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Semilinearity of Families of Languages*

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Techniques are developed for creating new and general language families of only semilinear languages, and for showing families only contain semilinear languages. It is shown that for language families \mathcal{L} that are semilinear full trios, the smallest full AFL containing \mathcal{L} that is also closed under intersection with languages in NCM (where NCM is the family of languages accepted by NFAs augmented with reversal-bounded counters), is also semilinear. If these closure properties are effective, this also immediately implies decidability of membership, emptiness, and infiniteness for these general families. From the general techniques, new grammar systems are given that are extensions of well-known families of semilinear full trios, whereby it is implied that these extensions must only describe semilinear languages. This also implies positive decidability properties for the new systems. Some characterizations of the new families are also given.

Keywords: semilinearity; closure properties; counter machines; pushdown automata; decidability.

1. Introduction

One-way nondeterministic reversal-bounded multicounter machines (NCM) operate like NFAs with λ transitions, where there are some number of stores that each can contain some non-negative integer. The transition function can detect whether each counter is zero or non-zero, and optionally increment or decrement each counter; however, there is a bound on the number of changes each counter can make between non-decreasing and non-increasing. These machines have been extensively studied in the literature, for example in [19], where it was shown that NCMs only accept

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semilinear languages (defined in Section 2). As the semilinear property is effective for NCM (in that, the proof consists of an algorithm for constructing a finite representation of the semilinear sets), this implies that NCMs have decidable membership, emptiness, and infiniteness properties, as emptiness and infiniteness can be decided easily on semilinear sets (and membership follows from emptiness by effective closure under intersection with regular languages). NCM machines have been applied extensively in the literature, for example, to model checking and verification [25, 20, 26, 21], often using the positive decidability properties of the family.

More general machine models have been studied with an unrestricted pushdown automaton augmented by some number of reversal-bounded counters (NPCM, [19]). Despite the unrestricted pushdown, the languages accepted are all semilinear, implying they have the same decidable properties. This family too has been applied to several verification problems [5, 22], including model checking recursive programs with numeric data types [12], synchronization- and reversal-bounded analysis of multithreaded programs [10], for showing decidable properties of models of integer-manipulating programs with recursive parallelism [11], and for decidability of problems on commutativity [23]. In these papers, the positive decidability properties — the result of the semilinearity — plus the use of the main store (the pushdown), plus the counters, played a key role. Hence, (effective) semilinearity is a crucial property for families of languages.

The ability to augment a machine model with reversal-bounded counters and to only accept semilinear languages is not unique to pushdown automata; in [13], it was found that many classes of machines \mathcal{M} accepting semilinear languages could be augmented with reversal-bounded counters, and the resulting family \mathcal{M}_c would also only accept semilinear languages. This includes models such as Turing machines with a one-way read-only input tape and a finite-crossing^a worktape. However, a precise formulation of which classes of machines this pertains to was not given.

Here, a precise formulation of families of languages that can be “augmented” with counters will be examined in terms of closure properties rather than machine models. This allows for application to families described by machine models, or grammatical models. It is shown that for any full trio (a family closed under homomorphism, inverse homomorphism, and intersection with regular languages) of semilinear languages \mathcal{L}_0 , then the smallest full AFL \mathcal{L} (a full trio also closed under union, concatenation, and Kleene-*) containing \mathcal{L}_0 that is closed under intersection with languages in NCM, must only contain semilinear languages. Furthermore, if the closure properties and semilinearity are effective in \mathcal{L}_0 , this implies a decidable membership, emptiness, and infiniteness problem in \mathcal{L} . Hence, this provides a new method for creating general families of languages with positive decidability properties.

Several specific models are created by adding counters. For example, indexed

^aA worktape is finite-crossing if there is a bound on the number of times the boundary of all neighboring cells on the worktape are crossed.

grammars are a well-studied general grammatical model like context-free grammars except where nonterminals keep stacks of “indices”. Although this system can generate non-semilinear languages, linear indexed grammars (indexed grammars with at most one nonterminal in the right hand side of every production) generate only semilinear languages [6]. Here, we define *linear indexed grammars with counters*, akin to linear indexed grammars, where every sentential form contains the usual sentential form, plus k counter values; each production operates as usual and can also optionally increase each counter by some amount; and a terminal word can be generated only if it can be produced with all counter values equal. It is shown that the family of languages generated must be semilinear since it is contained in the smallest full AFL containing the intersection of linear indexed languages and NCM languages. A characterization is also shown: linear indexed grammars with counters generate exactly those languages obtained by intersecting a linear indexed language with an NCM and then applying a homomorphism. Furthermore, it is shown that right linear indexed grammars (where terminals only appear to the left of nonterminals in productions) with counters coincide exactly with the machine model NPCM. Therefore, linear indexed grammars with counters are a natural generalization of NPCM containing only semilinear languages. This model is generalized once again as follows: an indexed grammar is uncontrolled finite-index if, there is a value k such that, for every derivation in the grammar, there are at most k occurrences of nonterminals in every sentential form. It is known that every uncontrolled finite-index indexed grammar generates only semilinear languages [3, 31]. It is shown here that uncontrolled finite-index indexed grammars with counters generate only semilinear languages, which is also a natural generalization of both linear indexed grammars with counters and NPCM. This immediately shows decidability of membership, emptiness, and infiniteness for this family.

Lastly, the closure property theoretic method of adding counters is found to often be more helpful than the machine model method of [13] in terms of determining whether the resulting family is semilinear, as here a machine model \mathcal{M} is constructed such that the language family accepted by \mathcal{M} is a semilinear full trio, but adding counters to the model to create \mathcal{M}_c accepts non-semilinear languages. This implies from our earlier results, that \mathcal{M}_c can accept languages that cannot be obtained from any accepted by \mathcal{M} by allowing any number of intersections with NCMs combined with any of the full AFL operations.

This paper therefore contains useful new techniques for creating new language families, and for showing existing language families only contain semilinear languages, which can then be used to immediately obtain decidable emptiness, membership, and infiniteness problems. Such families can perhaps also be applied to various areas, such as to verification, similarly to the use of NPCM. A preliminary version of this paper appeared in [18]. This version includes all missing proofs omitted due to space constraints, and the new Proposition 4 which allows for some of the other proposition statements to be more general. Section 5 is also new.

2. Preliminaries

In this section, preliminary background and notation is given.

Let \mathbb{N}_0 be the set of non-negative integers, and let \mathbb{N}_0^k be the set of all k -tuples of non-negative integers. A set $Q \subseteq \mathbb{N}_0^k$ is *linear* if there exists vectors $\vec{v}_0, \vec{v}_1, \dots, \vec{v}_l \in \mathbb{N}_0^k$ such that $Q = \{\vec{v}_0 + i_1\vec{v}_1 + \dots + i_l\vec{v}_l \mid i_1, \dots, i_l \in \mathbb{N}_0\}$. Here, \vec{v}_0 is called the *constant*, and $\vec{v}_1, \dots, \vec{v}_l$ are called the *periods*. A set Q is called *semilinear* if it is a finite union of linear sets.

