New counts for the number of triangulations of cyclic polytopes

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Abstract. We report on enumerating the triangulations of cyclic polytopes with the new software MPTOPCOM. This is relevant for its connection with higher Stasheff-Tamari orders, which occur in category theory and algebraic combinatorics.

1 Introduction

For an integer $d \ge 1$ the *d*-th moment curve is the map

$$\mu_d : \mathbb{R} \to \mathbb{R}^d, \ t \mapsto (t, \dots, t^d)$$
.

Picking n real numbers $t_1 < t_2 < \ldots < t_n$, where n > d, the convex hull

$$\mathcal{C}(n,d) = conv\{\mu_d(t_1), \mu_d(t_2), \dots, \mu_d(t_n)\}$$
(1)

is the d-dimensional cyclic polytope with n vertices. The combinatorics of $\mathcal{C}(n, d)$ is given by Gale's evenness criterion; cf. [14, Theorem 0.7]. In particular, the combinatorial type does not depend on the values t_1, t_2, \ldots, t_n but just on their number. The cyclic polytopes are neighborly, and hence their f-vectors attain McMullen's upper bound [14, Theorem 8.23]. The higher Stasheff-Tamari orders are certain poset structures on the set of all triangulations of $\mathcal{C}(n, d)$. Their study was initiated by Kapranov and Voevodsky [10] in the context of category theory; see also [6], [11] and [13]. Here we address the problem raised in [10, §5.2], which asks for determining the number of triangulations of $\mathcal{C}(n, d)$. We report on new computational results, obtained via the new software MPTOPCOM [9]. This verifies and extends previous results of Rambau and Reiner [13, Table 1], which were obtained with TOPCOM [12]. The general question remains wide open. Notice that the planar case d = 2 gives the Catalan numbers.

Triangulations of polytopes and of finite point configurations are the subject of the monograph [5] by De Loera, Rambau and Santos. Cyclic polytopes are discussed in [5, §6.1]. We are indebted to Jörg Rambau and Francisco Santos for

suggesting to apply MPTOPCOM to cyclic polytopes and for many useful comments on an earlier version of this text.

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2 The first higher Stasheff–Tamari order

Let $P \subset \mathbb{R}^d$ be a finite point configuration. A *circuit* of P is a minimally affinely dependent subconfiguration. A *triangulation* of P is a subdivision of the convex hull conv(P) whose vertices form a subset of the points in P. Two triangulations of P differ by a flip if they agree outside a circuit. Here we are interested in the triangulations of the point configuration given by the vertices of C(n, d) and their flips.

There is a canonical projection of the (d+1)-dimensional simplex $\mathcal{C}(d+2, d+1)$ onto $\mathcal{C}(d+2, d)$ by forgetting the last coordinate. There are precisely two triangulations of $\mathcal{C}(d+2, d)$, and these correspond to projecting the lower and the upper hull of $\mathcal{C}(d+2, d+1)$. Consequently, we call them the *lower* and the *upper triangulation* of $\mathcal{C}(d+2, d)$, respectively. From the construction (1) it is immediate that each circuit of $\mathcal{C}(n, d)$ looks like $\mathcal{C}(d+2, d)$. Combined with the observation on the two triangulations of $\mathcal{C}(d+2, d)$, this has far reaching consequences for the structure of the triangulations of $\mathcal{C}(n, d)$ for arbitrary n > d.

Let Δ and Δ' be two triangulations of $\mathcal{C}(n, d)$ which differ by a flip. Then there is subset C of the vertices of cardinality d+2 such that Δ and Δ' restricted to C look like the upper and the lower triangulations of $\mathcal{C}(d+2, d)$. If Δ is the lower and Δ' is the upper triangulation, then we call the flip $[\Delta \rightsquigarrow \Delta']$ from Δ to Δ' an *up-flip*. Conversely, the reverse flip $[\Delta' \rightsquigarrow \Delta]$ is a *down-flip*. In this case we write

$$\Delta \leq_1 \Delta' . \tag{2}$$

The partial ordering on the set of all triangulations of C(n, d) which is obtained as the transitive and reflexive closure of the relation (2) is the *first higher Stasheff-Tamari order*, denoted as $HST_1(n, d)$; cf. [5, Definition 6.1.18] and [11]. Figure 3 below shows $HST_1(6, 2)$ as an example.

