# New counts for the number of triangulations of cyclic polytopes 

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#### Abstract

We report on enumerating the triangulations of cyclic polytopes with the new software MPTOPCOM. This is relevant for its connection with higher Stasheff-Tamari orders, which occur in category theory and algebraic combinatorics.


## 1 Introduction

For an integer $d \geq 1$ the $d$-th moment curve is the map

$$
\mu_{d}: \mathbb{R} \rightarrow \mathbb{R}^{d}, t \mapsto\left(t, \ldots, t^{d}\right)
$$

Picking $n$ real numbers $t_{1}<t_{2}<\ldots<t_{n}$, where $n>d$, the convex hull

$$
\begin{equation*}
\mathcal{C}(n, d)=\operatorname{conv}\left\{\mu_{d}\left(t_{1}\right), \mu_{d}\left(t_{2}\right), \ldots, \mu_{d}\left(t_{n}\right)\right\} \tag{1}
\end{equation*}
$$

is the $d$-dimensional cyclic polytope with $n$ vertices. The combinatorics of $\mathcal{C}(n, d)$ is given by Gale's evenness criterion; cf. [14, Theorem 0.7]. In particular, the combinatorial type does not depend on the values $t_{1}, t_{2}, \ldots, t_{n}$ but just on their number. The cyclic polytopes are neighborly, and hence their $f$-vectors attain McMullen's upper bound [14, Theorem 8.23]. The higher Stasheff-Tamari orders are certain poset structures on the set of all triangulations of $\mathcal{C}(n, d)$. Their study was initiated by Kapranov and Voevodsky [10] in the context of category theory; see also [6, [11] and [13]. Here we address the problem raised in [10, §5.2], which asks for determining the number of triangulations of $\mathcal{C}(n, d)$. We report on new computational results, obtained via the new software MPTOPCOM [9]. This verifies and extends previous results of Rambau and Reiner [13, Table 1], which were obtained with TOPCOM [12]. The general question remains wide open. Notice that the planar case $d=2$ gives the Catalan numbers.

Triangulations of polytopes and of finite point configurations are the subject of the monograph [5] by De Loera, Rambau and Santos. Cyclic polytopes are discussed in [5, §6.1]. We are indebted to Jörg Rambau and Francisco Santos for
suggesting to apply MPTOPCOM to cyclic polytopes and for many useful comments on an earlier version of this text.

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## 2 The first higher Stasheff-Tamari order

Let $P \subset \mathbb{R}^{d}$ be a finite point configuration. A circuit of $P$ is a minimally affinely dependent subconfiguration. A triangulation of $P$ is a subdivision of the convex hull $\operatorname{conv}(P)$ whose vertices form a subset of the points in $P$. Two triangulations of $P$ differ by a flip if they agree outside a circuit. Here we are interested in the triangulations of the point configuration given by the vertices of $\mathcal{C}(n, d)$ and their flips.

There is a canonical projection of the $(d+1)$-dimensional simplex $\mathcal{C}(d+2, d+$ 1) onto $\mathcal{C}(d+2, d)$ by forgetting the last coordinate. There are precisely two triangulations of $\mathcal{C}(d+2, d)$, and these correspond to projecting the lower and the upper hull of $\mathcal{C}(d+2, d+1)$. Consequently, we call them the lower and the upper triangulation of $\mathcal{C}(d+2, d)$, respectively. From the construction (1) it is immediate that each circuit of $\mathcal{C}(n, d)$ looks like $\mathcal{C}(d+2, d)$. Combined with the observation on the two triangulations of $\mathcal{C}(d+2, d)$, this has far reaching consequences for the structure of the triangulations of $\mathcal{C}(n, d)$ for arbitrary $n>d$.

