# A Faster FPTAS for the Subset-Sums Ratio Problem 

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#### Abstract

The Subset-Sums Ratio problem (SSR) is an optimization problem in which, given a set of integers, the goal is to find two subsets such that the ratio of their sums is as close to 1 as possible. In this paper we develop a new FPTAS for the SSR problem which builds on techniques proposed in [D. Nanongkai, Simple FPTAS for the subset-sums ratio problem, Inf. Proc. Lett. 113 (2013)]. One of the key improvements of our scheme is the use of a dynamic programming table in which one dimension represents the difference of the sums of the two subsets. This idea, together with a careful choice of a scaling parameter, yields an FPTAS that is several orders of magnitude faster than the best currently known scheme of [C. Bazgan, M. Santha, Z. Tuza, Efficient approximation algorithms for the Subset-Sums Equality problem, J. Comp. System Sci. 64 (2) (2002)].


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## 1 Introduction

We study the optimization version of the following NP-hard decision problem which given a set of integers asks for two subsets of equal sum (but, in contrast to the Partition problem, the two subsets do not have to form a partition of the given set):

Equal Sum Subsets problem (ESS). Given a set $A=\left\{a_{1}, \ldots, a_{n}\right\}$ of $n$ positive integers, are there two nonempty and disjoint sets $S_{1}, S_{2} \subseteq\{1, \ldots, n\}$ such that $\sum_{i \in S_{1}} a_{i}=\sum_{j \in S_{2}} a_{j}$ ?

Our motivation to study the ESS problem and its optimization version comes from the fact that it is a fundamental problem closely related to problems appearing in many scientific areas. Some examples are the Partial Digest problem,
which comes from molecular biology (see [23]), the problem of allocating individual goods (see [8]), tournament construction (see [7), and a variation of the Subset Sum problem, namely the Multiple Integrated Sets SSP, which finds applications in the field of cryptography (see [10]).

The ESS problem has been proven NP-hard by Woeginger and Yu in 11 and several of its variations have been proven NP-hard by Cieliebak et al. in 4/56. The corresponding optimization problem is:

Subset-Sums Ratio problem (SSR). Given a set $A=\left\{a_{1}, \ldots, a_{n}\right\}$ of $n$ positive integers, find two nonempty and disjoint sets $S_{1}, S_{2} \subseteq\{1, \ldots, n\}$ that minimize the ratio

$$
\frac{\max \left\{\sum_{i \in S_{1}} a_{i}, \sum_{j \in S_{2}} a_{j}\right\}}{\min \left\{\sum_{i \in S_{1}} a_{i}, \sum_{j \in S_{2}} a_{j}\right\}} .
$$

The SSR problem was introduced by Woeginger and Yu [11. In the same work they present an 1.324 approximation algorithm which runs in $O(n \log n)$ time. The SSR problem received its first FPTAS by Bazgan et al. in [1, which approximates the optimal solution in time no less than $O\left(n^{5} / \varepsilon^{3}\right)$; to the best of our knowledge this is still the faster scheme proposed for SSR. A second, simpler but slower, FPTAS was proposed by Nanongkai in 9 .

The FPTAS we present in this paper makes use of some ideas proposed in [9, strengthened by certain key improvements that lead to a considerable acceleration: our algorithm approximates the optimal solution in $O\left(n^{4} / \varepsilon\right)$ time, several orders of magnitude faster than the best currently known scheme of 11 .

## 2 Preliminaries

We will first define two functions that will allow us to simplify several of the expressions that we will need throughout the paper.

Definition 1 (Ratio of two subsets). Given a set $A=\left\{a_{1}, \ldots, a_{n}\right\}$ of $n$ positive integers and two sets $S_{1}, S_{2} \subseteq\{1, \ldots, n\}$ we define $\mathcal{R}\left(S_{1}, S_{2}, A\right)$ as follows:

$$
\mathcal{R}\left(S_{1}, S_{2}, A\right)= \begin{cases}\frac{\sum_{i \in S_{1}} a_{i}}{\sum_{j \in S_{2}} a_{j}} & \text { if } S_{1} \cup S_{2} \neq \emptyset, \\ +\infty & \text { otherwise } .\end{cases}
$$

Definition 2 (Max ratio of two subsets). Given a set $A=\left\{a_{1}, \ldots, a_{n}\right\}$ of $n$ positive integers and two sets $S_{1}, S_{2} \subseteq\{1, \ldots, n\}$ we define $\mathcal{M R}\left(S_{1}, S_{2}, A\right)$ as follows:

$$
\mathcal{M R}\left(S_{1}, S_{2}, A\right)=\max \left\{\mathcal{R}\left(S_{1}, S_{2}, A\right), \mathcal{R}\left(S_{2}, S_{1}, A\right)\right\}
$$

Note that, in cases where at least one of the sets is empty, the Max Ratio function will return $\infty$. Using these functions, the SSR problem can be rephrased as shown below.

Subset-Sums Ratio problem (SSR) (equivalent definition). Given a set $A=\left\{a_{1}, \ldots, a_{n}\right\}$ of $n$ positive integers, find two disjoint sets $S_{1}, S_{2} \subseteq\{1, \ldots, n\}$ such that the value $\mathcal{M R}\left(S_{1}, S_{2}, A\right)$ is minimized.

In addition, from now on, whenever we have a set $A=\left\{a_{1}, \ldots, a_{n}\right\}$ we will assume that $0<a_{1}<a_{2}<\ldots<a_{n}$ (clearly, if the input contains two equal numbers then the problem has a trivial solution).

The FPTAS proposed by Nanonghai [9 approximates the SSR problem by solving a restricted version.