Introductory knowledge of formal language and automata theory is assumed such as nondeterministic finite automata (NFAs), pushdown automata (NPDA's), Turing machines, and closure properties [16]. An *alphabet* Σ is a finite set of symbols, a *word* w over Σ is a finite sequence of symbols from Σ , and Σ^* is the set of all words over Σ which includes the empty word λ . A *language* L over Σ is any $L \subseteq \Sigma^*$. The *complement* of a language $L \subseteq \Sigma^*$, denoted by \bar{L} , is $\Sigma^* - L$.

Given a word $w \in \Sigma^*$, the length of w is denoted by $|w|$. For $a \in \Sigma$, the number of a 's in w is denoted by $|w|_a$. Given a word w over an alphabet $\Sigma = \{a_1, \dots, a_k\}$, the Parikh image of w is $\psi(w) = (|w|_{a_1}, \dots, |w|_{a_k})$, and the Parikh image of a language L is $\{\psi(w) \mid w \in L\}$. The commutative closure of a language L is the language $\text{comm}(L) = \{w \in \Sigma^* \mid \psi(w) = \psi(v), v \in L\}$. Two languages are *letter-equivalent* if $\psi(L_1) = \psi(L_2)$.

A language L is *semilinear* if $\psi(L)$ is a semilinear set. Equivalently, a language is semilinear if and only if it is letter-equivalent to some regular language [14]. A family of languages is semilinear if all languages in it are semilinear, and it is said to be *effectively semilinear* if there is an algorithm to construct the constant and periods for each linear set from a representation of each language in the family. For example, it is well-known that all context-free languages are effectively semilinear [28].

We will only define NCM and NPCM informally here, and refer to [19] for a formal definition. A one-way nondeterministic counter machine can be defined equivalently to a one-way nondeterministic pushdown automaton [16] with only a bottom-of-pushdown marker plus one other symbol. Hence, the machine can add to the counter (by pushing), subtract from the counter (by popping), and can detect emptiness and non-emptiness of the pushdown. A k -counter machine has k independent counters. A k -counter machine M is l -reversal-bounded, if M makes at most l changes between non-decreasing and non-increasing of each counter in every accepting computation. Let NCM be the class of one-way nondeterministic l -reversal-bounded k -counter machines, for some k, l (DCM for deterministic machines). Let NPCM be the class of machines with one unrestricted pushdown plus some number of reversal-bounded counters. By a slight abuse of notation, we also use these names for the family of languages they accept.

Notation from AFL (abstract families of languages) theory is used from [7]. A *full trio* is any family of languages closed under homomorphism, inverse homomorphism, and intersection with regular languages. Furthermore, a *full AFL* is a full trio closed

under union, concatenation, and Kleene-*. Given a language family \mathcal{L} , the smallest family containing \mathcal{L} that is closed under arbitrary homomorphism is denoted by $\hat{\mathcal{H}}(\mathcal{L})$, the smallest full trio containing \mathcal{L} is denoted by $\hat{\mathcal{T}}(\mathcal{L})$, and the smallest full AFL containing \mathcal{L} is denoted by $\hat{\mathcal{F}}(\mathcal{L})$. Given families \mathcal{L}_1 and \mathcal{L}_2 , let $\mathcal{L}_1 \wedge \mathcal{L}_2 = \{L_1 \cap L_2 \mid L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2\}$. We denote by $\hat{\mathcal{F}}_{\text{NCM}}(\mathcal{L})$ the smallest full AFL containing \mathcal{L} that is closed under intersection with languages from NCM.

3. Full AFLs Containing Counter Languages

This section will start by showing that for every semilinear full trio \mathcal{L} , the smallest full AFL containing \mathcal{L} that is also closed under intersection with NCM is a semilinear full AFL (Proposition 4). First, an intermediate result is required.

Proposition 1. *If \mathcal{L} is a semilinear full trio, then $\hat{\mathcal{T}}(\mathcal{L} \wedge \text{NCM}) = \hat{\mathcal{H}}(\mathcal{L} \wedge \text{NCM})$ is a semilinear full trio.*

Proof. Let $\mathcal{C} = \hat{\mathcal{T}}(\mathcal{L} \wedge \text{NCM})$, and let $\hat{L} \in \mathcal{C}$ over alphabet $\Gamma = \{d_1, \dots, d_s\}$. By definition, \mathcal{C} is a full trio. It will be shown that \hat{L} is semilinear. Then \hat{L} can be obtained from a language L in $\mathcal{L} \wedge \text{NCM}$ via a finite sequence of operations involving homomorphisms, inverse homomorphisms, and intersections with regular sets. Theorem 3.2.3 of [7] shows that for all non-empty languages L , $\hat{\mathcal{T}}(L) = \{g_2(g_1^{-1}(L) \cap R_1) \mid R_1 \text{ is regular, } g_1, g_2 \text{ are decreasing homomorphisms}\}$. A homomorphism g is decreasing if and only if $|g(a)| \leq 1$, for all letters a . Such homomorphisms are called *weak codings*. Hence, it is enough to consider that \hat{L} is obtained from L via the following sequence: an application of an inverse weak coding homomorphism g_1 , followed by an intersection with a regular language R_1 , followed by an application of a weak coding homomorphism g_2 . Thus, $\hat{L} = g_2(g_1^{-1}(L) \cap R_1)$. Since L is in $\mathcal{L} \wedge \text{NCM}$, there are $L_1 \in \mathcal{L}$ and $L_2 \in \text{NCM}$ such that $L = L_1 \cap L_2$. Let L_2 be accepted by a k -counter reversal-bounded NCM M_2 , where, without loss of generality, all counters are 1-reversal-bounded [19], all counters increase at least once, and all counters decrease to zero before accepting.

Let $\Sigma = \{a_1, \dots, a_n\}$ be the alphabet of $L_1 \cup L_2$, and so $L, L_1, L_2 \subseteq \Sigma^*$, g_1 is from $\bar{\Sigma}^*$ to Σ^* for some alphabet $\bar{\Sigma}$, and so $g_1^{-1}(L) \subseteq \bar{\Sigma}^*$, $R_1 \subseteq \bar{\Sigma}^*$, and g_2 is from $\bar{\Sigma}^*$ to Γ^* . Introduce new symbols $\Delta = \{C_1, D_1, \dots, C_k, D_k\}$ (k is the number of counters).

