it is the disjoint union of vertex sets of some factor-components. For $G_1, G_2 \in \mathcal{G}(G)$, we say $G_1 \lhd G_2$ if there exists a separating set $X \subseteq V(G)$ with $V(G_1) \cup V(G_2) \subseteq X$ such that $G[X]/G_1$ is a factor-critical graph.

Theorem 3.2. For a graph G, the binary relation $\lhd$ is a partial order over $\mathcal{G}(G)$.

Definition 3.3. For $u, v \in V(G) \setminus D(G)$, we say $u \sim_G v$ if u and v are identical or if u and v are factor-connected and satisfy $\mathrm{def}(G - u - v) > \mathrm{def}(G)$.

Theorem 3.4. For a graph G, the binary relation $\sim_G$ is an equivalence relation.

We denote as $\mathcal{P}(G)$ the family of equivalence classes determined by $\sim_G$. This family is known as the *general Kotzig-Lovász decomposition* or just the *Kotzig-Lovász decomposition* of G. From the definition of $\sim_G$, for each $H \in \mathcal{G}(G)$, the family $\{S \in \mathcal{P}(G) : S \subseteq V(H)\}$ forms a partition of $V(H) \setminus D(G)$. We denote this family by $\mathcal{P}_G(H)$.

Let $H \in \mathcal{G}(G)$. The sets of strict and nonstrict upper bounds of H are denoted by $\mathcal{U}_G(H)$ and $\mathcal{U}^*_G(H)$, respectively. The sets of vertices $\bigcup\{V(I) : I \in \mathcal{U}_G(H)\}$ and $\bigcup\{V(I) : I \in \mathcal{U}^*_G(H)\}$ are denoted by $U_G(H)$ and $U^*_G(H)$, respectively.

Theorem 3.5. Let G be a graph, and let $H \in \mathcal{G}(G)$. Then, for each connected component K of $G[U_G(H)]$, there exists $S \in \mathcal{P}_G(H)$ such that $N_G(K) \cap V(H) \subseteq S$.

Under Theorem 3.5, for $S \in \mathcal{P}_G(H)$, we denote by $\mathcal{U}_G(S)$ the set of factor-components that are contained in a connected component K of $G[U_G(H)]$ with $N_G(K) \cap V(H) \subseteq S$. The set $\bigcup\{V(I) : I \in U_G(H)\}$ is denoted by $U_G(S)$. We denote $U_G(H) \setminus S \setminus U_G(S)$ by ${}^{\top}U_G(S)$.

Theorem 3.5 integrates the two structures given by Theorems 3.2 and 3.4 into a structure of graphs that is reminiscent of an architectural building. We call this integrated structure the *basilica decomposition* of a graph.

3.2. **Remark on Inconsistent Factor-Components.** Inconsistent factor-components in a graph have a trivial structure regarding the basilica decomposition. The next statement is easily confirmed from Fact 2.3 and the Gallai-Edmonds structure theorem.

Fact 3.6. Let G be a graph. Any inconsistent component is minimal in the poset $(\mathcal{G}(G), \lhd)$. For any $H \in \mathcal{G}^-(G)$, if $V(H) \cap A(G) \neq \emptyset$, then $\mathcal{P}_G(H) = \{V(H) \cap A(G)\}$; otherwise, $\mathcal{P}_G(H) = \emptyset$.

For simplicity, even for $H \in \mathcal{G}^-(G)$ with $V(H) \cap A(G) = \emptyset$, we treat as if $V(H) \cap A(G)$ is a member of $\mathcal{P}(G)$. That is, we let $\mathcal{P}_G(H) = \{V(H) \cap A(G)\}$ and ${}^{\top}U_G(V(H) \cap A(G)) = {}^{\top}U_G(\emptyset) = V(H) \cap D(G) = V(H)$.

Under Fact 3.6, the substantial information provided by the basilica decomposition lies in the consistent factor-components.

3.3. **Additional Properties.** In this section, we present some properties of the basilica decomposition that are used in later sections.

Lemma 3.7 (Kita [9])**.** Let G be a graph, and let M be a maximum matching of G. Let $H \in \mathcal{G}^+(G)$, $S \in \mathcal{P}_G(H)$, and $s \in S$.