Restricted Subset-Sums Ratio problem. Given a set $A=\left\{a_{1}, \ldots, a_{n}\right\}$ of $n$ positive integers and two integers $1 \leq p<q \leq n$, find two disjoint sets $S_{1}, S_{2}$ $\subseteq\{1, \ldots, n\}$ such that $\left\{\max S_{1}, \max S_{2}\right\}=\{p, q\}$ and the value $\mathcal{M R}\left(S_{1}, S_{2}, A\right)$ is minimized.

Inspired by this idea, we define a less restricted version. The new problem requires one additional input integer, instead of two, which represents the smallest of the two maximum elements of the sought optimal solution.

Semi-Restricted Subset-Sums Ratio problem. Given a set $A=\left\{a_{1}, \ldots\right.$, $\left.a_{n}\right\}$ of $n$ positive integers and an integer $1 \leq p<n$, find two disjoint sets $S_{1}$, $S_{2} \subseteq\{1, \ldots, n\}$ such that $\max S_{1}=p<\max S_{2}$ and the value $\mathcal{M} \mathcal{R}\left(S_{1}, S_{2}, A\right)$ is minimized.

Let $A=\left\{a_{1}, \ldots, a_{n}\right\}$ be a set of $n$ positive integers and $p \in\{1, \ldots, n-1\}$. Observe that, if $S_{1}^{*}, S_{2}^{*}$ is the optimal solution of SSR problem of instance $A$ and $S_{1}^{p}, S_{2}^{p}$ the optimal solution of Semi-Restricted SSR problem of instance $A$, $p$ then:

$$
\mathcal{M R}\left(S_{1}^{*}, S_{2}^{*}, A\right)=\min _{p \in\{1, \ldots, n-1\}} \mathcal{M} \mathcal{R}\left(S_{1}^{p}, S_{2}^{p}, A\right)
$$

Thus, we can find the optimal solution of SSR problem by solving the SSR Semi-Restricted SSR problem for all $p \in\{1, \ldots, n-1\}$.

## 3 Pseudo-polynomial time algorithm for Semi-Restricted SSR problem

Let the $A, p$ be an instance of the Semi-Restricted SSR problem where $A=$ $\left\{a_{1}, \ldots, a_{n}\right\}$ and $1 \leq p<n$. For solving the problem we have to check two cases for the maximum element of the optimal solution. Let $S_{1}^{*}, S_{2}^{*}$ be the optimal solution of this instance and $\max S_{2}^{*}=q$. We define $B=\left\{a_{i} \mid i>p, a_{i}<\right.$ $\left.\sum_{j=1}^{p} a_{j}\right\}$ and $C=\left\{a_{i} \mid a_{i} \geq \sum_{j=1}^{p} a_{j}\right\}$ from which we have that either $a_{q} \in B$ or $a_{q} \in C$. Note that $A=\left\{a_{1}, \ldots, a_{p}\right\} \cup B \cup C$.

Case $1\left(a_{q} \in C\right)$. It is easy to see that if $a_{q} \in C$, then $a_{q}=\min C$ and the optimal solution will be ( $S_{1}=\{1, \ldots, p\}, S_{2}=\{q\}$ ). We describe below a function that returns this pair of sets, thus computing the optimal solution if Case 1 holds.

Definition 3 (Case 1 solution). Given a set $A=\left\{a_{1}, \ldots, a_{n}\right\}$ of $n$ positive integers and an integer $1 \leq p<n$ we define the function $\mathcal{S O} \mathcal{L}_{1}(A, p)$ as follows:

$$
\mathcal{S O} \mathcal{L}_{1}(A, p)= \begin{cases}(\{1, \ldots, p\},\{\min C\}) & \text { if } C \neq \emptyset \\ (\emptyset, \emptyset) & \text { otherwise }\end{cases}
$$

where $C=\left\{a_{i} \mid a_{i}>\sum_{j=1}^{p} a_{j}\right\}$.

Case $2\left(a_{q} \in B\right)$. This second case is not trivial. Here, we define an integer $m=\max \left\{j \mid a_{j} \in A \backslash C\right\}$ and a matrix $T$, where $T[i, d], 0 \leq i \leq m,-2$. $\sum_{k=1}^{p} a_{k} \leq d \leq \sum_{k=1}^{p} a_{k}$, is a quadruple to be defined below. A cell $T[i, d]$ is nonempty if there exist two disjoint sets $S_{1}, S_{2}$ with sums sum $_{1}$, sum ${ }_{2}$ such that $s u m_{1}-s u m_{2}=d$, $\max S_{1}=p$, and $S_{1} \cup S_{2} \subseteq\{1, \ldots, i\} \cup\{p\}$; if $i>p$, we require in addition that $p<\max S_{2}$. In such a case, cell $T[i, d]$ consists of the two sets $S_{1}, S_{2}$, and two integers $\max \left(S_{1} \cup S_{2}\right)$ and $s u m_{1}+s u m_{2}$. A crucial point in our algorithm is that if there exist more than one pairs of sets which meet the required conditions, we keep the one that maximize the value $s u m_{1}+s u m_{2}$; for convenience, we make use of a function to check this property and select the appropriate sets. The algorithm for this case (Algorithm 1) finally returns the pair $S_{1}, S_{2}$ which, among those that appear in some $T[m, d] \neq \emptyset$, has the smallest ratio $\mathcal{M} \mathcal{R}\left(S_{1}, S_{2}, A\right)$.