Let h_Σ be a homomorphism from $(\Sigma \cup \Delta)^*$ to Σ^* that fixes each letter of Σ and erases all letters of Δ , and let $h_{\bar{\Sigma}}$ be a homomorphism $(\bar{\Sigma} \cup \Delta)^*$ to $\bar{\Sigma}^*$ that fixes each letter of $\bar{\Sigma}$ and erases all letters of Δ . There exists $R_2 \subseteq (\Sigma \cup \Delta)^*$, a regular set, accepted by a nondeterministic finite automaton M'_2 that “encodes” the computation of M_2 (without doing the counting), as follows:

- M'_2 switches states as in M_2 ; M'_2 starts by simulating transitions on each counter being zero;
- every time M_2 adds to counter i , M'_2 instead reads the input letter C_i ; this

- is forced to happen at least once for each i , and after it reads the first C_i , it simulates transitions of M_2 where counter i is positive;
- every time M_2 subtracts from counter i , M'_2 reads D_i ; M'_2 verifies that at least one D_i is read, and that no C_i is read afterwards;
- for each i , $1 \leq i \leq k$, at some nondeterministically guessed spot after reading some D_i symbol, M'_2 guesses that counter i has hit zero, and it no longer reads any D_i symbol and simulates only transitions on counter i being zero;
- M'_2 must end in a final state.

Let $L' = h_{\Sigma}^{-1}(L_1) \cap R_2$ (here, $h_{\Sigma}^{-1}(L_1)$ has symbols of Δ “shuffled in” to L_1). Let L'' be those words in L' with the same number of C_i ’s as D_i ’s, for each i . Then it is evident that $h_{\Sigma}(L'') = L$ as the NFA M'_2 that accepts R_2 is behaving like M_2 but without counting; however the counting is occurring using the intersection in L'' , and then the counter symbols of Δ are erased with h_{Σ} . But looking only at L' , it must be that $L' \in \mathcal{L}$ since \mathcal{L} is a full trio. Let \bar{g}_1 be the extension of g_1 to be a homomorphism from $(\bar{\Sigma} \cup \Delta)^*$ to $(\Sigma \cup \Delta)^*$, where each letter of Δ is mapped to itself, and let \bar{g}_2 be the extension of g_2 to be a homomorphism from $(\bar{\Sigma} \cup \Delta)^*$ to $(\Gamma \cup \Delta)^*$, where each letter of Δ is mapped to itself. Let $R'_1 = h_{\Sigma}^{-1}(R_1)$; that is, it has symbols of Δ^* “shuffled in”. Note that R'_1 is regular since the regular languages are closed under inverse homomorphism.

Then $L''' = \bar{g}_2(\bar{g}_1^{-1}(L') \cap R'_1)$, which is also in \mathcal{L} , over $(\Gamma \cup \Delta)^*$. Note that since g_1 is a weak coding homomorphism, \bar{g}_1 is as well, and therefore \bar{g}_1^{-1} simply operates as g_1^{-1} does while fixing all letters of Δ . If we then take L''' and intersect it with all words where the number of C_i ’s is equal to the number of D_i ’s for each i , and then erase all C_i ’s and D_i ’s, we obtain \hat{L} . Let the Parikh image order the letters of this alphabet $d_1, \dots, d_s, C_1, D_1, \dots, C_k, D_k$. This Parikh image of L''' gives a set $Q''' \subseteq \mathbb{N}_0^{s+2k}$, which is semilinear since \mathcal{L} is semilinear. Let Q'''' be the set obtained from Q''' by enforcing that the number of C_i ’s is equal to D_i ’s for each i , which is also semilinear since the intersection of two semilinear sets is again semilinear [8]. Then \bar{Q} , the set obtained from Q'''' by projection on the first s coordinates, is also semilinear and this is the Parikh image of \hat{L} . Hence, \hat{L} is semilinear.

By definition, $\hat{\mathcal{H}}(\mathcal{L} \wedge \text{NCM}) \subseteq \hat{\mathcal{T}}(\mathcal{L} \wedge \text{NCM})$. To show $\hat{\mathcal{T}}(\mathcal{L} \wedge \text{NCM}) \subseteq \hat{\mathcal{H}}(\mathcal{L} \wedge \text{NCM})$, let $\hat{L} \in \hat{\mathcal{T}}(\mathcal{L} \wedge \text{NCM})$. Using the proof that $\hat{\mathcal{T}}(\mathcal{L} \wedge \text{NCM})$ is semilinear above, from $L''' \in \mathcal{L}$, it is possible to then intersect this language with an NCM that verifies that the number of C_i ’s is equal to the number of D_i ’s, for each i . And then, a homomorphism that erases elements of Δ can be applied to obtain \hat{L} . \square

The next result is relatively straightforward from results in [7, 9], however we have not seen it explicitly stated as we have done. From Corollary 2, Section 3.4 of [7], for any full trio \mathcal{L} , the smallest full AFL containing \mathcal{L} is the substitution of the regular languages into \mathcal{L} . And from [9], the substitution closure of one semilinear family into another is semilinear. Therefore, we obtain:

Lemma 2. *If \mathcal{L} is a semilinear full trio, then $\hat{\mathcal{F}}(\mathcal{L})$ is semilinear.*

For semilinear full trios \mathcal{L} , $\hat{\mathcal{T}}(\mathcal{L} \wedge \text{NCM})$ is a semilinear full trio by Proposition 1, and starting with this family and applying Lemma 2, the smallest full AFL containing intersections of languages in \mathcal{L} with **NCM** is semilinear.

Proposition 3. *If \mathcal{L} is a semilinear full trio, then $\hat{\mathcal{F}}(\mathcal{L} \wedge \text{NCM})$ is semilinear.*

It is worth noting that this procedure can be iterated, as therefore $\hat{\mathcal{F}}(\hat{\mathcal{F}}(\mathcal{L} \wedge \text{NCM}) \wedge \text{NCM})$ must also be a semilinear full AFL, etc. for additional levels. However, it is an interesting open question as to whether there is a strict hierarchy with respect to this iteration.

One could also consider the smallest full AFL containing \mathcal{L} that is closed under intersection with **NCM**. Here, the intersections with **NCM** can occur arbitrarily many times, even after or in between applying the other full AFL operations.

Proposition 4. *If \mathcal{L} is a semilinear full trio, then $\hat{\mathcal{F}}_{\text{NCM}}(\mathcal{L})$ is semilinear.*

Proof. Let $\hat{L} \in \hat{\mathcal{F}}_{\text{NCM}}(\mathcal{L})$. Then \hat{L} is obtained from some language $L \in \mathcal{L}$ via some sequence of the full AFL operations, plus some number, n say, of intersections with **NCMs**. Hence,

$$\hat{L} \in \mathcal{C} = \overbrace{\hat{\mathcal{F}}(\hat{\mathcal{F}}(\cdots \hat{\mathcal{F}}(\mathcal{L} \wedge \text{NCM}) \wedge \text{NCM}) \wedge \cdots \wedge \text{NCM})}^n.$$

By iterating Proposition 3 n times, \mathcal{C} is semilinear, hence \hat{L} is semilinear. \square

In contrast, it is shown in [7] that $\hat{\mathcal{H}}(\hat{\mathcal{F}}(\{a^n b^n \mid n > 0\}) \wedge \hat{\mathcal{F}}(\{a^n b^n \mid n > 0\}))$ is equal to the family of recursively enumerable languages. Therefore, $\hat{\mathcal{H}}(\hat{\mathcal{F}}(\text{NCM}) \wedge \hat{\mathcal{F}}(\text{NCM}))$ is also equal to the family of recursively enumerable languages (which is not semilinear). But in Proposition 4, only intersections with languages in **NCMs** are allowed, and not intersections with languages in $\hat{\mathcal{F}}(\text{NCM})$, thereby creating the large difference.