(i) For any $t \in S$, there is an M-forwarding path from s to t, whose vertices are contained in $S \cup {}^{\top}U_G(S)$; however, there is no M-saturated path between s and t.

(ii) For any $t \in {}^{\top}U_G(S)$, there exists an M-saturated path between s and t whose vertices are contained in $S \cup {}^{\top}U_G(S)$.

(iii) For any $t \in U_G(S)$, there is an M-forwarding path from t to s, whereas there is no M-forwarding path from s to t or M-saturated path between s and t.

The first part of the next lemma is provided in Kita [10], and the second part can be easily proved from Lemma 3.7.

Lemma 3.8. Let G be a graph, and let M be a maximum matching of G. Let $S \in \mathcal{P}(G)$. If there is an M-ear relative to $S \cup {}^{\top}U_G(S)$ that has internal vertices, then the ends of this ear are contained in S.

4. Poset with Transitive Forbidden Relation

We now introduce the new concept of *posets with a transitive forbidden relation*, which serves as a language to describe the nonbipartite DM decomposition.

Definition 4.1. Let X be a set, and let $\preceq$ be a partial order over X. Let $\smile$ be a binary relation over X such that,

(i) for each $x, y, z \in X$, if $x \preceq y$ and $y \smile z$ hold, then $x \smile z$ holds (transitivity);
(ii) for each $x \in X$, $x \smile x$ does not hold (nonreflexivity); and,
(iii) for each $x, y \in X$, if $x \smile y$ holds, then $y \smile x$ also holds (symmetry).

We call this poset endowed with this additional binary relation a *poset with a transitive forbidden relation* or *TFR poset* in short, and denote this by $(X, \preceq, \smile)$. We call a pair of two elements x and y with $x \smile y$ *forbidden*.

Let $(X, \preceq, \smile)$ be a TFR poset. For two elements $x, y \in X$ with $x \smile y$, we say that $x \overset{\star}{\smile} y$ if, there is no $z \in X \setminus \{x, y\}$ with $x \preceq z$ and $z \smile y$. We call such a forbidden pair of x and y *immediate*. A TFR poset can be visualized in a similar way to an ordinary posets. We represent $\preceq$ just in the same way as the Hasse diagrams and depict $\smile$ by indicating every immediate forbidden pairs.

Definition 4.2. Let P be a TFR poset $(X, \preceq, \smile)$. A lower or upper ideal Y of P is *legitimate* if no elements $x, y \in Y$ satisfy $x \smile y$. Otherwise, we say that Y is *illegitimate*. Let Y be a consistent lower or upper ideal, and let Z be the subset of $X \setminus Y$ such that, for each $x \in Z$, there exists $y \in Y$ with $x \smile y$. We say that Y is *spanning* if $Y \cup Z = X$.

5. Dulmage-Mendelsohn Decomposition for General Graphs

We now provide our new results of the DM decomposition for general graphs. In this section, unless otherwise stated, let G be a graph.

Definition 5.1. A *Dulmage-Mendelsohn component*, or a *DM component* in short, is a subgraph of the form $G[S \cup {}^{\top}U_G(S)]$, where $S \in \mathcal{P}(G)$, endowed with S as an attribute known as the *base*. For a DM component C, the base of C is denoted by $\pi(C)$. Conversely, for $S \in \mathcal{P}(G)$, $K(S)$ denotes the DM components whose base is S. We denote by $\mathcal{D}(G)$ the set of DM components of G.

Hence, distinct DM components can be equivalent as a subgraph of G. A base $S \in \mathcal{P}(G)$ uniquely determines a DM component.

Definition 5.2. A DM component C is said to be *inconsistent* if $\pi(C) \in \mathcal{P}_G(H)$ for some $H \in \mathcal{G}^-(G)$; otherwise, C is said to be *consistent*. The sets of consistent and inconsistent DM components are denoted by $\mathcal{D}^+(G)$ and $\mathcal{D}^-(G)$, respectively.