Definition 4 (Larger total sum tuple selection). Given two tuples $\boldsymbol{v}_{\mathbf{1}}=$ $\left(S_{1}, S_{2}, q, x\right)$ and $\boldsymbol{v}_{\mathbf{2}}=\left(S_{1}^{\prime}, S_{2}^{\prime}, q^{\prime}, x^{\prime}\right)$ we define the function $\mathcal{L T S T}\left(\boldsymbol{v}_{\mathbf{1}}, \boldsymbol{v}_{\mathbf{2}}\right)$ as follows:

$$
\mathcal{L T S T}\left(\boldsymbol{v}_{\mathbf{1}}, \boldsymbol{v}_{\mathbf{2}}\right)= \begin{cases}\boldsymbol{v}_{\mathbf{2}} & \text { if } \boldsymbol{v}_{\mathbf{1}}=\emptyset \text { or } x^{\prime}>x \\ \boldsymbol{v}_{\mathbf{1}} & \text { otherwise }\end{cases}
$$

```
Algorithm 1 Case 2 solution [ \(\mathcal{S O} \mathcal{L}_{2}(A, p)\) function]
Input: a strictly sorted set \(A=\left\{a_{1}, \ldots, a_{n}\right\}, a_{i} \in \mathbb{Z}^{+}\), and an integer \(p, 1 \leq p<n\).
Output: the sets of an optimal solution for Case 2.
    \(S_{1}^{\prime} \leftarrow \emptyset, S_{2}^{\prime} \leftarrow \emptyset\)
    \(Q \leftarrow \sum_{i=1}^{p} a_{i}, m \leftarrow \max \left\{i \mid a_{i}<Q\right\}\)
    if \(m>p\) then
        for all \(i \in\{0, \ldots, m\}, d \in\{-2 \cdot Q, \ldots, Q\}\) do
            \(T[i, d] \leftarrow \emptyset\)
        end for
```

```
\(T\left[0, a_{p}\right] \leftarrow\left(\{p\}, \emptyset, p, a_{p}\right)\)
for \(i \leftarrow 1\) to \(m\) do
    if \(i<p\) then
            for all \(T[i-1, d] \neq \emptyset\) do
                    \(\left(S_{1}, S_{2}, q, x\right) \leftarrow T[i-1, d]\)
                    \(T[i, d] \leftarrow \mathcal{L T S T}(T[i, d], T[i-1, d])\)
                \(T\left[i, d+a_{i}\right] \leftarrow \mathcal{L T S T}\left(T\left[i, d+a_{i}\right],\left(S_{1} \cup\{i\}, S_{2}, q, x+a_{i}\right)\right)\)
                \(T\left[i, d-a_{i}\right] \leftarrow \mathcal{L T S T}\left(T\left[i, d-a_{i}\right],\left(S_{1}, S_{2} \cup\{i\}, q, x+a_{i}\right)\right)\)
            end for
        else if \(i=p\) then \(\quad \triangleright p\) is already placed in \(S_{1}\)
            for all \(T[i-1, d] \neq \emptyset\) do
                \(T[i, d] \leftarrow T[i-1, d]\)
            end for
        else
            for all \(T[i-1, d] \neq \emptyset\) do
                \(\left(S_{1}, S_{2}, q, x\right) \leftarrow T[i-1, d]\)
                if \(i>p+1\) then
                    \(T[i, d] \leftarrow \mathcal{L} \mathcal{T} \mathcal{S T}(T[i, d], T[i-1, d])\)
            end if
            if \(d-a_{i} \geq-2 \cdot Q\) then
                    \(T\left[i, d-a_{i}\right] \leftarrow \mathcal{L T S T}\left(T\left[i, d-a_{i}\right],\left(S_{1}, S_{2} \cup\{i\}, i, x+a_{i}\right)\right)\)
                end if
        end for
        for all \(T[p, d] \neq \emptyset\) do
            \(\left(S_{1}, S_{2}, q, x\right) \leftarrow T[p, d]\)
            if \(d-a_{i} \geq-2 \cdot Q\) then
                \(T\left[i, d-a_{i}\right] \leftarrow \mathcal{L T S T}\left(T\left[i, d-a_{i}\right],\left(S_{1}, S_{2} \cup\{i\}, i, x+a_{i}\right)\right)\)
            end if
        end for
        end if
    end for
    for \(d \leftarrow-2 \cdot Q\) to \(Q\) do
        \(\left(S_{1}, S_{2}, q, x\right) \leftarrow T[m, d]\)
        if \(\mathcal{M R}\left(S_{1}, S_{2}, A\right)<\mathcal{M R}\left(S_{1}^{\prime}, S_{2}^{\prime}, A\right)\) then
        \(S_{1}^{\prime} \leftarrow S_{1}, S_{2}^{\prime} \leftarrow S_{2}\)
        end if
    end for
end if
return \(S_{1}^{\prime}, S_{2}^{\prime}\)
```

We next present the complete algorithm for Semi-Restricted SSR (Algorithm 2) which simply returns the best among the two solutions obtained by solving the two cases. Algorithm 2 runs in time polynomial in $n$ and $Q$ (where $Q=\sum_{i=1}^{p} a_{i}$ ), therefore it is a pseudo-polynomial time algorithm. More precisely, by using appropriate data structures we can store the sets in the matrix cells in $O(1)$ time (and space) per cell, which implies that the time complexity of the algorithm is $O(n \cdot Q)$.

```
Algorithm 2 Exact solution for Semi-Restricted \(\operatorname{SSR}\left[\mathcal{S O} \mathcal{L}_{e x}(A, p)\right.\) function]
Input: a strictly sorted set \(A=\left\{a_{1}, \ldots, a_{n}\right\}, a_{i} \in \mathbb{Z}^{+}\), and an integer \(p, 1 \leq p<n\).
Output: the sets of an optimal solution of Semi-Restricted SSR.
    \(\left(S_{1}, S_{2}\right) \leftarrow \mathcal{S O} \mathcal{L}_{1}(A, p)\)
    \(\left(S_{1}^{\prime}, S_{2}^{\prime}\right) \leftarrow \mathcal{S O} \mathcal{L}_{2}(A, p)\)
    if \(\operatorname{MR}\left(S_{1}, S_{2}, A\right) \leq \mathcal{M R}\left(S_{1}^{\prime}, S_{2}^{\prime}, A\right)\) then
        return \(S_{1}, S_{2}\)
    else
            return \(S_{1}^{\prime}, S_{2}^{\prime}\)
    end if
```


## 4 Correctness of the Semi-Restricted SSR algorithm

In this section we will prove that Algorithm 2 solves exactly the Semi-Restricted SSR problem. Let $S_{1}^{*}, S_{2}^{*}$ be the sets of an optimal solution for input ( $A=$ $\left.\left\{a_{1}, \ldots, a_{n}\right\}, p\right)$.