Many acceptor and grammar systems are known to be semilinear full trios, such as finite-index **ETOL** systems [29], indexed grammars with a bound on the number of variables appearing in every sentential form (called uncontrolled finite-index) [3], multi-push-down machines (which have k pushdowns that can simultaneously be written to, but they can only pop from the first non-empty pushdown) [2], a Turing machine variant with one finite-crossing worktape [13], and pushdown machines that can flip their pushdown up to k times [15].

Corollary 5. *Let \mathcal{L} be any of the following families:*

- languages generated by context-free grammars,
- languages generated by finite-index **ETOL**,
- languages generated by uncontrolled finite-index indexed languages,
- languages accepted by one-way multi-push-down machine languages,

- languages accepted by one-way read-only input nondeterministic Turing machines with a two-way finite-crossing read/write worktape,
- languages accepted by one-way k -flip pushdown automata.

Then $\hat{\mathcal{F}}_{\text{NCM}}(\mathcal{L})$ is a semilinear full AFL.

A simplified analogue to this result is known for certain types of machines [13], although the new result here is defined entirely using closure properties rather than machines. Furthermore, the results in [13] do not allow Kleene-* type closure as part of the full AFL properties. For the machine models \mathcal{T} above, it is an easy exercise to show that augmenting them with reversal-bounded counters to produce \mathcal{T}_c , the languages accepted by \mathcal{T}_c are a subset of the smallest full AFL closed under intersection with NCM containing languages in \mathcal{T} . Hence, these models augmented by counters only accept semilinear languages. Similarly, this type of technique also works for grammar systems, as we will see in Section 6.

In addition, in [9], it was shown that if \mathcal{L} is a semilinear family, then the smallest AFL containing the commutative closure of languages in \mathcal{L} is a semilinear AFL. It is known that the commutative closure of every semilinear language is in NCM [23], and we know now that if we have a semilinear full trio \mathcal{L} , then the smallest full AFL containing \mathcal{L} is also semilinear. So, we obtain an alternate proof that is an immediate corollary since we know that the smallest full AFL containing NCM is a semilinear full AFL.

For any semilinear full trio \mathcal{L} where the semilinearity and the intersection with regular language properties are effective, the membership and emptiness problems in \mathcal{L} are decidable. Indeed, to decide emptiness, it suffices to check if the semilinear set is empty. And to decide if a word w is in L , one constructs the language $L \cap \{w\}$, then emptiness is decided.

Corollary 6. *For any semilinear full trio \mathcal{L} where the semilinearity and intersection with regular language properties are effective, then the membership, emptiness, and infiniteness problems are decidable for languages in $\hat{\mathcal{F}}_{\text{NCM}}(\mathcal{L})$. In these cases, $\hat{\mathcal{F}}_{\text{NCM}}(\mathcal{L})$ are a proper subset of the recursive languages.*

As membership is decidable, the family must only contain recursive languages, and the inclusion must be strict as the recursive languages are not closed under homomorphism.

As another consequence, we provide an interesting decomposition theorem of semilinear languages into linear parts. Consider any semilinear language L , where its Parikh image is a finite union of linear sets A_1, \dots, A_k , and the constant and periods for each linear set can be constructed. Then we can effectively create languages in perhaps another semilinear full trio separately accepting those words in $L_i = \{w \in L \mid \psi(w) \in A_i\}$, for each $1 \leq i \leq k$.

Proposition 7. *Let \mathcal{L} be a semilinear full trio, where semilinearity is effective. Given $L \in \mathcal{L}$, we can determine representations of disjoint simple sets (ie. disjoint*

linear sets where the periods form a basis) A_1, \dots, A_k such that the Parikh image of L is $A = A_1 \cup \dots \cup A_k$, and $L_i = \{w \in L \mid \psi(w) \in A_i\} \in \hat{\mathcal{F}}_{\text{NCM}}(\mathcal{L})$, for $1 \leq i \leq k$.

Proof. Since semilinearity is effective, we can construct a representation of linear sets A_1, \dots, A_k . Moreover, it is known that given any set of constants and periods generating a semilinear set Q , it is possible to effectively construct another set of constants and periods that forms a disjoint finite union of simple sets also generating Q [4, 30]. Therefore, we can assume A_1, \dots, A_k are of this form. An NCM M_i can be created to accept $\psi^{-1}(A_i)$, for each i , $1 \leq i \leq k$ as follows: if $L \subseteq \{a_1, \dots, a_n\}^*$, then M_i has n counters. If (x_1, \dots, x_n) is the constant of A_i , then M_i adds x_j to counter j for each j . Then, for each period, (y_1, \dots, y_n) , M_i nondeterministically guesses some number c and adds cy_j to counter j for each j . At this point, the counters can contain any value from A_i . From here, for every a_j read as input, M_i subtracts one from counter j , and accepts at the end of the input if all counters are empty. Hence, $L_i = L \cap L(M_i) \in \hat{\mathcal{F}}_{\text{NCM}}(\mathcal{L})$, for each i , $1 \leq i \leq k$. \square

Therefore, by moving to a more general full trio (contained in the recursive languages), it is possible to decompose a language into separate (disjoint) languages such that each has one of the linear sets as its Parikh image.

4. Application to General Multi-Store Machine Models

In [7], a generalized type of multitape automata was studied, called multitape abstract families of automata (multitape AFAs). We will not define the notation used there, but in Theorem 4.6.1 (and Exercise 4.6.3), it is shown that if we have two types of automata \mathcal{M}_1 and \mathcal{M}_2 (defined using the AFA formalism), accepting language families \mathcal{L}_1 and \mathcal{L}_2 respectively, then the languages accepted by automata combining together the stores of \mathcal{M}_1 and \mathcal{M}_2 , accepts exactly the family $\hat{\mathcal{H}}(\mathcal{L}_1 \wedge \mathcal{L}_2)$. This is shown for machines accepting full AFLs in Theorem 4.6.1 of [7], and for union-closed full trios mentioned in Exercise 4.6.3. We will show that this is tightly coupled with this precise definition of AFAs, as we will define two simple types of automata where each on their own accept a semilinear family, but combining the two stores together to form one multitape model accepts non-semilinear languages.

Given a family of one-way acceptors \mathcal{M} , let \mathcal{M}_c be those acceptors augmented by reversal-bounded counters. A checking stack automaton (NCSA) M is a one-way NFA with a store tape, called a stack. At each move, M pushes a string (possibly λ) on the stack, but M cannot pop. And, M can enter and read from the inside of the stack in a two-way read-only fashion. But once the machine enters the stack, it can no longer change the contents. The checking stack automaton is said to be *restricted* (or *no-read* using the terminology of [24]), if it does not read from the inside of the stack until the end of the input. We denote by RNCSA the family of machines, as well as the family of languages described by the machines above. A preliminary investigation of RNCSA_c was done in [24].