Under Fact 3.6, any $H \in \mathcal{D}^-(G)$ is equal to an inconsistent factor-component as a subgraph of G, and $\pi(H) = V(H) \cap A(G)$ and $V(H) \setminus \pi(H) = V(H) \cap D(G)$.

Definition 5.3. We define binary relations $\preceq^\circ$ and $\preceq$ over $\mathcal{D}(G)$ as follows: for $D_1, D_2 \in \mathcal{D}(G)$, we let $D_1 \preceq^\circ D_2$ if $D_1 = D_2$ or if $N_G({}^\top U_G(S_1)) \cap S_2 \neq \emptyset$; we let $D_1 \preceq D_2$ if there exist $C_1, \ldots, C_k \in \mathcal{D}(G)$, where $k \geq 1$, such that $\pi(C_1) = \pi(D_1)$, $\pi(C_k) = \pi(D_2)$, and $C_i \preceq^\circ C_{i+1}$ for each $i \in \{1, \ldots, k\} \setminus \{k\}$.

Definition 5.4. We define binary relations $\smile^\circ$ and $\smile$ over $\mathcal{D}(G)$ as follows: for $D_1, D_2 \in \mathcal{D}(G)$, we let $D_1 \smile^\circ D_2$ if $\pi(D_2) \subseteq V(D_1) \setminus \pi(D_1)$ holds; we let $D_1 \smile D_2$ if there exists $D' \in \mathcal{D}(G)$ with $D_1 \preceq D'$ and $D' \smile^\circ D_2$.

In the following, we prove that $(\mathcal{D}(G), \preceq, \smile)$ is a TFR poset, which gives a generalization of the DM decomposition. The next lemma is easily observed from Facts 2.3 and 3.6.

Lemma 5.5. *If C is an inconsistent DM component of a graph G, then there is no $C' \in \mathcal{D}(G) \setminus \{C\}$ with $C \preceq C'$ or $C \smile C'$.*

We first prove that $\preceq$ is a partial order over $\mathcal{D}(G)$. We provide Lemmas 5.6 and 5.7 and thus prove Theorem 5.9.

Lemma 5.6. *Let G be a graph, let M be a maximum matching of G, and let $D_1, \ldots, D_k \in \mathcal{D}(G)$, where $k \geq 1$, be DM components with $D_1 \preceq^\circ \cdots \preceq^\circ D_k$ no two of which share vertices. Then, for any $s \in \pi(D_1)$ and for any $t \in \pi(D_k)$, there is an M-forwarding path from s to t whose vertices are contained in $V(D_1) \dot\cup \cdots \dot\cup V(D_k)$. If $D_k \in \mathcal{D}^+(G)$ holds and t is a vertex from $V(D_k) \setminus \pi(D_k)$, then there is an M-saturated path between s and t whose vertices are contained in $V(D_1) \dot\cup \cdots \dot\cup V(D_k)$.*

Proof. For each $i \in \{1, \ldots, k\} \setminus \{k\}$, let $t_i \in {}^\top U_G(\pi(D_i))$ and $s_{i+1} \in \pi(D_{i+1})$ be vertices with $t_i s_{i+1} \in E(G)$. Let $s_1 := s$ and $t_k := t$. According to Lemma 3.7, for each $i \in \{1, \ldots, k\} \setminus \{k\}$, there is an M-saturated path P_i between s_i and t_i with $V(P_i) \subseteq V(D_i)$; additionally, there is an M-forwarding path P_k from s_k to t with $V(P_k) \subseteq V(D_k)$. Thus, $P_1 + \cdots + P_k + \{t_i s_{i+1} : i = 1, \ldots, k-1\}$ is a desired M-forwarding path from s to t. The claim for $t \in V(D_k) \setminus \pi(D_k)$ can be also proved in a similar way using Lemma 3.7. □

Lemma 5.6 yields Lemma 5.7:

Lemma 5.7. *Let G be a graph, let M be a maximum matching of G, and let $D_1, \ldots, D_k$, where $k \geq 2$, be DM components with $D_1 \preceq^\circ \cdots \preceq^\circ D_k$ such that $\pi(D_i) \neq \pi(D_{i+1})$ for any $i \in \{1, \ldots, k-1\}$. Then, for any $i, j \in \{1, \ldots, k\}$ with $i \neq j$, $V(D_i) \cap V(D_j) = \emptyset$.*