Starting with the case 1 (where $\max S_{2}^{*} \in\left\{i \mid a_{i} \geq \sum_{j=1}^{p} a_{j}\right\}$ ), is easy to see that:

Observation 1. The sets $S_{1}^{*}=\{1, \ldots, p\}, S_{2}^{*}=\left\{\min \left\{i \mid a_{i} \geq \sum_{j=1}^{p} a_{j}\right\}\right\}$ give the optimal ratio.

Those are the sets which the function $\mathcal{S O} \mathcal{L}_{1}(A, p)$ returns.
For the case $2\left(\right.$ where $\left.\max S_{2}^{*} \in\left\{i \mid i>p, a_{i}<\sum_{j=1}^{p} a_{j}\right\}\right)$ we have to show that the cell $T[m, d]$ (where $d=\sum_{i \in S_{1}^{*}} a_{i}-\sum_{j \in S_{2}^{*}} a_{j}$ ) contains two sets $S_{1}, S_{2}$ with ratio equal to optimum. Before that we will show a lemma for the sums of the sets of the optimal solution.

Lemma 1. Let $Q=\sum_{i=1}^{p} a_{i}$ then we have $\sum_{i \in S_{1}^{*}} a_{i} \leq Q$ and $\sum_{i \in S_{2}^{*}} a_{i}<2 \cdot Q$.
Proof. Observe that max $S_{1}^{*}=p$. This gives us $\sum_{i \in S_{1}^{*}} a_{i} \leq \sum_{i=1}^{p} a_{i}$ so it remains to prove $\sum_{i \in S_{2}^{*}} a_{i}<2 \cdot Q$. Suppose that $\sum_{i \in S_{2}^{*}} a_{i} \geq 2 \cdot Q$. We can define the set $S_{2}$ as $S_{2}^{*} \backslash\left\{\min S_{2}^{*}\right\}$. Note that, for all $i \in S_{2}^{*}$, we have that the $a_{i}<\sum_{i=1}^{p} a_{i}$. Because of that,

$$
\sum_{i \in S_{1}^{*}} a_{i} \leq \sum_{i=1}^{p} a_{i}<\sum_{i \in S_{2}} a_{i}<\sum_{i \in S_{2}^{*}} a_{i}
$$

which means that the pair $\left(S_{1}^{*}, S_{2}\right)$ is a feasible solution with smaller max ratio than the optimal, which is a contradiction.

The next two lemmas describe same conditions which guarantee that the cells of $T$ are nonempty. Furthermore, they secure that we will store the appropriate sets to return an optimal solution.

Lemma 2. If there exist two disjoint sets $\left(S_{1}, S_{2}\right)$ such that
$-\max S_{2}<\max S_{1}=p$
$-\sum_{i \in S_{1}} a_{i}-\sum_{j \in S_{2}} a_{j}=d$
then $T[i, d] \neq \emptyset$ for all $p \geq i \geq \max \left(S_{1} \cup S_{2} \backslash\{p\}\right)$. Furthermore for the sets $\left(S_{1}^{\prime}, S_{2}^{\prime}\right)$ which are stored in $T[i, d]$ it holds that

$$
\sum_{i \in S_{1}^{\prime}} a_{i}+\sum_{j \in S_{2}^{\prime}} a_{j} \geq \sum_{i \in S_{1}} a_{i}+\sum_{j \in S_{2}} a_{j}
$$

Proof. Note that, for all pairs $\left(S_{1}, S_{2}\right)$ which meet the conditions, their sums are smaller than $Q$ because $\max \left(S_{1} \cup S_{2}\right)=p$ so for the value $d=\sum_{i \in S_{1}} a_{i}-$ $\sum_{j \in S_{2}} a_{j}$ we have

$$
-Q \leq d \leq Q
$$

The same clearly holds for every pair of subsets of $S_{1}, S_{2}$.
We will prove the lemma by induction on $q=\max \left(S_{1} \cup S_{2} \backslash\{p\}\right)$. For convenience if $S_{1} \cup S_{2} \backslash\{p\}=\emptyset$ we let $q=0$.

- $q=0$ (base case).

The only pair which meets the conditions for $q=0$ is the $(\{p\}, \emptyset)$. Observe that cell $T\left[0, a_{p}\right]$ is nonempty by the construction of the table and the same holds for $T\left[i, a_{p}\right], 1 \leq i \leq p$ (by line 12 . In this case the pair of sets which meets the conditions and the pair which is stored are exactly the same, so the lemma statement is obviously true.