Here, we will show the following:

- (1) RNCSA is a full trio of semilinear languages equal to the regular languages,
- (2) $\hat{\mathcal{F}}(\text{RNCSA} \wedge \text{NCM})$ and $\hat{\mathcal{F}}_{\text{NCM}}(\text{RNCSA})$ are semilinear full AFLs,
- (3) every language in $\text{RNCSA} \wedge \text{NCM}$ is accepted by some machine in RNCSA_c ,
- (4) there are non-semilinear languages accepted by machines in RNCSA_c .

Therefore, RNCSA_c contains some languages not in the smallest full AFL containing RNCSA closed under intersection with NCM, and the multitape automata and results from [7] and [13] do not apply to this type of automata.

Proposition 8. *RNCSA accepts exactly the regular languages, which is a full trio of semilinear languages.*

Proof. It is clear that all regular languages are in RNCSA. For the other direction, take an RNCSA machine M , and assume without loss of generality that the input alphabet Σ and the stack alphabet Γ are disjoint. Construct a two-way NFA (2NFA) M' over $(\Sigma \cup \Gamma)^*$ whose input is divided into segments $u_1v_1 \cdots u_nv_n$, where $u_i \in (\Sigma \cup \{\lambda\})$ and $v_i \in \Gamma^*$. M' simulates M by first verifying that M , when reading u_i , writes v_i on the stack. When M' simulates the two-way read-only phase of M (which only occurs in M after reaching the end of the input), it does so by using the two-way NFA and skipping over the segments of Σ . Since this language accepted by the 2NFA M' is regular, the language obtained by erasing all letters of Γ via homomorphism is also regular, which is exactly $L(M)$. \square

From Proposition 3, the following is true:

Corollary 9. $\hat{\mathcal{F}}(\text{RNCSA} \wedge \text{NCM}) = \hat{\mathcal{F}}_{\text{NCM}}(\text{RNCSA}) = \hat{\mathcal{F}}(\text{NCM})$ is a semilinear full AFL.

Since RNCSA is equal to the family of regular languages, and NCM is closed under intersection with regular languages, the following is true:

Proposition 10. $\hat{\mathcal{F}}_{\text{NCM}}(\text{RNCSA}) = \text{NCM} \subsetneq \text{RNCSA}_c$. Furthermore, the latter family contains non-semilinear languages.

Proof. Containment is immediate since RNCSA_c has reversal-bounded counters. The non-semilinear $L = \{a^ib^j \mid i, j \geq 1, j \text{ is divisible by } i\}$ can be accepted by an RDSCA_c M with one counter that makes only one reversal. M , on input x checks that $x = a^ib^j$ for some $i, j \geq 1$, copies a^i onto the stack, and increments the counter to j . Then M makes multiple left-to-right and right-to-left sweeps on a^i with the stack while in parallel decrementing the counter to check that j is divisible by i . \square

It is concluded that RNCSA_c contains some languages not in $\hat{\mathcal{F}}_{\text{NCM}}(\text{RNCSA}) = \text{NCM}$, since NCM is semilinear. Then it is clear that combining together the stores of RNCSA and NCM accepts significantly more than $\hat{\mathcal{H}}(\text{RNCSA} \wedge \text{NCM})$ as is the case for multitape AFA [7]. The reason for the discrepancy between this result and Ginsburg's result is that the definition of multitape AFA allows for reading the

input while performing instructions (like operating in two-way read-only mode in the stack). In contrast, RNCSA does not allow this behavior. And if this behavior is added into the definition, the full capability of checking stack automata is achieved which accepts non-semilinear languages.

A similar analysis can be done using the method developed in [13] for augmenting the machine models with counters. Let \mathcal{M} be a family of one-way acceptors with some type of store structure X . For example, if the storage X is a pushdown stack, then \mathcal{M} is the family of nondeterministic pushdown automata (NPDAs). In [13], the following was shown for many families \mathcal{M} :

- (*) If \mathcal{M} is a semilinear family (i.e, the languages accepted by the machines in \mathcal{M} have a semilinear Parikh image), then \mathcal{M}_c is also a semilinear family.

It was not clear in [13] whether the result above is true for all types of one-way acceptors, in general or for which types (*) holds. However, the family RNCSA (equal to the regular languages) is semilinear (Proposition 8), but RDCSA_c is not semilinear (Proposition 10).

5. Properties of semilinear language families

This section investigates certain properties of semilinear language families.

Definition 11. *Given a language family \mathcal{L} , define the following families:*

$$\begin{aligned}\overline{\mathcal{L}} &= \{\overline{L} \mid L \in \mathcal{L}\}, \\ \mathcal{L}_D &= \{L_1 - L_2 \mid L_1, L_2 \in \mathcal{L}\}, \\ \mathcal{L}_\cup &= \{L_1 \cup L_2 \mid L_1, L_2 \in \mathcal{L}\}, \\ \mathcal{L}_\cap &= \{L_1 \cap L_2 \mid L_1, L_2 \in \mathcal{L}\}, \\ \mathcal{L}\mathcal{L} &= \{L_1 L_2 \mid L_1, L_2 \in \mathcal{L}\}, \\ \mathcal{L}^* &= \{L^* \mid L \in \mathcal{L}\}, \\ \mathcal{L}_{RQ} &= \{L_1 L_2^{-1} \mid L_1, L_2 \in \mathcal{L}\}, \text{ (right quotient)}, \\ \mathcal{L}_{LQ} &= \{L_1^{-1} L_2 \mid L_1, L_2 \in \mathcal{L}\}, \text{ (left quotient)}, \\ \mathcal{H}(\mathcal{L}) &= \{h(L) \mid L \in \mathcal{L}, h \text{ a homomorphism}\}, \\ \mathcal{H}^{-1}(\mathcal{L}) &= \{h^{-1}(L) \mid L \in \mathcal{L}, h \text{ a homomorphism}\}.\end{aligned}$$

If \mathcal{L} is semilinear, an interesting question is whether the defined families above must also be semilinear. In [9], it is shown that the substitution of one semilinear family into another is again semilinear. This immediately implies that if \mathcal{L} is a semilinear family, then all of \mathcal{L}^* , \mathcal{L}_\cup , $\mathcal{L}\mathcal{L}$, and $\mathcal{H}(\mathcal{L})$ are also semilinear. For the remaining properties, we have not seen proofs in the literature, and therefore include short proofs here.

Proposition 12. *If \mathcal{L} is semilinear, then all of the following need not be semilinear: $\overline{\mathcal{L}}$, \mathcal{L}_D , \mathcal{L}_{RQ} , \mathcal{L}_{LQ} , \mathcal{L}_\cap , $\mathcal{H}^{-1}(\mathcal{L})$.*

Proof. First, it will be shown for $\overline{\mathcal{L}}$. Let $L = \{a^1 \# a^2 \# \cdots \# a^k \# \mid k \geq 1\}$ where a is a letter. Then the complement of L , \overline{L} can easily be accepted by an NCM with one 1-reversal counter which, when given an input w , nondeterministically selects (1) or (2) below:

- (1) accepts, if w is not in a valid format, i.e., not of the form $(a^+ \#)^+$. (M does not need the counter.)
- (2) accepts w if it is of the form $a^{i_1} \# \cdots \# a^{i_k}$ but $i_r + 1 \neq i_{r+1}$ for some r . (M uses a 1-reversal counter.)

