Multiresolutional Techniques in Finite Element Method Solution of Eigenvalue Problem

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Abstract. Computational analysis of unidirectional transient problems in multiscale heterogeneous media using specially adopted homogenization technique and the Finite Element Method is described below. Multiresolutional homogenization being the extension of the classical micro-macro traditional approach is used to calculate effective parameters of the composite. Effectiveness of the method is compared against previous techniques thanks to the FEM solution of some engineering problems with real material parameters and with their homogenized values. Further computational studies are necessary in this area, however application of the multiresolutional technique is justified by the natural multiscale character of composites.

1 Introduction

Wavelet analysis [1] perfectly reflects the very demanding needs of composite materials computational modeling. It is due to the fact that wavelet functions like Haar, sinusoidal (harmonic), Gabor, Morlet or Daubechies, for instance, relating neighboring scales in the medium analysed can efficiently model a variety of heterogeneities preserving composites periodicity, for instance. It is evident now that wavelet techniques may serve for analysis in the finest scale by various numerical techniques [2,4,5] as well as using multiresolutional analysis (MRA) [3,5,6,8]. The first method leads to the exponential increase of the total number of degrees of freedom in the model, because each new decomposition level almost doubles this number, while an application of the homogenization method is connected with determination of effective material parameters.

Both methodologies are compared here in the context of eigenvalue problem solution for a simply supported linear elastic Euler-Bernoulli beam using the Finite Element Method (FEM) computational procedures. The corresponding comparison made for a transient heat transfer has been discussed before in [5]. Homogenization of a composite is performed here through (1) simple spatial averaging of composite properties, (2) two-scale classical approach [7] as well as (3) thanks to the multiresolutional technique based on the Haar wavelets. An application of the symbolic package MAPLE guarantees an efficient integration of algebraic formulas defining effective properties for a composite with material properties given by some wavelet functions.

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2 Multiresolutional Homogenization Scheme

MRA approach uses the algebraic transformation between various scales provided by the wavelet analysis to determine the fine-scale behavior and introduce it explicitly into the macroscopic equilibrium equations. The following relation:

$$\Omega_0 \subset \Omega_{-1} \subset \Omega_{-2} \subset \ldots \tag{1}$$

defines the hierarchical geometry of the scales and this chain of subspaces is so defined that Ω_j is "finer" than Ω_{j+1}. Further, let us note that the main assumption on general homogenization procedure for transient problems is a separate averaging of the coefficients from the governing partial differential equation responsible for a static behavior and of the unsteady component. The problem can be homogenized only if its equilibrium can be expressed by the following operator equation:

$$BT + u + \lambda = L(AT + v) \tag{2}$$

This equation in the multiscale notation can be rewritten at the given scale j as

$$B_0^{(j)} T_0^{(j)} + u_0^{(j)} + \lambda = L_0 \left(A_0^{(j)} T_0^{(j)} + v_0^{(j)} \right), \tag{3}$$

with the recurrence relations used j times to compute $B_0^{(j)}, A_0^{(j)}, u_0^{(j)}, v_0^{(j)}$. MRA homogenization theorem is obtained as a limit for $j \to -\infty$

$$B_0^{(-\infty)} T_0^{(-\infty)} + u_0^{(-\infty)} + \lambda = L_0 \left(A_0^{(-\infty)} T_0^{(-\infty)} + v_0^{(-\infty)} \right), \tag{4}$$

which enables to eliminate infinite number of the geometrical scales with the reduced coefficients $B_0^{(-\infty)}, A_0^{(-\infty)}$. If the limits defining the matrices $B_0^{(-\infty)}$ and $A_0^{(-\infty)}$ exist, then there exist constant matrices B^h, A^h and forcing terms u^h, v^h, such that the reduced coefficients and forcing terms are given by $B_0^{(-\infty)}, A_0^{(-\infty)}, u_0^{(-\infty)}, v_0^{(-\infty)}$. The homogenized coefficients are equal to

$$A^h = A_0^{(-\infty)}, \; B^h = A^h \tilde{A}^{-1} - I, \tag{5}$$

$$u^h = u_0^{(-\infty)} + \left(I - \tfrac{1}{2}\tilde{A} - \tilde{A}\left(\exp\left(\tilde{A} - I\right)^{-1} A^h \right)^{-1} v^h \right), \tag{6}$$

where

$$\tilde{A} = \log\left(I + \left(I + B_0^{(-\infty)} - \tfrac{1}{2} A^h \right)^{-1} A^h \right). \tag{7}$$

As the example let us review the static equilibrium of elastic Euler-Bernoulli beam

$$-\frac{d}{dx}\left(E(x) \frac{d}{dx} u(x) \right) = f(x)\, ; \; x \in [0,1], \tag{8}$$

where E(x), defining material properties of the heterogeneous medium, varies arbitrarily on many scales. The unit interval denotes here the Representative Volume Element (RVE), called also the periodicity cell. This equation can represent linear elastic behavior of unidirectional structures as well as unidirectional heat conduction and other related physical fields. A periodic structure with a small parameter $\varepsilon>0$, tending to 0, relating the lengths of the periodicity cell and the entire composite, is considered in a classical approach. The displacements are expanded as

$$u(x) = u^{(0)}(x,y) + \varepsilon^1 u^{(1)}(x,y) + \varepsilon^2 u^{(2)}(x,y) + \ldots, \tag{9}$$

where $u^{(i)}(x,y)$ are also periodic; the coordinate x is introduced for macro scale, while y - in micro scale. Introducing these expansions into classical Hooke's law, the homogenized elastic modulus is obtained as [6]

$$E^{(eff)} = \left(\int_\Omega \frac{dy}{E(y)} \right)^{-1}. \tag{10}$$

The method called multiresolutional starts from the following decomposition:

$$\begin{cases} \dfrac{d}{dx} u(x) = \dfrac{v(x)}{E(x)} \\ \dfrac{d}{dx} v(x) = -f(x) \end{cases} \tag{11}$$

to determine the homogenized coefficient $E^{(eff)}$ constant for $x \in [0,1]$. Therefore