- Assume that the lemma statement holds for $q=k \leq p-1$; we will prove it for $q=k+1$ as well.
Let $\left(S_{1}, S_{2}\right)$ be a pair of sets which meets the conditions. Either $q \in S_{1}$ or $q \in S_{2}$; therefore either $\left(S_{1} \backslash\{q\}, S_{2}\right)$ or $\left(S_{1}, S_{2} \backslash\{q\}\right)$ (respectively) meets the conditions. By the inductive hypothesis, we know that
- either $T\left[q-1, d-a_{q}\right]$ or $T\left[q-1, d+a_{q}\right]$ (resp.) is nonempty
- in any of the above cases for the stored pair $\left(S_{1}^{\prime}, S_{2}^{\prime}\right)$ it holds that: $\sum_{i \in S_{1}^{\prime}} a_{i}+\sum_{j \in S_{2}^{\prime}} a_{j} \geq \sum_{i \in S_{1}} a_{i}+\sum_{j \in S_{2}} a_{j}-a_{q}$
In particular, if $\left(S_{1} \backslash\{q\}, S_{2}\right)$ meets the conditions then $T\left[q-1, d-a_{q}\right]$ is nonempty. In line $13 q$ is added to the first set and therefore $T[q, d]$ is nonempty and the stored pair is $\left(S_{1}^{\prime} \cup\{q\}, S_{2}^{\prime}\right)$ (or some other with larger total sum). Hence, the total sum of the pair in $T[q, d]$ is at least

$$
\sum_{i \in S_{1}^{\prime}} a_{i}+\sum_{j \in S_{2}^{\prime}} a_{j}+a_{q} \geq \sum_{i \in S_{1}} a_{i}+\sum_{j \in S_{2}} a_{j}
$$

If on the other hand $\left(S_{1}, S_{2} \backslash\{q\}\right)$ is the pair that meets the conditions then $T\left[q-1, d+a_{q}\right]$ is nonempty. In line $14 q$ is added to the second set and therefore $T[q, d]$ is nonempty and the stored pair is $\left(S_{1}^{\prime}, S_{2}^{\prime} \cup\{q\}\right)$ (or other with larger total sum). Hence, the total sum of the pair in $T[q, d]$ is at least

$$
\sum_{i \in S_{1}^{\prime}} a_{i}+\sum_{j \in S_{2}^{\prime}} a_{j}+a_{q} \geq \sum_{i \in S_{1}} a_{i}+\sum_{j \in S_{2}} a_{j}
$$

The same holds for cells $T[i, d]$ with $q<i \leq p$ (due to line 12 ).
This concludes the proof.
A similar lemma can be proved for sets with maximum element index greater than $p$.

Lemma 3. If there exist two disjoint sets $\left(S_{1}, S_{2}\right)$ such that

$$
\begin{aligned}
& -\max S_{2}=q>p=\max S_{1} \\
& -\sum_{i \in S_{1}} a_{i} \leq Q, \sum_{j \in S_{2}} a_{j}<2 \cdot Q \\
& -\sum_{i \in S_{1}} a_{i}-\sum_{j \in S_{2}} a_{j}=d
\end{aligned}
$$

then $T[i, d] \neq \emptyset$ for all $i \geq q$. Furthermore for the sets $\left(S_{1}^{\prime}, S_{2}^{\prime}\right)$ which are stored in $T[i, d]$ it holds that

$$
\sum_{i \in S_{1}^{\prime}} a_{i}+\sum_{j \in S_{2}^{\prime}} a_{j} \geq \sum_{i \in S_{1}} a_{i}+\sum_{j \in S_{2}} a_{j}
$$

Proof. Note that, for all pairs $\left(S_{1}, S_{2}\right)$ which meet the conditions, the value $d=\sum_{i \in S_{1}} a_{i}-\sum_{j \in S_{2}} a_{j}$ it holds that

$$
-2 \cdot Q \leq d \leq Q
$$

The same clearly holds for every pair of subsets of $S_{1}, S_{2}$.
We will prove the lemma by induction. Let $\left(S_{1}, S_{2}\right)$ meet the conditions and $q=\max S_{2}$.

- $q=p+1$ (base case)

Clearly $\max S_{2}=p+1$ so the sets $\left(S_{1}, S_{2} \backslash\{p+1\}\right)$ meet the conditions of the Lemma 2 which gives us that
$-T\left[p, d+a_{p+1}\right]$ is nonempty

- for the stored pair $\left(S_{1}^{\prime}, S_{2}^{\prime}\right)$ it holds that:
$\sum_{i \in S_{1}^{\prime}} a_{i}+\sum_{j \in S_{2}^{\prime}} a_{j} \geq \sum_{i \in S_{1}} a_{i}+\sum_{j \in S_{2}} a_{j}-a_{p+1}$
Having the $T\left[p, d+a_{p+1}\right] \neq \emptyset$ the algorithm uses it in lines $31+34$ and adds $p+1$ to the second (stored) set so, we have that $T[p+1, d]$ is nonempty and the stored sets have total sum (at least):

$$
\sum_{i \in S_{1}^{\prime}} a_{i}+\sum_{j \in S_{2}^{\prime}} a_{j}+a_{p+1} \geq \sum_{i \in S_{1}} a_{i}+\sum_{j \in S_{2}} a_{j}
$$

Furthermore, because $T[p+1, d]$ is nonempty the above hold, additionally, for all $T[i, d], i>p+1$ (because the condition at line 23 is met, the algorithm fills those cells). The above conclude the base case.

- Assuming that the lemma statement holds for $q=k>p$, we will prove it for $q=k+1$.
Here we have to check two cases. Either $\max \left(S_{2} \backslash\{q\}\right)>p$ or not.
Case $1\left(\max \left(S_{2} \backslash\{q\}\right)>p\right)$. The pair of sets $\left(S_{1}, S_{2} \backslash\{q\}\right)$ meets the conditions; by the inductive hypothesis, we have
$-T\left[q-1, d-a_{q}\right]$ is nonempty
- for the stored pair $\left(S_{1}^{\prime}, S_{2}^{\prime}\right)$ it holds that:
$\sum_{i \in S_{1}^{\prime}} a_{i}+\sum_{j \in S_{2}^{\prime}} a_{j} \geq \sum_{i \in S_{1}} a_{i}+\sum_{j \in S_{2}} a_{j}-a_{q}$
Having the $T\left[q-1, d+a_{p+1}\right] \neq \emptyset$ the algorithm uses it in line 27 and adds $q$ to the second (stored) set so we have that $T[q, d]$ is nonempty and the stored sets have total sum (at least):

$$
\sum_{i \in S_{1}^{\prime}} a_{i}+\sum_{j \in S_{2}^{\prime}} a_{j}+a_{q} \geq \sum_{i \in S_{1}} a_{i}+\sum_{j \in S_{2}} a_{j}
$$