Since all NCM languages are semilinear, \overline{L} is semilinear, but L is not semilinear (if it were semilinear, then projecting onto a would be semilinear, but all unary semilinear languages are regular [14] and this language is not regular by the pumping lemma). This also implies non-closure for \mathcal{L}_D .

Next, for right quotient, it is known that there is a non-recursively enumerable unary language $L \subseteq a^*$ (that is not semilinear) [27]. Let $L' = cLd \cup da^*c$. Then L' is semilinear since it has the same Parikh image as the regular language da^*c . But the right quotient of L' with d is cL , which is not semilinear.

Similarly, the left quotient of L' with c is Ld , which is not semilinear. The result for intersection is also similar.

For inverse homomorphism, take a homomorphism h that maps b to ca , e to a , and f to d , and g to ac . Then $h^{-1}(L') = bL''f \cup fa^*g$ where $L'' = La^{-1}$. The language $h^{-1}(L')$ is clearly not semilinear. \square

In contrast, it can be seen that for inverse homomorphisms where the homomorphisms are *weak codings* (that is, $|h(a)| \leq 1$ for all $a \in \Sigma$), then the resulting family is semilinear, as inverse homomorphisms act just as substitutions (as mentioned, the substitution closure of a semilinear family is semilinear) with the additional arbitrary insertion of characters erased by h (which can be added in by placing another period in each linear set of the semilinear set with all 0's except for a 1 for the position of the character erased by the homomorphism).

These closure properties will motivate the next notion that can help define “well-behaved” semilinear languages.

Definition 13. A semilinear language L is well-behaved if $\hat{\mathcal{T}}(L)$ is semilinear; that is, it is well-behaved if closing it under all full trio operations only give semilinear languages.

Some basic facts are in order.

Proposition 14. The following are true:

- if $L \in \mathcal{L}$, a semilinear full trio, then L is well-behaved,
- not all semilinear languages are well-behaved.

Proof. The first property is immediate since $\hat{\mathcal{T}}(L) \subseteq \mathcal{L}$.

Consider a non-recursively enumerable unary language $L \subseteq a^*$. Therefore, this language is not semilinear (all unary semilinear languages are regular). Consider $L' = bLc \cup ca^*b$. Then L' is semilinear since it has the same Parikh image as the regular language ca^*b . But $L' \cap ba^*c = bLc$, which is not semilinear. Thus, the closure of L' by intersection with regular languages gives non-semilinear languages. \square

Consider the following language family:

$$\mathcal{L}_{WB} = \{L \mid L \text{ is well-behaved}\}.$$

Proposition 15. \mathcal{L}_{WB} is the largest semilinear full trio. That is, all semilinear full trios are contained in \mathcal{L}_{WB} .

Proof. First, it is a semilinear full trio since all languages in it are semilinear, and the closure of each under the full trio properties are in it.

Furthermore, it is the largest since any language not in it must either not be semilinear, or closing it under the full trio operations produces languages that are not semilinear. \square

This is similar to the known result that there is a largest semilinear AFL [9]. This is an interesting language family, as properties that hold for this single language family hold for all semilinear full trios.

For example, consider the following. A bounded language $L \subseteq w_1^* \cdots w_k^*$ is called *bounded Ginsburg semilinear* (often just called bounded semilinear in the literature) if the set $\{(i_1, \dots, i_k) \mid w_1^{i_1} \cdots w_k^{i_k} \in L\}$ is a semilinear set. The following is true from [17]:

Proposition 16. All bounded languages in \mathcal{L}_{WB} are bounded Ginsburg semilinear languages.

Next, it follows from Theorem 3.2.3 of [7], that for all non-empty languages L , $\hat{\mathcal{T}}(L) = \{h_2(h_1^{-1}(L) \cap R) \mid R \text{ is regular}, h_1, h_2 \text{ are decreasing homomorphisms}\}$. The homomorphisms are both weak codings. Also, semilinear languages are closed under homomorphisms. Hence, the following is true:

Proposition 17. L is well-behaved if and only if the family

$$\{h^{-1}(L) \cap R \mid R \text{ is regular}, h \text{ is a weak coding homomorphism}\}$$

are all semilinear.

It is evident that $h^{-1}(L)$ must be semilinear since it was previously noted that the family obtained from any semilinear family via inverse weak coding homomorphisms must be semilinear. Hence, if one examines the family of semilinear languages $\mathcal{L} = \{h^{-1}(L) \mid h \text{ is a weak coding homomorphism}\}$, then L is well-behaved if and only if $\mathcal{L} \wedge \mathcal{L}(\text{NFA})$.

6. Applications to Indexed Grammars with Counters

In this section, we describe some new types of grammars obtained from existing grammars generating a semilinear language family \mathcal{L} , by adding counters. The languages generated by these new grammars are then shown to be contained in $\hat{\mathcal{F}}(\mathcal{L} \wedge \text{NCM})$, and by an application of Proposition 3, are all semilinear with positive decidability properties.

We need the definition of an indexed grammar introduced in [1] by following the notation of [16], Section 14.3.

Definition 18. An indexed grammar is a 5-tuple $G = (V, \Sigma, I, P, S)$, where V, Σ, I are finite pairwise disjoint sets: the set of nonterminals, terminals, and indices, respectively, S is the start nonterminal, and P is a finite set of productions, each of the form either

$$1) A \rightarrow \nu, \quad 2) A \rightarrow Bf, \quad \text{or} \quad 3) Af \rightarrow \nu,$$

where $A, B \in V$, $f \in I$ and $\nu \in (V \cup \Sigma)^*$.

Let ν be an arbitrary sentential form of G , which is of the form

$$\nu = u_1 A_1 \alpha_1 u_2 A_2 \alpha_2 \cdots u_k A_k \alpha_k u_{k+1},$$

where $A_i \in V, \alpha_i \in I^*, u_i \in \Sigma^*, 1 \leq i \leq k, u_{k+1} \in \Sigma^*$. For a sentential form $\nu' \in (VI^* \cup \Sigma)^*$, we write $\nu \Rightarrow_G \nu'$ if one of the following three conditions holds:

- (1) There exists a production in P of the form (1) $A \rightarrow w_1 C_1 \cdots w_\ell C_\ell w_{\ell+1}$, $C_j \in V, w_j \in \Sigma^*$, and there exists i with $1 \leq i \leq k$, $A_i = A$ and

$$\nu' = u_1 A_1 \alpha_1 \cdots u_i (w_1 C_1 \alpha_i \cdots w_\ell C_\ell \alpha_i w_{\ell+1}) u_{i+1} A_{i+1} \alpha_{i+1} \cdots u_k A_k \alpha_k u_{k+1}.$$
- (2) There exists a production in P of the form (2) $A \rightarrow Bf$ and there exists i , $1 \leq i \leq k$, $A_i = A$ and $\nu' = u_1 A_1 \alpha_1 \cdots u_i (Bf \alpha_i) u_{i+1} A_{i+1} \alpha_{i+1} \cdots u_k A_k \alpha_k u_{k+1}$.
- (3) There exists a production in P of the form (3) $Af \rightarrow w_1 C_1 \cdots w_\ell C_\ell w_{\ell+1}$, $C_j \in V, w_j \in \Sigma^*$, and an i , $1 \leq i \leq k$, $A_i = A$, $\alpha_i = f \alpha'_i$, $\alpha'_i \in I^*$, with

$$\nu' = u_1 A_1 \alpha_1 \cdots u_i (w_1 C_1 \alpha'_i \cdots w_\ell C_\ell \alpha'_i w_{\ell+1}) u_{i+1} A_{i+1} \alpha_{i+1} \cdots u_k A_k \alpha_k u_{k+1}.$$

Then, \Rightarrow_G^* denotes the reflexive and transitive closure of \Rightarrow_G . The language $L(G)$ generated by G is the set $L(G) = \{u \in \Sigma^* \mid S \Rightarrow_G^* u\}$.

This type of grammar can be generalized to include monotonic counters as follows:

Definition 19. An indexed grammar with k counters is defined as in indexed grammars, except where rules (1), (2), (3) above are modified so that a rule $\alpha \rightarrow \beta$ now becomes:

$$\alpha \rightarrow (\beta, c_1, \dots, c_k), \tag{1}$$

where $c_i \geq 0$, $1 \leq i \leq k$. Sentential forms are of the form (ν, n_1, \dots, n_k) , and \Rightarrow_G operates as do indexed grammars on ν , and for a production in Equation 1, adds

c_i to n_i , for $1 \leq i \leq k$. The language generated by G with terminal alphabet Σ and start nonterminal S is, $L(G) = \{u \mid u \in \Sigma^*, (S, 0, \dots, 0) \Rightarrow_G^* (u, n_1, \dots, n_k), n_1 = \dots = n_k\}$.

Given an indexed grammar with counters, the underlying grammar is the indexed grammar obtained by removing the counter components from productions.

Although indexed grammars generate non-semilinear languages, restrictions will be studied that only generate semilinear languages.

An indexed grammar G is *linear* [6] if the right side of every production of G has at most one variable. Furthermore, G is *right linear* if it is linear, and terminals can only appear to the left of a nonterminal in productions. Let L-IND be the family of languages generated by linear indexed grammars, and let RL-IND be the family of languages generated by right linear indexed grammars.

Similarly, indexed grammars with counters can be restricted to be linear. An indexed grammar with k -counters is said to be *linear indexed* (resp. *right linear*) with k counters, if the underlying grammar is linear (resp. right linear). Let L-IND_c (resp. RL-IND_c) be the family of languages generated by linear (resp. right linear) indexed grammars with counters.

Example 20. Consider the language $L = \{v\$w \mid v, w \in \{a, b, c\}^*, |v|_a = |v|_b = |v|_c, |w|_a = |w|_b = |w|_c\}$ which can be generated by a linear indexed grammar with counters $G = (V, \Sigma, I, P, S)$ where P contains

$$\begin{aligned} S &\rightarrow (S, 1, 1, 1, 0, 0, 0) \mid (S, 0, 0, 0, 1, 1, 1) \mid (T, 0, 0, 0, 0, 0, 0) \\ T &\rightarrow (aT, 1, 0, 0, 0, 0, 0) \mid (bT, 0, 1, 0, 0, 0, 0) \mid (cT, 0, 0, 1, 0, 0, 0) \mid (\$R, 0, 0, 0, 0, 0, 0) \\ R &\rightarrow (aR, 0, 0, 0, 1, 0, 0) \mid (bR, 0, 0, 0, 0, 1, 0) \mid (cR, 0, 0, 0, 0, 0, 1) \mid (\lambda, 0, 0, 0, 0, 0, 0). \end{aligned}$$

This language cannot be generated by a linear indexed grammar [3].

Next, a characterization of languages generated by these grammars will be given with a sequence of results used towards the proof of Proposition 25.

In the following, Σ is a terminal alphabet, $C = \{c_1, \dots, c_k\}$ (for some $k \geq 1$) is an alphabet distinct from Σ , and h_c is a homomorphism on $\Sigma \cup C$ defined by $h_c(a) = a$ for each a in Σ , and $h_c(c_i) = \lambda$ for each c_i in C .

Lemma 21. If L is in NCM (resp., NPCM), there is regular language (resp., NPDA) R over the alphabet $\Sigma \cup C$ such that $L = h_c(\{w \mid w \in R, |w|_{c_1} = \dots = |w|_{c_k}\})$.

Proof. Let $L \subseteq \Sigma^*$ be accepted by an NCM (resp., NPCM) with n 1-reversal counters. Let $C = \{b_1, c_1, \dots, b_n, c_n\}$ be an alphabet distinct from Σ . (Thus $k = 2n$.) It follows from the constructions in [19], that there is a regular language R (resp., NPDA) over alphabet $\Sigma \cup C$ such that $L = h_c(\{w \mid w \in R, |w|_{b_1} = |w|_{c_1}, \dots, |w|_{b_n} = |w|_{c_n}\})$. Now let $R' = (b_1 c_1)^* \dots (b_n c_n)^* R$. Clearly, R' is also regular (resp., NPDA), and $L = h_c(\{w \mid w \in R', |w|_{b_1} = |w|_{c_1} = \dots = |w|_{b_n} = |w|_{c_n}\})$. \square

Lemma 22. *If $L_1 \subseteq \Sigma^*$ is in L-IND, and $L_2 \subseteq \Sigma^*$ is in NCM, then $L_1 \cap L_2 \in \text{L-IND}_c$.*

Proof. By Lemma 21, since L_2 is in NCM, there is regular set R over alphabet $\Sigma \cup C$ such that $L_2 = h_c(\{w \mid w \in R, |w|_{c_1} = \dots = |w|_{c_k}\})$. Also, $L'_1 = h_c^{-1}(L_1)$ is also a linear indexed language since the family is a full trio [6], and $L_3 = L'_1 \cap R$ is also a linear indexed language. Let G_3 be a linear indexed grammar generating L_3 .

We can now construct from G_3 a linear indexed grammar with counters generating $L = L_1 \cap L_2$, such that, if $\alpha \rightarrow \beta$ is a production in G_3 , then $\alpha \rightarrow (h_c(\beta), |\beta|_{c_1}, \dots, |\beta|_{c_k})$ is a production in G_4 . Then $L(G_4) = L_1 \cap L_2$. \square

Since languages generated by linear indexed grammars with counters are clearly closed under homomorphism, the following is true:

Corollary 23. *Let h be a homomorphism, $L_1 \in \text{L-IND}$, and $L_2 \in \text{NCM}$. Then $h(L_1 \cap L_2) \in \text{L-IND}_c$.*

Lemma 24. *If $L \in \text{L-IND}_c$, then $L = h(L_1 \cap L_2)$ for some homomorphism h , $L_1 \in \text{L-IND}$, and $L_2 \in \text{NCM}$.*

Proof. Let L be generated by G . Construct a linear indexed grammar (without counters) G_1 as follows:

If $\alpha \rightarrow (\beta, d_1, \dots, d_k)$ is a rule in G , then $\alpha \rightarrow \beta'$ is a rule in G_1 , where $\beta' = c_1^{d_1} \dots c_k^{d_k} \beta$, (i.e., we append to the left of β a terminal string representing the increments in the counters).