Let $\Delta$ and $\Delta^{\prime}$ be two triangulations of $\mathcal{C}(n, d)$ which differ by a flip. Then there is subset $C$ of the vertices of cardinality $d+2$ such that $\Delta$ and $\Delta^{\prime}$ restricted to $C$ look like the upper and the lower triangulations of $\mathcal{C}(d+2, d)$. If $\Delta$ is the lower and $\Delta^{\prime}$ is the upper triangulation, then we call the flip [ $\Delta \rightsquigarrow \Delta^{\prime}$ ] from $\Delta$ to $\Delta^{\prime}$ an up-flip. Conversely, the reverse flip $\left[\Delta^{\prime} \rightsquigarrow \Delta\right.$ ] is a down-flip. In this case we write

$$
\begin{equation*}
\Delta \leq_{1} \Delta^{\prime} \tag{2}
\end{equation*}
$$

The partial ordering on the set of all triangulations of $\mathcal{C}(n, d)$ which is obtained as the transitive and reflexive closure of the relation (2) is the first higher StasheffTamari order, denoted as $H_{S T}(n, d)$; cf. [5, Definition 6.1.18] and [11]. Figure 3 below shows $H S T_{1}(6,2)$ as an example.

On the same set of triangulations of $\mathcal{C}(n, d)$ there is a second natural partial ordering, the second higher Stasheff-Tamari order, $\operatorname{HST}_{2}(n, d)$; cf. [5, Definition 6.1.16]. It is known that $\operatorname{HST}_{1}(n, d)$ is a weaker partial order than $H S T_{2}(n, d)$. Moreover, these two orders coincide for $d \leq 3$ and $n-d \in\{1,2,3\}$; cf. [13]. In general, it is open whether or not they agree.

## 3 GKZ-vectors

For a triangulation $\Delta$ of an affine spanning point configuration $P \subset \mathbb{R}^{d}$ the GKZ-vector is

$$
g k z_{\Delta}=\left(g k z_{\Delta}(p) \mid p \in P\right)
$$



Fig. 1. Lower (left) and upper (right) triangulations of $C(4,2)$ with their GKZ-vectors. In the lower triangulation the gaps are 4 and 2, i.e., even; whereas in the upper triangulation the gaps are 3 and 1 . Here $d=2$ is even.


1234, 1245, 2345
$(84,96,24,96,84)$


1235, 1345
$(96,48,96,48,96)$

Fig. 2. Lower (left) and upper (right) triangulations of $C(5,3)$ with their GKZ-vectors. In the lower triangulation the gaps are 4 and 2, i.e., even; whereas in the upper triangulation the gaps are 5,3 and 1 . Here $d=3$ is odd.
where $g k z_{\Delta}(p)$ is the sum of the normalized volumes of those simplices in $\Delta$ which contain $p$ as a vertex. The normalized volume is the Euclidean volume multiplied by $d!$.

In order to determine the GKZ-vectors of triangulations of cyclic polytopes, we need to choose coordinates. To keep it simple we can take the lattice points $\mu_{d}(1), \mu_{d}(2), \ldots, \mu_{d}(n)$ as the vertices of $\mathcal{C}(n, d)$. The normalized volume of any $d$-simplex spanned by $\mu_{d}\left(i_{1}\right), \mu_{d}\left(i_{2}\right), \ldots, \mu_{d}\left(i_{d+1}\right)$ with $i_{1}<i_{2}<\ldots<i_{d+1}$, is the Vandermonde determinant

$$
\operatorname{det}\left(\begin{array}{cccc}
1 & i_{1} & \cdots & i_{1}^{d}  \tag{3}\\
1 & i_{2} & \cdots & i_{2}^{d} \\
\vdots & \vdots & \ddots & \vdots \\
1 & i_{d+1} & \cdots & i_{d+1}^{d}
\end{array}\right)=\prod_{1 \leq k<\ell \leq d+1}\left(i_{\ell}-i_{k}\right)
$$

In particular, these values do not change when we replace the standard parameters $1,2, \ldots, n$ for the moment curve by any other set of $n$ consecutive integers; cf. Figure [1. where we chose the parameters $-1,0,1$ and 2 , while we keep the labels $1,2,3,4$. We fix the natural ordering of the vertices on the moment curve in order to identify GKZ-vectors of triangulations of $\mathcal{C}(n, d)$ with vectors in $\mathbb{R}^{n}$. The following basic observation is crucial.