As before, the same holds for the cells $T[i, d]$ with $i>p+1$ because the condition at line 23 is met.
Case $2\left(\max \left(S_{2} \backslash\{q\}\right)<p\right)$. The sets $\left(S_{1}, S_{2} \backslash\{q\}\right)$ meets the conditions of the Lemma 2 (because $\max S_{1}=p$ ) which gives that
$-T\left[p, d+a_{q}\right]$ is nonempty

- for the stored pair $\left(S_{1}^{\prime}, S_{2}^{\prime}\right)$ it holds that:

$$
\sum_{i \in S_{1}^{\prime}} a_{i}+\sum_{j \in S_{2}^{\prime}} a_{j} \geq \sum_{i \in S_{1}} a_{i}+\sum_{j \in S_{2}} a_{j}-a_{q}
$$

Having $T\left[p, d+a_{q}\right] \neq \emptyset$ the algorithm uses it in lines 31.34 and adds $q$ to the second (stored) set so we have that $T[q, d]$ is nonempty and the stored sets have total sum (at least):

$$
\sum_{i \in S_{1}^{\prime}} a_{i}+\sum_{j \in S_{2}^{\prime}} a_{j}+a_{q} \geq \sum_{i \in S_{1}} a_{i}+\sum_{j \in S_{2}} a_{j}
$$

Furthermore, because $T[q, d]$ is nonempty the previous hold for all $T[i, d], i>$ $q \geq p+1$ (because the condition at line 23 is met).

Now we can prove that, in the second case, the pair of sets which the algorithm returns and the pair of sets of an optimal solution have the same ratio.

Lemma 4. If $\left(S_{1}^{\prime}, S_{2}^{\prime}\right)$ is the pair of sets that Algorithm 1 returns, then:

$$
\mathcal{M R}\left(S_{1}^{\prime}, S_{2}^{\prime}, A\right)=\mathcal{M \mathcal { R }}\left(S_{1}^{*}, S_{2}^{*}, A\right)
$$

Proof. Let $m$ be the size of the first dimension of the matrix $T$. Observe that for all $i, p+1 \leq i \leq m$, the sets $S_{1}, S_{2}$ of the nonempty cells $T[i, d]$ are constructed (lines 21.35 of Algorithm 1) such that $\max S_{1}=p$ and $i \geq \max S_{2}>p$. Therefore the pair $\left(S_{1}^{\prime}, S_{2}^{\prime}\right)$ returned by the algorithm is a feasible solution. We can see that the sets $S_{1}^{*}, S_{2}^{*}$ meet the conditions of Lemma 3 (the conditions for the sums are met because of Lemma 1) which give us that the cell $T[m, d]$ (where $\left.d=\sum_{i \in S_{1}^{*}} a_{i}-\sum_{j \in S_{2}^{*}} a_{j}\right)$ is non empty and contains two sets with total sum non less than $\sum_{i \in S_{1}^{*}} a_{i}+\sum_{j \in S_{2}^{*}} a_{j}$. Let $S_{1}, S_{2}$ be the sets which are stored to the cell $T[m, d]$. Then we have

$$
\begin{equation*}
\mathcal{M R}\left(S_{1}^{\prime}, S_{2}^{\prime}, A\right) \leq \mathcal{M R}\left(S_{1}, S_{2}, A\right) \leq \mathcal{M R}\left(S_{1}^{*}, S_{2}^{*}, A\right) \tag{1}
\end{equation*}
$$

where the second inequality is because

$$
\sum_{i \in S_{1}^{*}} a_{i}-\sum_{j \in S_{2}^{*}} a_{j}=\sum_{i \in S_{1}} a_{i}-\sum_{j \in S_{2}} a_{j}
$$

and

$$
\sum_{i \in S_{1}^{*}} a_{i}+\sum_{j \in S_{2}^{*}} a_{j} \leq \sum_{i \in S_{1}} a_{i}+\sum_{j \in S_{2}} a_{j} .
$$

By the Eq 1 and because the $S_{1}^{*}, S_{2}^{*}$ have the smallest Max Ratio we have

$$
\mathcal{M R}\left(S_{1}^{\prime}, S_{2}^{\prime}, A\right)=\mathcal{M \mathcal { R }}\left(S_{1}^{*}, S_{2}^{*}, A\right)
$$

Now, we can write the next theorem, which follows by the previous cases.
Theorem 1. Algorithm 2 returns an optimal solution for Semi-Restricted SSR.

## 5 FPTAS for Semi-Restricted SSR and SSR

Algorithm 2, which we presented at Section 3, is an exact pseudo-polynomial time algorithm for the Semi-Restricted SSR problem. In order to derivee a $(1+\varepsilon)$ approximation algorithm we will define a scaling parameter $\delta=\frac{\varepsilon \cdot a_{p}}{3 \cdot n}$ which we will use to make a new set $A^{\prime}=\left\{a_{1}^{\prime}, \ldots, a_{n}^{\prime}\right\}$ with $a_{i}^{\prime}=\left\lfloor\frac{a_{i}}{\delta}\right\rfloor$. The approximation algorithm solves the problem optimally on input $\left(A^{\prime}, p\right)$ and returns the sets of this exact solution. The ratio of those sets is a $(1+\varepsilon)$-approximation of the optimal ratio of the original input.

```
Algorithm 3 FPTAS for Semi-Restricted SSR [ \(\mathcal{S O} \mathcal{L}_{a p x}(A, p, \varepsilon)\) function]
Input: a strictly sorted set \(A=\left\{a_{1}, \ldots, a_{n}\right\}, a_{i} \in \mathbb{Z}^{+}\), an integer \(p, 1 \leq p<n\), and
    an error parameter \(\varepsilon \in(0,1)\).
Output: the sets of a \((1+\varepsilon)\)-approximation solution for Semi-Restricted SSR.
    \(\delta \leftarrow \frac{\varepsilon \cdot a_{p}}{3 \cdot n}\)
    \(A^{\prime} \leftarrow \emptyset\)
    for \(i \leftarrow 1\) to \(n\) do
        \(a_{i}^{\prime} \leftarrow\left\lfloor\frac{a_{i}}{\delta}\right\rfloor\)
        \(A^{\prime} \leftarrow A^{\prime} \cup\left\{a_{i}^{\prime}\right\}\)
    end for
    \(\left(S_{1}, S_{2}\right) \leftarrow \mathcal{S O} \mathcal{L}_{e x}\left(A^{\prime}, p\right)\)
    return \(S_{1}, S_{2}\)
```