Proof. Let q be the minimum number from $\{1, \ldots, k-1\}$ such that D_{q+1} shares vertice with some D_i, where $i \in \{1, \ldots, q-1\}$. Let p be the maximum number from $\{1, \ldots, q-1\}$ such that $V(D_{q+1}) \cap V(D_p) \neq \emptyset$. Then, $D_p, \ldots, D_q$ are mutually disjoint. Additionally, from Lemma 5.5, we have $D_p, \ldots, D_q \in \mathcal{D}^+(G)$. Either $\pi(D_{q+1}) \subseteq V(D_p)$ or ${}^\top U_G(\pi(D_{q+1})) \cap V(D_p) \neq \emptyset$ holds. In the first case, let t_{q+1} be an arbitrary vertex from ${}^\top U_G(\pi(D_{q+1}))$, and, in the second case, let t_{q+1} be a vertex from ${}^\top U_G(\pi(D_{q+1})) \cap V(D_p)$. Let $t_q \in {}^\top U_G(\pi(D_q))$ and $s_{q+1} \in \pi(D_{q+1})$ be vertices with $t_q s_{q+1} \in E(G)$. From Lemma 3.7, there is an M-saturated path P between s_{q+1} and t_{q+1} with $V(P) \subseteq V(D_{q+1})$.

Let $t_p \in {}^\top U_G(\pi(D_p))$ and s_{p+1} be vertices with $t_p s_{p+1} \in E(G)$. From Lemma 5.6, there is an M-saturated path Q between s_{p+1} and t_q with $V(Q) \subseteq V(D_p) \cup \cdots \cup V(D_q)$.

Trace P from s_{q+1}, and let x be the first encountered vertex in D_p, for which $x = s_{q+1}$ is allowed. Then, by letting $R := t_p s_{p+1} + Q + t_q s_{q+1} + s_{q+1} P x$, R is an M-ear relative to D_p whose ends are x and t_p. Note that R contains internal vertices, e.g., s_{p+1}.

If $x \in \pi(D_p)$ holds, then, under Lemma 3.7, let L be an M-saturated path between x and t_p with $V(L) \subseteq V(D_p)$. Then, $R + L$ is an M-alternating circuit containing non-allowed edges of G. This is a contradiction. If $x \in {}^{\top}U_G(\pi(D_p))$ holds, then this contradicts Lemma 3.8. This completes the proof. $\square$

Combining Lemmas 5.6 and 5.7, the next lemma can be stated, which we do not use for proving Theorem 5.9.

Lemma 5.8. Let G be a graph. Let $C_1, C_2 \in \mathcal{D}(G)$ with $C_1 \preceq C_2$, and let $D_1, \ldots, D_k \in \mathcal{D}(G)$, where $k \geq 1$, be DM components with $C_1 = D_1$, $C_2 = D_k$, and $D_1 \preceq^{\circ} \cdots \preceq^{\circ} D_k$. Then, for any $s \in \pi(D_1)$ and for any $t \in \pi(D_k)$ (resp. $t \in V(D_k) \setminus \pi(D_k)$), there is an M-forwarding path from s to t (resp. M-saturated path between s and t) whose vertices are contained in $V(D_1) \dot\cup \cdots \dot\cup V(D_k)$.

We now obtain Theorem 5.9.

Theorem 5.9. Let G be a graph. Then, $\preceq$ is a partial order over $\mathcal{D}(G)$.

Proof. Reflexivity and transitivity are obvious from the definition. Antisymmetry is obviously implied by Lemma 5.7. $\square$

In the following, we prove the properties required for $\smile$ to form a TFR poset $(\mathcal{D}(G), \preceq, \smile)$. We provide Lemmas 5.10, 5.11, and 5.13, and thus prove Theorem 5.14

Lemma 5.10. Let G be a graph, and let M be a maximum matching of G. Let $s, t \in V(G)$, and let S be the member of $\mathcal{P}(G)$ with $s \in S$. Let P be an M-forwarding path P from s to t or an M-saturated path between s and t. If $t \in S \cup {}^{\top}U_G(S)$ holds, then $P - E(G[S \cup {}^{\top}U_G(S)])$ is empty; otherwise, $P - E(G[S \cup {}^{\top}U_G(S)])$ is a path.