On the same set of triangulations of C(n, d) there is a second natural partial ordering, the second higher Stasheff-Tamari order, $HST_2(n, d)$; cf. [5, Definition 6.1.16]. It is known that $HST_1(n, d)$ is a weaker partial order than $HST_2(n, d)$. Moreover, these two orders coincide for $d \leq 3$ and $n - d \in \{1, 2, 3\}$; cf. [13]. In general, it is open whether or not they agree.

3 GKZ-vectors

For a triangulation Δ of an affine spanning point configuration $P \subset \mathbb{R}^d$ the GKZ-vector is

$$gkz_{\Delta} = (gkz_{\Delta}(p) \mid p \in P)$$

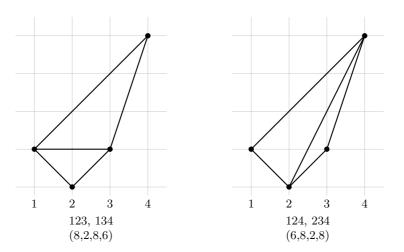


Fig. 1. Lower (left) and upper (right) triangulations of C(4, 2) with their GKZ-vectors. In the lower triangulation the gaps are 4 and 2, i.e., even; whereas in the upper triangulation the gaps are 3 and 1. Here d = 2 is even.

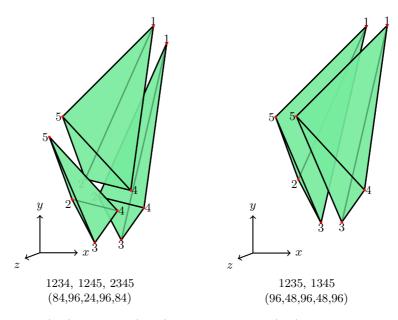


Fig. 2. Lower (left) and upper (right) triangulations of C(5,3) with their GKZ-vectors. In the lower triangulation the gaps are 4 and 2, i.e., even; whereas in the upper triangulation the gaps are 5, 3 and 1. Here d = 3 is odd.

where $gkz_{\Delta}(p)$ is the sum of the normalized volumes of those simplices in Δ which contain p as a vertex. The *normalized volume* is the Euclidean volume multiplied by d!.

In order to determine the GKZ-vectors of triangulations of cyclic polytopes, we need to choose coordinates. To keep it simple we can take the lattice points $\mu_d(1), \mu_d(2), \ldots, \mu_d(n)$ as the vertices of $\mathcal{C}(n, d)$. The normalized volume of any *d*-simplex spanned by $\mu_d(i_1), \mu_d(i_2), \ldots, \mu_d(i_{d+1})$ with $i_1 < i_2 < \ldots < i_{d+1}$, is the Vandermonde determinant

$$\det \begin{pmatrix} 1 & i_1 & \cdots & i_1^d \\ 1 & i_2 & \cdots & i_2^d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & i_{d+1} & \cdots & i_{d+1}^d \end{pmatrix} = \prod_{1 \le k < \ell \le d+1} (i_\ell - i_k) .$$
(3)

In particular, these values do not change when we replace the standard parameters $1, 2, \ldots, n$ for the moment curve by any other set of n consecutive integers; cf. Figure 1, where we chose the parameters -1, 0, 1 and 2, while we keep the labels 1, 2, 3, 4. We fix the natural ordering of the vertices on the moment curve in order to identify GKZ-vectors of triangulations of $\mathcal{C}(n, d)$ with vectors in \mathbb{R}^n . The following basic observation is crucial.

Proposition 1. Let Δ and Δ' be two triangulations of $\mathcal{C}(n,d)$ related by a flip $[\Delta \rightsquigarrow \Delta']$. Then we have

$$\Delta \leq_1 \Delta' \iff \begin{cases} gkz_\Delta >_{lex} gkz_{\Delta'} & \text{if } d \text{ even,} \\ gkz_\Delta <_{lex} gkz_{\Delta'} & \text{if } d \text{ odd.} \end{cases}$$

Proof. Since each circuit looks like C(d + 2, d) it suffices to consider the case n = d + 2. We exploit the relationship of the triangulations of C(d + 2, d) with the upper and lower hull of C(d + 2, d + 1) previously explained.