Now, we will prove that the algorithm approximates the optimal solution by factor $(1+\varepsilon)$. Our proof follows closely the proof of Theorem 2 in (9].

Let $S_{A}, S_{B}$ be the pair of sets returned by Algorithm 3 on input $A=$ $\left\{a_{1}, \ldots, a_{n}\right\}, p$ and $\varepsilon$ and $\left(S_{1}^{*}, S_{2}^{*}\right)$ be an optimal solution to the problem.

Lemma 5. For any $S \in\left\{S_{A}, S_{B}, S_{1}^{*}, S_{2}^{*}\right\}$

$$
\begin{align*}
\sum_{i \in S} a_{i}-n \cdot \delta & \leq \sum_{i \in S} \delta \cdot a_{i}^{\prime} \leq \sum_{i \in S} a_{i}  \tag{2}\\
n \cdot \delta & \leq \frac{\varepsilon}{3} \cdot \sum_{i \in S} a_{i} \tag{3}
\end{align*}
$$

Proof. For Eq. (2) notice that for all $i \in\{1, \ldots, n\}$ we define $a_{i}^{\prime}=\left\lfloor\frac{a_{i}}{\delta}\right\rfloor$. This gives us

$$
\frac{a_{i}}{\delta}-1 \leq a_{i}^{\prime} \leq \frac{a_{i}}{\delta} \Rightarrow a_{i}-\delta \leq \delta \cdot a_{i} \leq a_{i}
$$

In addition, for any $S \in\left\{S_{A}, S_{B}, S_{1}^{*}, S_{2}^{*}\right\}$ we have $|S| \leq n$, which means that

$$
\sum_{i \in S} a_{i}-n \cdot \delta \leq \sum_{i \in S} \delta \cdot a_{i}^{\prime} \leq \sum_{i \in S} a_{i}
$$

For the Eq. (3) observe that max $S \geq p$ for any $S \in\left\{S_{A}, S_{B}, S_{1}^{*}, S_{2}^{*}\right\}$. By this observation, we can show the second inequality

$$
n \cdot \delta \leq n \cdot \frac{\varepsilon \cdot a_{p}}{3 \cdot n} \leq \frac{\varepsilon}{3} \cdot \sum_{i \in S} a_{i}
$$

Lemma 6. $\mathcal{M} \mathcal{R}\left(S_{A}, S_{B}, A\right) \leq \mathcal{M} \mathcal{R}\left(S_{A}, S_{B}, A^{\prime}\right)+\frac{\varepsilon}{3}$
Proof.

$$
\begin{aligned}
\mathcal{R}\left(S_{A}, S_{B}, A\right)=\frac{\sum_{i \in S_{A}} a_{i}}{\sum_{j \in S_{B}} a_{j}} & \leq \frac{\sum_{i \in S_{A}} \delta \cdot a_{i}^{\prime}+\delta \cdot n}{\sum_{j \in S_{B}} a_{j}} & & \text { [by Eq. (2)] } \\
& \leq \frac{\sum_{i \in S_{A}} a_{i}^{\prime}}{\sum_{j \in S_{B}} a_{j}^{\prime}}+\frac{\delta \cdot n}{\sum_{j \in S_{B}} a_{j}} & & \text { [by Eq. (2)] } \\
& \leq \mathcal{M R}\left(S_{A}, S_{B}, A^{\prime}\right)+\frac{\varepsilon}{3} & & \text { [by Eq. (3)] }
\end{aligned}
$$

The same way, we have

$$
\mathcal{R}\left(S_{B}, S_{A}, A\right) \leq \mathcal{M} \mathcal{R}\left(S_{A}, S_{B}, A^{\prime}\right)+\frac{\varepsilon}{3}
$$

thus the lemma holds.

Lemma 7. For any $\varepsilon \in(0,1), \mathcal{M} \mathcal{R}\left(S_{1}^{*}, S_{2}^{*}, A^{\prime}\right) \leq\left(1+\frac{\varepsilon}{2}\right) \cdot \mathcal{M} \mathcal{R}\left(S_{1}^{*}, S_{2}^{*}, A\right)$.