Proof. Suppose that the claim fails. The connected components of $P - E(G[S \cup {}^{\top}U_G(S)])$ except for the one that contains s are M-ears relative to $S \cup {}^{\top}U_G(S)$ with internal vertices. Let S' be the set of the ends of these M-ears. From Lemma 3.8, we have $S' \subseteq S$. Trace P from s, and let s' be the first vertex in S'. Then, sPr is an M-saturated path between s and s', which contradicts $s \sim_G s'$. This proves the claim. $\square$

Lemma 5.10 derives the next lemma with Lemmas 3.7 and 3.8.

Lemma 5.11. Let G be a graph, and let M be a maximum matching of G. Let $s, t \in V(G)$, and let S and T be the members from $\mathcal{P}(G)$ with $s \in S$ and $t \in T$, respectively.

(i) If there is no M-saturated path between s and t, whereas there is an M-forwarding path from s to t, then $K(S) \preceq K(T)$ holds.
(ii) If there is an M-saturated path between s and t, then $K(S) \smile K(T)$ holds.

Proof. Let P be an M-forwarding path from s to t or an M-saturated between s and t. We proceed by induction on the number of edges in P. If $V(P) \subseteq S \cup {}^{\top}U_G(S)$ holds, then Lemma 3.7 proves the statements. Hence, let $V(P) \setminus V(K(S)) \neq \emptyset$, and assume that the statements hold for every case where $|E(P)|$ is fewer. By Lemma 5.10, $P - E(K(S))$ is an M-exposed path one of whose end is t; let x be the other end of P. Let $y \in V(P)$ be the vertex with $xy \in E(P)$. The subpath sPx is obviously M-saturated between s and x, for which $x \in V(K(S))$ holds. Hence, Lemma 3.7 implies $x \in {}^{\top}U_G(S)$. Let R be the member of $\mathcal{P}(G)$ with $y \in R$. Then, we have $K(S) \preceq^{\circ} K(R)$.

If P is an M-saturated path, then yPt is an M-saturated path between y and t. Therefore, the induction hypothesis implies $K(R) \smile K(T)$. Thus, $K(S) \smile K(T)$ is obtained, and (ii) is proved.

Consider now the case where P is an M-forwarding path from s to t. The subpath yPt is an M-forwarding path from y to t.

Claim 5.12. There is no M-saturated path between y and t in G.

Proof. Suppose the contrary, and let Q be an M-saturated path between y and t. First, suppose that Q shares vertices with $S \cup {}^{\top}U_G(S)$. Trace Q from y, and let z be the first vertex in $S \cup {}^{\top}U_G(S)$. Then, $zQy + yx$ is an M-ear relative to $S \cup {}^{\top}U_G(S)$ with internal vertices, e.g., y. This contradicts Lemma 3.8. Hence, Q is disjoint from $S \cup {}^{\top}U_G(S)$. This however implies that $sPx + xy + Q$ is an M-saturated path between s and t, which contradicts the assumption. □

Therefore, under the induction hypothesis, $K(R) \preceq K(T)$. We thus obtain $K(S) \preceq K(T)$, and (i) is proved. □

The symmetry of $\smile$ now can be proved from Lemmas 5.8 and 5.11.

Lemma 5.13. For a graph G, the binary relation $\smile$ is symmetric, that is, if $D_1 \smile D_2$ holds for $D_1, D_2 \in \mathcal{D}(G)$, then $D_2 \smile D_1$ holds.

Proof. Let M be a maximum matching of G, and let $x_1 \in \pi(D_1)$ and $x_2 \in \pi(D_2)$. From Lemma 5.8, $D_1 \smile D_2$ implies that there is an M-saturated path P between x_1 and x_2. From Lemma 5.11, this implies $D_2 \smile D_1$. □

We now prove Theorem 5.14 from Theorem 5.9 and Lemma 5.13:

Theorem 5.14. For a graph G, the triple $(\mathcal{D}(G), \preceq, \smile)$ is a TFR poset.