The Oriented Gale's Evenness Criterion from [5, Corollary 6.1.9] describes the upper and lower facets of C(d+2, d+1). Let $F \subseteq C(d+2, d+1)$ be a facet, then F can be written as a subset of [d+2], the set of indices of vertices in F. The gaps of F are the elements of $[n] \setminus F$. A gap i of F is even if the number of elements in F that are larger than i is even. It is called odd otherwise. Correspondingly, a facet is called odd/even if all its gaps are odd/even. The odd facets correspond to the upper triangulation of C(d+2, d) and the even facets give rise to the lower triangulation of C(d+2, d).

Assume that 1 is a gap of F. Since every facet of C(d+2, d+1) is a simplex, F must be $\{2, 3, \ldots, d+2\}$ and 1 is the only gap of F. Hence, if d is odd, then F is even. Conversely, if d is even, then F must be odd. We conclude that, if d is odd, then all odd facets contain 1. However, if d is even, then only the even facets contain 1.

Assume now that d is even. The odd case is similar.

Let Δ and Δ' be the lower and upper triangulations of C(d+2,d), i.e. $\Delta \leq_1 \Delta'$. Then Δ contains all the even facets of C(d+2,d+1). But any even facet contains 1, thus the first entry of gkz_{Δ} is the entire normalized volume

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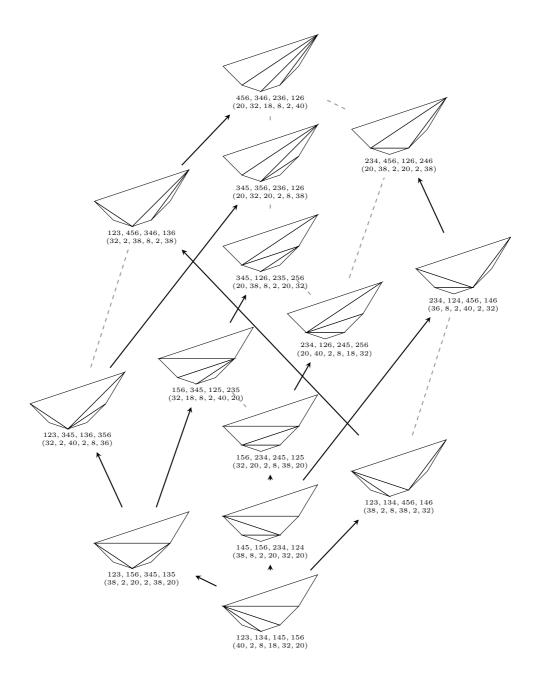


Fig. 3. First higher Stasheff-Tamari order $HST_1(6, 2)$ with reverse search tree marked. The lowest triangulation has the lexicographically largest GKZ-vector.

of $\mathcal{C}(d+2,d)$. The facet $\{2,3,\ldots,d+2\}$ is odd, and hence it belongs to Δ' . Since it does not contain 1, we infer that $gkz_{\Delta}(1) > gkz_{\Delta'}(1)$. Hence we obtain $gkz_{\Delta} >_{lex} gkz_{\Delta'}$.

This argument can be reversed, and this completes the proof.

Figure 1 and Figure 2 depicts the situation considered in the proof above for (n, d) = (4, 2) and (n, d) = (5, 3), respectively. The interest in Proposition 1 comes from the following.

In [8] Imai et al. described an algorithm for computing all (regular) triangulations of a given point configurations, which is based on the reverse search enumeration scheme of Avis and Fukuda [2]. That algorithm, which we call *down-flip reverse search*, was improved and implemented by Skip Jordan with the authors of this extended abstract [9]. The basic idea is to orient each flip according to lexicographic ordering of the GKZ-vectors. Then down-flip reverse search produces a directed spanning tree of those triangulations which can be obtained from some seed triangulation by monotone flipping; cf. [5, §5.3.2]. For the cyclic polytopes we arrive at two choices for orienting the flips, one by GKZ-vectors, one according to the first higher Stasheff–Tamari order. Now Proposition 1 says that these two choices fortunately agree.