Let L_1 be the language generated by G_1 . Let $L_2 = \{w \mid w \in (\Sigma \cup C)^*, |w|_{c_1} = \dots = |w|_{c_k}\}$. Clearly L_2 is an NCM language, and $L = h_c(L_1 \cap L_2)$. \square

Proposition 25. *$L \in \text{L-IND}_c$ if and only if there is a homomorphism h , $L_1 \in \text{L-IND}$, and $L_2 \in \text{NCM}$ such that $L = h(L_1 \cap L_2)$.*

Proof. This follows immediately from Corollary 23 and Lemma 24. \square

Implied from the above result and Proposition 3 and since L-IND is an effectively semilinear trio [6] is that $\text{L-IND}_c \subseteq \mathcal{F}(\text{L-IND} \wedge \text{NCM})$, and therefore L-IND_c is effectively semilinear.

Corollary 26. *The languages generated by linear indexed grammar with counters are effectively semilinear, with decidable emptiness, membership, and infiniteness problems.*

Next, a machine model characterization of right linear indexed grammars with counters will be provided. Recall that an NPCM is a pushdown automaton augmented by reversal-bounded counters. The proof uses the fact that every context-free language can be generated by a right-linear indexed grammar [6].

Proposition 27. $\text{RL-IND}_c = \text{NPCM}$.

Proof. First, it will be show that $\text{NPCM} \subseteq \text{RL-IND}_c$. Let $L \in \text{NPCM}$. Then, by Lemma 21, there is an NPDA L_1 over $\Sigma \cup C$ such that $L = h_c(\{w \mid w \in L_1, |w|_{c_1} = \dots = |w|_{c_k}\})$. It is known that every context-free language can be generated by a right-linear indexed grammar [6], and hence there is a right-linear indexed grammar G_1 generating L_1 . Construct from G_1 , a right-linear indexed grammar G with counters generating L , such that, if $\alpha \rightarrow \beta$ is a production in G_1 , then $\alpha \rightarrow (h_c(\beta), |\beta|_{c_1}, \dots, |\beta|_{c_k})$ is a production in G . Then $L(G) = L$.

Next, the converse will be shown. Let G be a right-linear indexed grammar with counters. We first construct a right-linear grammar (without counters) G_1 generating a language L_1 as in the proof of Lemma 24. Then $L(G_1)$ is a context-free language, and can be accepted by an NPDA M_1 . We then construct an NPCM M which, when given an input w , simulates M_1 and checks that $|w|_{c_1} = \dots = |w|_{c_k}$ using 1-reversal counters. Finally, we construct from M another NPCM M' which erases the c_i 's. Clearly, $L(M') = L$. \square

We conjecture that the family of languages generated by right-linear indexed grammars with counters (the family of NPCM languages) is properly contained in the family of languages generated by linear indexed grammars with counters. Candidate witness languages are $L = \{w\$w \mid w \in \{a, b, c\}^*, |w|_a + |w|_b = |w|_c\}$ and $L' = \{w\$w \mid w \in \{a, b\}^*\}$. It is known that L' is generated by a linear indexed grammar [6], and hence L can be generated by such a grammar with two counters. But, both L' and L seem unlikely to be accepted by any NPCM. Therefore, indexed grammars with counters form quite a general semilinear family as it seems likely to be more general than NPCM.

Next, another subfamily of indexed languages is studied that are even more expressive than linear indexed grammars but still only generate semilinear languages.

An indexed grammar $G = (V, \Sigma, I, P, S)$ is said to be *uncontrolled index- r* if, every sentential form in every successful derivation has at most r nonterminals. G is uncontrolled finite-index if G is uncontrolled index- r , for some r . Let U-IND be the languages generated by uncontrolled finite-index indexed grammars.

Uncontrolled finite-index indexed grammars have also been studied under the name of breadth-bounded indexed grammars in [31, 3], where it was shown that the languages generated by these grammars are a semilinear full trio.

This concept can then be carried over to indexed grammars with counters.

Definition 28. *An indexed grammar with k -counters is uncontrolled index- r (resp. uncontrolled finite-index) if the underlying grammar is uncontrolled index- r (resp. uncontrolled finite-index). Let U-IND_c be the languages generated by uncontrolled finite-index indexed grammar with k -counters, for some k .*

One can easily verify that Proposition 25 also applies to uncontrolled finite-index indexed grammars with counters. Hence, we have:

Proposition 29. *$L \in \text{U-IND}_c$ if and only if there is a homomorphism h , $L_1 \in \text{U-IND}$, $L_2 \in \text{NCM}$ such that $L = h(L_1 \cap L_2)$.*

Implied from the above proposition and Proposition 3 also is that these new languages are all semilinear.

Corollary 30. *U-IND_c is effectively semilinear, with decidable emptiness, membership, and infiniteness problems.*

Hence, $\text{RL-IND}_c \subseteq \text{L-IND}_c \subseteq \text{U-IND}_c$. We conjecture that both containments are strict; the first was discussed previously, and the second is likely true since $\text{L-IND} \subsetneq \text{U-IND}$ [3]. Hence, U-IND_c forms quite a general semilinear family, containing NPCM with positive decidability properties.

7. Conclusions and Future Directions

It has been previously shown that certain types of machine models accepting only semilinear languages can be augmented by reversal-bounded counters to create a more general machine model, while maintaining semilinearity and positive decision properties. However, this approach did not clearly define what types of models would work with this augmentation, and it did not work with other mechanisms for describing languages. Here, a closure property theoretic method is developed, and it is shown that, for every semilinear full trio \mathcal{L} , the smallest full AFL containing \mathcal{L} also closed under intersection with reversal-bounded multicounter languages (NCM) is semilinear. Furthermore, the semilinearity is effective in the resulting family if it is effective (with other properties) in \mathcal{L} .

This can be applied in numerous ways. For example, it is shown that if certain subclasses of indexed grammars (linear indexed, or uncontrolled finite-index) are augmented by counters with additional components of the grammars that function like counters, then the resulting families are more general, yet they remain semilinear and have decidable emptiness, membership, and infiniteness problems. There are also other applications, such as to analyzing definitions of abstract automata with multitape stores.

Several open problems remain. It is open whether the application of Proposition 3 creates a strict hierarchy. With respect to indexed grammars with counters, it is open as to whether right-linear grammars are strictly weaker than linear grammars, and whether those are weaker than uncontrolled finite-index grammars.

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