Proof. Under Theorem 5.9, it now suffices prove the conditions for $\smile$. Nonreflexivity and transitivity are obvious from the definition. Symmetry is proved by Lemma 5.13. □

For a graph G, the TFR poset $(\mathcal{D}(G), \preceq, \smile)$ is uniquely determined. We denote this TFR poset by $\mathcal{O}(G)$. We call this canonical structure of a graph G that the TIP $\mathcal{O}(G)$ describes the *nonbipartite Dulmage-Mendelsohn (DM) decomposition* of G. We show in Section 8 that this is a generalization of the classical DM decomposition for bipartite graphs.

Remark 5.15. As mentioned previously, a DM component is identified by its base. Therefore, the nonbipartite DM decomposition is essentially the relations between the members of $\mathcal{P}(G)$.

6. Preliminaries on Maximal Barriers

6.1. **Classical Properties of Maximal Barriers.** We now present some preliminary properties of maximal barriers to be used in Section 7. A barrier is *maximal* if it is inclusion-wise maximal. A barrier X is *odd-maximal* if it is maximal with respect to D_X; that is, for no $Y \subseteq D_X$, $X \cup Y$ is a barrier. A maximal barrier is an odd-maximal barrier.

The next two propositions are classical facts. See Lovász and Plummer [15].

Proposition 6.1. Let G be a graph, and let $X \subseteq V(G)$ be a barrier. Then, X is an odd-maximal barrier if and only if every odd component of $G - X$ are factor-critical.

Proposition 6.2. Let G be a graph. An odd-maximal barrier is a maximal barrier if and only if $C_X = \emptyset$.

6.2. **Generalization of Lovász's Canonical Partition Theorem.** In this section, we explain a known theorem about the structure of a given odd-maximal barrier [9]. This theorem is a generalization of *Lovász's canonical partition theorem* [9, 14, 15] for general graphs, which is originally for factor-connected graphs. This theorem contains the classical result about the relationship between maximal barriers and the Gallai-Edmonds decomposition, which states that $A(G)$ of a graph G is the intersection of all maximal barriers [15].

Theorem 6.3 (Kita [9])**.** Let G be a graph and $X \subseteq V(G)$ be an odd-maximal barrier of G. Then, there exist $S_1, \dots, S_k \in \mathcal{P}(G)$, where $k \geq 1$, such that $X = S_1 \dot\cup \cdots \dot\cup S_k$ and $D_X = {}^{\top}U_G(S_1) \dot\cup \cdots \dot\cup {}^{\top}U_G(S_k)$. The odd components of $G - X$ are the connected components of $G[{}^{\top}U_G(S_i)]$, where i is taken over all $\{1, \dots, k\}$.

The next statement can be derived from Theorem 6.3 as a corollary.

Corollary 6.4. Let G be a graph. For each $S \in \mathcal{P}(G)$, $G[{}^{\top}U_G(S)]$ consists of $|S| + \mathrm{def}(G[S \cup {}^{\top}U_G(S)])$ connected components, which are factor-critical. If $S \in \mathcal{P}_G(H)$ holds for some $H \in \mathcal{G}^+(G)$, then $\mathrm{def}(G[S \cup {}^{\top}U_G(S)]) = 0$; otherwise, $\mathrm{def}(G[S \cup {}^{\top}U_G(S)]) > 0$. Let $\mathcal{S} := \bigcup\{S \in \mathcal{P}_G(H) : H \in \mathcal{G}^-(G) \text{ and } V(H) \cap X \neq \emptyset\}$. Then, $\Sigma_{S \in \mathcal{S}} \mathrm{def}(G[S \cup {}^{\top}U_G(S)]) = \mathrm{def}(G)$.

7. Canonical Characterization of Maximal Barriers

We now derive the characterization of the family of maximal barriers in general graphs, using the nonbipartite DM decomposition. In this section, unless otherwise stated, let G be a graph. It is a known fact that a graph has an exponentially many number of maximal barriers, however the family of maximal barriers can be fully characterized in terms of ideals of $\mathcal{O}(G)$.