Proposition 1. Let $\Delta$ and $\Delta^{\prime}$ be two triangulations of $\mathcal{C}(n, d)$ related by a flip [ $\left.\Delta \rightsquigarrow \Delta^{\prime}\right]$. Then we have

$$
\Delta \leq_{1} \Delta^{\prime} \Longleftrightarrow \begin{cases}g k z_{\Delta}>_{\text {lex }} g k z_{\Delta^{\prime}} & \text { if } d \text { even }, \\ g k z_{\Delta}<_{\text {lex }} g k z_{\Delta^{\prime}} & \text { if } d \text { odd } .\end{cases}
$$

Proof. Since each circuit looks like $C(d+2, d)$ it suffices to consider the case $n=d+2$. We exploit the relationship of the triangulations of $C(d+2, d)$ with the upper and lower hull of $C(d+2, d+1)$ previously explained.

The Oriented Gale's Evenness Criterion from [5, Corollary 6.1.9] describes the upper and lower facets of $C(d+2, d+1)$. Let $F \subseteq C(d+2, d+1)$ be a facet, then $F$ can be written as a subset of $[d+2]$, the set of indices of vertices in $F$. The gaps of $F$ are the elements of $[n] \backslash F$. A gap $i$ of $F$ is even if the number of elements in $F$ that are larger than $i$ is even. It is called odd otherwise. Correspondingly, a facet is called odd/even if all its gaps are odd/even. The odd facets correspond to the upper triangulation of $C(d+2, d)$ and the even facets give rise to the lower triangulation of $C(d+2, d)$.

Assume that 1 is a gap of $F$. Since every facet of $C(d+2, d+1)$ is a simplex, $F$ must be $\{2,3, \ldots, d+2\}$ and 1 is the only gap of $F$. Hence, if $d$ is odd, then $F$ is even. Conversely, if $d$ is even, then $F$ must be odd. We conclude that, if $d$ is odd, then all odd facets contain 1 . However, if $d$ is even, then only the even facets contain 1.

Assume now that $d$ is even. The odd case is similar.
Let $\Delta$ and $\Delta^{\prime}$ be the lower and upper triangulations of $C(d+2, d)$, i.e. $\Delta \leq_{1} \Delta^{\prime}$. Then $\Delta$ contains all the even facets of $C(d+2, d+1)$. But any even facet contains 1, thus the first entry of $g k z_{\Delta}$ is the entire normalized volume


Fig. 3. First higher Stasheff-Tamari order $H S T_{1}(6,2)$ with reverse search tree marked.
The lowest triangulation has the lexicographically largest GKZ-vector.
of $\mathcal{C}(d+2, d)$. The facet $\{2,3, \ldots, d+2\}$ is odd, and hence it belongs to $\Delta^{\prime}$. Since it does not contain 1, we infer that $g k z_{\Delta}(1)>g k z_{\Delta^{\prime}}(1)$. Hence we obtain $g k z_{\Delta}>_{\text {lex }} g k z_{\Delta^{\prime}}$.

This argument can be reversed, and this completes the proof.
Figure 1 and Figure 2 depicts the situation considered in the proof above for $(n, d)=(4,2)$ and $(n, d)=(5,3)$, respectively. The interest in Proposition 1 comes from the following.

In [8] Imai et al. described an algorithm for computing all (regular) triangulations of a given point configurations, which is based on the reverse search enumeration scheme of Avis and Fukuda [2]. That algorithm, which we call down-flip reverse search, was improved and implemented by Skip Jordan with the authors of this extended abstract 9. The basic idea is to orient each flip according to lexicographic ordering of the GKZ-vectors. Then down-flip reverse search produces a directed spanning tree of those triangulations which can be obtained from some seed triangulation by monotone flipping; cf. [5, §5.3.2]. For the cyclic polytopes we arrive at two choices for orienting the flips, one by GKZ-vectors, one according to the first higher Stasheff-Tamari order. Now Proposition 1 says that these two choices fortunately agree.

Corollary 1. Down-flip reverse search computes a directed spanning tree of the first Stasheff-Tamari poset $\operatorname{HST}_{1}(n, d)$, rooted at the triangulation with the lexicographically largest GKZ-vector. For d even, the root is the lowest triangulation of $\mathcal{C}(n, d)$, whereas, for $d$ odd, the root is the highest triangulation. In particular, each triangulation of a cyclic polytope can be obtained by monotone flipping from the respective roots.