Proof. If $\mathcal{R}\left(S_{1}^{*}, S_{2}^{*}, A^{\prime}\right) \geq 1$, let $\left(S_{1}, S_{2}\right)=\left(S_{1}^{*}, S_{2}^{*}\right)$, otherwise $\left(S_{1}, S_{2}\right)=\left(S_{2}^{*}\right.$, $\left.S_{1}^{*}\right)$. That gives us

$$
\begin{aligned}
& \mathcal{M R}\left(S_{1}^{*}, S_{2}^{*}, A^{\prime}\right)=\mathcal{R}\left(S_{1}, S_{2}, A^{\prime}\right)=\frac{\sum_{i \in S_{1}} a_{i}^{\prime}}{\sum_{j \in S_{2}} a_{j}^{\prime}} \\
& \leq \frac{\sum_{i \in S_{1}} a_{i}}{\sum_{j \in S_{2}} a_{j}-n \cdot \delta} \\
&=\frac{\sum_{i \in S_{2}} a_{i}}{\sum_{j \in S_{2}} a_{j}-n \cdot \delta} \cdot \frac{\sum_{i \in S_{1}} a_{i}}{\sum_{j \in S_{2}} a_{j}} \\
&=\left(1+\frac{n \cdot \delta}{\sum_{j \in S_{2}} a_{j}-n \cdot \delta}\right) \cdot \frac{\sum_{i \in S_{1}} a_{i}}{\sum_{j \in S_{2}} a_{j}} .
\end{aligned}
$$

Because $S_{2} \in\left\{S_{1}^{*}, S_{2}^{*}\right\}$ by Eq. (3) it follows that

$$
\begin{aligned}
\mathcal{M R}\left(S_{1}^{*}, S_{2}^{*}, A^{\prime}\right) & \leq\left(1+\frac{1}{\frac{3}{\varepsilon}-1}\right) \cdot \frac{\sum_{i \in S_{1}} a_{i}}{\sum_{j \in S_{2}} a_{j}} \\
& =\left(1+\frac{\varepsilon}{3-\varepsilon}\right) \cdot \frac{\sum_{i \in S_{1}} a_{i}}{\sum_{j \in S_{2}} a_{j}} \\
& \left.\leq\left(1+\frac{\varepsilon}{2}\right) \cdot \frac{\sum_{i \in S_{1}} a_{i}}{\sum_{j \in S_{2}} a_{j}} \quad \text { [because } \varepsilon \in(0,1)\right] \\
& \leq\left(1+\frac{\varepsilon}{2}\right) \cdot \mathcal{M R}\left(S_{1}^{*}, S_{2}^{*}, A\right) .
\end{aligned}
$$

This concludes the proof.
Now we can prove that Algorithm 3 is a $(1+\varepsilon)$ approximation algorithm.
Theorem 2. Let $S_{A}, S_{B}$ be the pair of sets returned by Algorithm 3 on input $\left(A=\left\{a_{1}, \ldots, a_{n}\right\}, p, \varepsilon\right)$ and $S_{1}^{*}, S_{2}^{*}$ be an optimal solution, then:

$$
\mathcal{M \mathcal { R }}\left(S_{A}, S_{B}, A\right) \leq(1+\varepsilon) \cdot \mathcal{M} \mathcal{R}\left(S_{1}^{*}, S_{2}^{*}, A\right)
$$

Proof. The theorem follows from a sequence of inequalities:

$$
\begin{array}{rlrl}
\mathcal{M R}\left(S_{B}, S_{A}, A\right) & \leq \mathcal{M \mathcal { R }}\left(S_{A}, S_{B}, A^{\prime}\right)+\frac{\varepsilon}{3} & & {[\text { by Lemma } 6]} \\
& \leq \mathcal{M} \mathcal{R}\left(S_{1}^{*}, S_{2}^{*}, A^{\prime}\right)+\frac{\varepsilon}{3} & \\
& \leq\left(1+\frac{\varepsilon}{2}\right) \cdot \mathcal{M \mathcal { R }}\left(S_{1}^{*}, S_{2}^{*}, A\right)+\frac{\varepsilon}{3} & & \text { by Lemma } 7] \\
& \leq(1+\varepsilon) \cdot \mathcal{M \mathcal { R }}\left(S_{1}^{*}, S_{2}^{*}, A\right) &
\end{array}
$$

It remains to show that the complexity of Algorithm 3 is $\mathcal{O}(\operatorname{poly}(n, 1 / \varepsilon))$. Like we said at 3 the algorithm solves the Semi-Restricted SSR problem in $O(n \cdot Q)$
(where $Q=\sum_{i=1}^{p} a_{i}^{\prime}$ ). We have to bound the value of $Q$. By the definition of $a_{i}^{\prime}$ we have,

$$
Q=\sum_{i=1}^{p} a_{i}^{\prime} \leq n \cdot a_{p}^{\prime} \leq \frac{n \cdot a_{p}}{\delta}=\frac{3 \cdot n^{2}}{\varepsilon}
$$

which means that Algorithm 3 runs in $O\left(n^{3} / \varepsilon\right)$.
Clearly, it suffices to perform $n-1$ executions of the FPTAS for Semi-Restricted SSR (Algorithm 3), and pick the best of the returned solutions, in order to obtain an FPTAS for the (unrestricted) SSR problem. Therefore, we obtain the following.

Theorem 3. The above described algorithm is an FPTAS for SSR that runs in $O\left(n^{4} / \varepsilon\right)$ time.

## 6 Conclusion

In this paper we provide an FPTAS for the Subset-Sums Ratio (SSR) problem that is much faster than the best currently known scheme of Bazgan et al. [1]. There are two novel ideas that provide this improvement. The first comes from observing that in [9] the proof of correctness essentially relies only on the value of the smallest of the two maximum elements; this led to the idea to use only that information in order to solve the problem by defining and solving a new variation which we call Semi-Restricted SSR. A key ingredient in our approximation scheme is the use, in the scaling parameter $\delta$, of a value smaller than the sums of the sets of both optimal and approximate solutions (which in our case is the value of the smallest of the two maximum elements). We believe that this technique can be used in several other partition problems, e.g. such as those described in 810.

The second idea was to use one dimension only, for the difference of the sums of the two sets, instead of two dimensions, one for each sum. This idea, combined with the observation that between two pairs of sets with the same difference, the one with the largest total sum has ratio closer to 1 , is the key to obtain an optimal solution in much less time. It's interesting to see whether and how this technique could be used to problems that seek more than two subsets.

A natural open question is whether our techniques can be applied to obtain approximation results for other variations of the SSR problem 56.

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