Definition 7.1. For $\mathcal{I} \subseteq \mathcal{D}(G)$, the *normalization* of $\mathcal{I}$ is the set $\mathcal{I} \cup \mathcal{D}^-(G)$. A set $\mathcal{I}' \subseteq \mathcal{D}(G)$ is said to be *normalized* if $\mathcal{I}' = \mathcal{I} \cup \mathcal{D}^-(G)$ for some $\mathcal{I} \subseteq \mathcal{D}(G)$.

From Lemma 5.5, the next statement can be easily observed.

Observation 7.2. The normalization of an upper ideal is an upper ideal. The normalization of a legitimate upper ideal is legitimate.

From Theorem 6.3, the next lemma characterizes the family of odd-maximal barriers.

Lemma 7.3. Let G be a graph. A set of vertices $X \subseteq V(G)$ is an odd-maximal barrier if and only if there exists a legitimate normalized upper ideal $\mathcal{I}$ of the TFR poset $\mathcal{O}(G)$ such that $X = \bigcup\{\pi(C) : C \in \mathcal{I}\}$.

Proof. We first prove the sufficiency. Let X be an odd-maximal barrier, and, under Theorem 6.3, let $S_1, \dots, S_k$, where $k \geq 1$, be the members of $\mathcal{P}(G)$ such that $X = S_1 \cup \cdots \cup S_k$. Let $\mathcal{I} := \{K(S_i) : i = 1, \dots, k\}$. We prove that $\mathcal{I}$ is a legitimate normalized upper ideal of $\mathcal{O}(G)$. For proving $\mathcal{I}$ is an upper ideal, it suffices to prove that, for any $C \in \mathcal{D}(G)$, $K(S_i) \preceq^\circ C$ implies $\pi(C) \subseteq X$; and, this is obviously confirmed from Theorem 6.3. It is also confirmed by Theorem 6.3 that this upper ideal is normalized and legitimate.

Next, we prove the necessity. Let $\mathcal{I}$ be a legitimate normalized upper ideal of $\mathcal{O}(G)$, and let $X = \bigcup\{\pi(C) : C \in \mathcal{I}\}$. From the definition of $\preceq^\circ$, $\mathcal{I}$ being an upper ideal implies that, for each $C \in \mathcal{I}$, $N_G({}^{\top}U_G(\pi(C))) \subseteq X$; $\mathcal{I}$ being legitimate implies that ${}^{\top}U_G(\pi(C)) \cap X = \emptyset$. Hence, each connected component of $G[{}^{\top}U_G(\pi(C))]$ is also a connected component of $G - X$ that is factor-critical. Therefore, Corollary 6.4 implies that $G - X$ has $|X| + \mathrm{def}(G)$ odd components, and accordingly X is a barrier. By Theorem 6.3, these odd components are factor-critical, and therefore, Proposition 6.1 implies that X is odd-maximal. $\square$

From Lemma 7.3 and Proposition 6.2, the family of maximal barriers is now characterized:

Theorem 7.4. Let G be a graph. A set of vertices $X \subseteq V(G)$ is a maximal barrier if and only if there exists a spanning legitimate normalized upper ideal $\mathcal{I}$ of the TFR poset $\mathcal{O}(G)$ such that $X = \bigcup\{\pi(C) : C \in \mathcal{I}\}$.

8. Original DM Decomposition for Bipartite Graphs

In this section, we explain the original DM decomposition for bipartite graphs, and prove this from our result in Section 5. In the remainder of this section, unless stated otherwise, let G be a bipartite graph with color classes A and B, and let $W \in \{A, B\}$.

Definition 8.1. The binary relations $\leq_W^\circ$ and $\leq_W$ over $\mathcal{G}(G)$ are defined as follows: for $G_1, G_2 \in \mathcal{G}(G)$, let $G_1 \leq_W^\circ G_2$ if $G_1 = G_2$ or if $E_G[W \cap V(G_2), V(G_1) \setminus W] \neq \emptyset$; let $G_1 \leq_W G_2$ if there exist $H_1, \ldots, H_k \in \mathcal{G}(G)$, where $k \geq 1$, such that $H_1 = G_1$, $H_k = G_2$, and $H_1 \leq_W^\circ \cdots \leq_W^\circ H_k$.