The first higher Stasheff-Tamari order $H S T_{1}(6,2)$ with GKZ-vectors is shown in Figure 3

## 4 Computations with MPTOPCOM

The open source software MPTOPCOM is designed for computing triangulations in a massively parallel setup. Its algorithm is the down-flip reverse search method of Imai et al. [8] with several improvements as described in 9. As its key feature reverse search is output sensitive, and this makes it attractive for extremely large enumeration problems. Our parallelization, based on the MPI protocol, employs budgeting for load balancing; cf. 31]. In this way MPTOPCOM can enumerate the (regular) triangulations of much larger point sets than other software before; extensive experiments are described in [9, §7]. MPTOPCOM uses linear algebra and basic data types from polymake [7, triangulations and flips from TOPCOM [12] and the budgeted parallel reverse search from mts 3.

The most recent census of triangulations of cyclic polytopes that we are aware of is by Rambau and Reiner [13, Table 1]; we use their notation and introduce the parameter $c:=n-d$. Note that there are two rather obvious typos in the rows $c \in\{10,11\}$ of the column $d=1$ in [13, Table 1]. Apart from

Table 1. The number of triangulations of $\mathcal{C}(c+d, d)$. The column $d=2$ contains the Catalan numbers, while the row $c=4$ is known by results of Azaola and Santos 4. The rows $c \in\{1,2,3\}$ are trivial and only listed for completeness. The row $c=5$ and the column $d=2$ are marked for their relevance to Question Our new results are written in blue; the rest of the table agrees with [13) Table 1].

| $\mathrm{c} \backslash \mathrm{d}:$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 4 | 14 | 25 | 40 | 67 | 102 | 165 | 244 |
| 5 | 42 | 138 | 357 | 1233 | 3278 | 12589 | 35789 |
| 6 | 132 | 972 | 4824 | 51676 | 340560 | 6429428 | 68007706 |
| 7 | 429 | 8477 | 96426 | 5049932 | 132943239 |  |  |
| 8 | 1430 | 89405 | 2800212 | 1171488063 |  |  |  |
| 9 | 4862 | 1119280 | 116447760 |  |  |  |  |
| 10 | 16796 | 16384508 |  |  |  |  |  |
| 11 | 58786 | 276961252 |  |  |  |  |  |
| 12 | 208012 | 5349351298 | 10 | 11 | 12 | 13 | 14 |
| $\mathrm{c} \backslash \mathrm{d}:$ | 9 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 2 | 2 | 2 | 2 | 2 |  |
| 2 | 2 | 13 | 14 | 15 | 16 | 17 |  |
| 3 | 12 | 562 | 881 | 1264 | 1967 | 2798 |  |
| 4 | 387 | 499900 | 2677865 | 9421400 | 62226044247567074 |  |  |
| 5 | 159613 |  |  |  |  |  |  |

that we can confirm their results; cf. Table 1 Our new results are the values for $(c, d) \in\{(12,3),(8,5),(6,8),(5,14)\}$.

Our experiments used MPTOPCOM, version 1.0, on a cluster with four nodes, each of which comes with $2 \times 8$-Core Xeon E5-2630v3 ( 2.4 GHz ) and 64GB per node. We ran MPTOPCOM with 40 threads. The operating system is SMP Linux 4.4.121. For instance, the computation for $c=5$ and $d=14$, i.e., $n=19$ took 71191 seconds, i.e., less than 20 hours.

Azaola and Santos [4, p. 30] implicitly raised the following question.
Question 1. Is there an absolute constant $\beta>1$ such that, for all $n \geq 7$ :

$$
\begin{equation*}
\frac{1}{\beta} \leq \frac{\#\{\text { triangulations of } C(n, n-5)\}}{\#\{\text { triangulations of } C(n, 2)\}} \leq \beta ? \tag{4}
\end{equation*}
$$

This relates the row $c=5$ with the column $d=2$; these are marked in Table 1 . From MPTOPCOM's results we can derive the series (4) for $n \in\{7,8, \ldots, 19\}$ :

$$
\begin{gathered}
1,1.045,0.832,0.862,0.674,0.750,0.609 \\
0.767,0.673,1.001,0.972,1.760,1.910
\end{gathered}
$$

Note that the sequence in [4, p. 30] lists the reciprocals of the above; moreover, that sequence contains two more (trivial) values for $n \in\{5,6\}$, which we omit.

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