Corollary 1. Down-flip reverse search computes a directed spanning tree of the first Stasheff-Tamari poset $HST_1(n, d)$, rooted at the triangulation with the lexicographically largest GKZ-vector. For d even, the root is the lowest triangulation of C(n, d), whereas, for d odd, the root is the highest triangulation. In particular, each triangulation of a cyclic polytope can be obtained by monotone flipping from the respective roots.

The first higher Stasheff–Tamari order $HST_1(6,2)$ with GKZ-vectors is shown in Figure 3.

4 Computations with MPTOPCOM

The open source software MPTOPCOM is designed for computing triangulations in a massively parallel setup. Its algorithm is the down-flip reverse search method of Imai et al. [8] with several improvements as described in [9]. As its key feature reverse search is output sensitive, and this makes it attractive for extremely large enumeration problems. Our parallelization, based on the MPI protocol, employs *budgeting* for load balancing; cf. [3,1]. In this way MPTOPCOM can enumerate the (regular) triangulations of much larger point sets than other software before; extensive experiments are described in [9, §7]. MPTOPCOM uses linear algebra and basic data types from polymake [7], triangulations and flips from TOPCOM [12] and the budgeted parallel reverse search from mts [3].

The most recent census of triangulations of cyclic polytopes that we are aware of is by Rambau and Reiner [13, Table 1]; we use their notation and introduce the parameter c := n - d. Note that there are two rather obvious typos in the rows $c \in \{10, 11\}$ of the column d = 1 in [13, Table 1]. Apart from

Table 1. The number of triangulations of C(c + d, d). The column d = 2 contains the Catalan numbers, while the row c = 4 is known by results of Azaola and Santos [4]. The rows $c \in \{1, 2, 3\}$ are trivial and only listed for completeness. The row c = 5 and the column d = 2 are marked for their relevance to Question 1. Our new results are written in blue; the rest of the table agrees with [13, Table 1].

$c \setminus d$:	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	5	6	7	8	9	10	11
4	14	25	40	67	102	165	244
5	42	138	357	1233	3278	12589	35789
6	132	972	4824	51676	340560	6429428	68 007 706
7	429	8477	96426	5049932	132943239		
8	1430	89405	2800212	1171488063			
9	4862	1119280	116447760				
10	16796	16384508					
11	58786	276961252					
12	208012	5349351298					
$c \setminus d$:	9	10	11	12	13	14	
1	1	1	1	1	1	1	
2	2	2	2	2	2	2	
3	12	13	14	15	16	17	
4	387	562	881	1264	1967	2798	
5	159613	499900	2677865	9421400	62226044	247567074	

that we can confirm their results; cf. Table 1. Our new results are the values for $(c, d) \in \{(12, 3), (8, 5), (6, 8), (5, 14)\}.$

Our experiments used MPTOPCOM, version 1.0, on a cluster with four nodes, each of which comes with 2 x 8-Core Xeon E5-2630v3 (2.4 GHz) and 64GB per node. We ran MPTOPCOM with 40 threads. The operating system is SMP Linux 4.4.121. For instance, the computation for c = 5 and d = 14, i.e., n = 19 took 71191 seconds, i.e., less than 20 hours.

Azaola and Santos [4, p. 30] implicitly raised the following question.

Question 1. Is there an absolute constant $\beta > 1$ such that, for all $n \ge 7$:

$$\frac{1}{\beta} \leq \frac{\#\{\text{triangulations of } C(n, n-5)\}}{\#\{\text{triangulations of } C(n, 2)\}} \leq \beta ?$$
(4)

This relates the row c = 5 with the column d = 2; these are marked in Table 1. From MPTOPCOM's results we can derive the series (4) for $n \in \{7, 8, ..., 19\}$:

Note that the sequence in [4, p. 30] lists the reciprocals of the above; moreover, that sequence contains two more (trivial) values for $n \in \{5, 6\}$, which we omit.

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