Note that $G_1 \leq_A G_2$ holds if and only if $G_2 \leq_B G_1$ holds. The next theorem determines the classical DM decomposition.

Theorem 8.2 (Dulmage and Mendelsohn [2–4, 15])**.** Let G be a bipartite graph with color classes A and B, and let $W \in \{A, B\}$. Then, the binary relation $\leq_W$ is a partial order over $\mathcal{G}(G)$.

We call the poset $(\mathcal{G}(G), \leq_W)$ proved by Theorem 8.2 the *Dulmage-Mendelsohn decomposition* of a bipartite graph G.

In the following, we demonstrate how our nonbipartite DM decomposition derives Theorem 8.2 under the special properties of bipartite graphs regarding

(i) inconsistent factor-components (Observation 8.3) and
(ii) the basilica decomposition (Observation 8.4).

The set of inconsistent factor-components with some vertices from $D(G) \setminus W$ is denoted by $\mathcal{G}_W^-(G)$. The next statement about $\mathcal{G}_W^-(G)$ can be easily confirmed from the Gallai-Edmonds structure theorem. This statement can also be proved from first principles by a simple discussion on alternating paths, which is employed in original proof. As is also the case in the basilica and nonbipartite DM decomposition, the substantial part of the bipartite DM decomposition lies in $\mathcal{G}^+(G)$.

Observation 8.3. The sets $\mathcal{G}_A^-(G)$ and $\mathcal{G}_B^-(G)$ are disjoint. Any $C \in \mathcal{G}_B^-(G)$ is minimal with respect to $\leq_A$.

Bipartite graphs have a trivial structure regarding the basilica decomposition:

Observation 8.4. Let G be a bipartite graph with color classes A and B, and let $W \in \{A, B\}$.

(i) Then, for each $H \in \mathcal{G}^+(G)$, $\mathcal{P}_G(H) = \{V(H) \cap A, V(H) \cap B\}$. For each $H \in \mathcal{G}_W^-(G)$, $\mathcal{P}_G(H) = \{V(H) \cap W\}$.
(ii) For any $H_1, H_2 \in \mathcal{G}(G)$ with $H_1 \neq H_2$, $H_1 \triangleleft H_2$ does not hold.

Under Observation 8.4, we define $\mathcal{D}^W(G)$ as the set $\{C \in \mathcal{D}(G) : \pi(C) \subseteq W\}$. Define a mapping $f_W : \mathcal{G}^+(G) \cup \mathcal{G}_W^-(G) \to \mathcal{D}^W(G)$ as $f_W(C) := K(V(C) \cap W)$ for $C \in \mathcal{G}^+(G)$. The next statement is obvious from Observation 8.4.

Observation 8.5. The mapping f_W is a bijection; and, for any $C_1, C_2 \in \mathcal{G}(G)$, $C_1 \leq_W C_2$ holds if and only if $f(C_1) \preceq f(C_2)$ holds.

According to Theorem 5.14 and Observation 8.5, the system $(\mathcal{G}^+(G) \cup \mathcal{G}_W^-(G), \leq_W$) is a poset. Observations 8.3 and 8.5 now prove Theorem 8.2.

9. Computational Properties

Given a graph G, its basilica decomposition can be computed in $O(|V(G)| \cdot |E(G)|)$ time [7, 8]. Assume that the basilica decomposition of G is given. From the definition of $\preceq^\circ$, the poset $(\mathcal{D}(G), \preceq)$ can be computed in $O(|\mathcal{P}(G)| \cdot |E(G)|)$ time, and accordingly, in $O(|V(G)| \cdot |E(G)|)$ time. According to the definition of $\smile^\circ$, given the poset $(\mathcal{D}(G), \preceq)$, the TFR poset $\mathcal{O}(G)$ can be obtained in $O(|V(G)|)$ time. Therefore, the next thereom can be stated.

Theorem 9.1. Given a graph G, the TFR poset $\mathcal{O}(G)$ can be computed in $O(|V(G)| \cdot |E(G)|)$ time.

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