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} - \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \int_0^x \left(\begin{pmatrix} 0 & E(t)^{-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} 0 \\ -f(t) \end{pmatrix} \right) dt. \tag{12}$$

The reduction algorithm between multiple scales of the composite consists in determination of such effective tensors $B^{(eff)}$, $A^{(eff)}$, $p^{(eff)}$ and $q^{(eff)}$, such that

$$\left(\mathbf{I} + B^{(eff)} \right) \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} + q^{(eff)} + \lambda = \int_0^x \left(A^{(eff)} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + p^{(eff)} \right) dt, \tag{13}$$

where $\mathbf{I}$ is an identity matrix. In our case we apply

$$B^{(eff)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; A^{(eff)} = \begin{pmatrix} 0 & C_1 - 2C_2 \\ 0 & 0 \end{pmatrix}, \quad C_1 = \int_0^1 \frac{dt}{E(t)}; C_2 = \int_0^1 \frac{\left(t - \frac{1}{2}\right) dt}{E(t)} \tag{14}$$

Furthermore, for f(x)=0 there holds $p^{(eff)} = q^{(eff)} = 0$, while, in a general case, $B^{(eff)}$ and $A^{(eff)}$ do not depend on p and q.

3 Multiresolutional Finite Element Method

Let us consider the governing equation

$$-e\nabla^2 u = f, \quad x \in \Omega \tag{15}$$

with

$$u = 0, \quad x \in \Gamma_u \subset \partial\Omega. \tag{16}$$

Variational formulation of this problem for the multiscale medium at the scale k is given as

$$\int_\Omega e \nabla u_k \nabla \varphi_k d\Omega + \int_\Omega \gamma u_k \varphi_k d\Omega = \int_\Gamma f \varphi_k d\Gamma, \quad x \in \Omega. \tag{17}$$

Solution of the problem must be found recursively by using some transformation between the neighboring scales. Hence, the following nonsingular n x n wavelet transform matrix $\mathbf{W}_k$ is introduced [2]:

$$\mathbf{W}_k = \mathbf{T}_k \begin{bmatrix} \mathbf{T}_{k-1} & 0 \\ 0 & \mathbf{I}_{k-1} \end{bmatrix}, \tag{18}$$

and

$$\boldsymbol{\psi}_k = \mathbf{W}_k^T \boldsymbol{\varphi}_k . \tag{19}$$

$\mathbf{T}_k$ is a two-scale transform between the scales k-1 and k, such that

$$\begin{Bmatrix} \boldsymbol{\varphi}_{k-1} \\ \boldsymbol{\psi}_k \end{Bmatrix} = \mathbf{T}_k^T \boldsymbol{\varphi}_k , \tag{20}$$

with

$$\psi_k^j = \varphi_k^{2j-1} , \; j=1,\ldots,N_k \tag{21}$$

N_k denotes here the total number of the FEM nodal points at the scale k. Let us illustrate the wavelet-based FEM idea using the example of 1D linear two-noded finite element with the shape functions [9]

$$\mathbf{N}^T = \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{Bmatrix}, \tag{22}$$

where N_1 is valid for ξ=-1 and N_2 – for ξ=1 in local coordinates system of this element. The scale effect is introduced on the element level by inserting new extra degrees of freedom at each new scale. Then, the scale 1 corresponds to first extra multiscale DOF per the original finite element, scale 2 – next two additional multiscale DOFs and etc. It may be generally characterized as

$$\psi_k(\xi) = \psi_k\left(2^{k-1}(1+\xi) - 2j - 1\right) \tag{23}$$

where

$$\begin{cases} 2^{2-k} j - 1 \le \xi \le 2^{2-k} j + 2^{1-k} - 1 \\ 2^{2-k} j + 2^{1-k} - 1 \le \xi \le 2^{2-k} j + 2^{2-k} - 1 \end{cases} \tag{24}$$

The value of k defines the actual scale. The reconstruction algorithm starts from the original solution for the original mesh. Next, the new scales are introduced using the formula

$$\mathbf{u}_k^{2+2^{k-1}+j} = \sum_{i=1}^{N_{old}} \mathbf{N}_i \mathbf{u}_0^i + \sum_{i=1}^{N_{new}} \boldsymbol{\psi}_k^{2+2^{k-1}+j} \Delta\mathbf{u}_k^{2+2^{k-1}+j} . \tag{25}$$

The wavelet algorithm for stiffness matrix reconstruction starts at scale 0 with the smallest rank stiffness matrix

$$\mathbf{K}_0 = \frac{e}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{26}$$

where h is the node spacing parameter. Then, the diagonal components of the stiffness matrix for any k>0 are equal to

$$K_k^{2+2^{K-1}+J} = \frac{2^{k+1}e}{h}. \tag{27}$$

It should be underlined that the FEM so modified reflects perfectly the needs of computational modeling of multiscale heterogeneous media. The reconstruction algorithm can be applied for such n, which assure a sufficient mesh zoom on the smallest scale in the composite.

4 Finite Element Method Equations of the Problem

The following variational equation is proposed to study dynamic equilibrium for the linear elastic system:

$$\int_\Omega \rho \ddot{u}_i \delta u_i d\Omega + \int_\Omega C_{ijkl} \varepsilon_{ij} \delta\varepsilon_{kl} d\Omega = \int_\Omega \rho f_i \delta u_i d\Omega + \int_{\partial\Omega_\sigma} \hat{t}_i \delta u_i d(\partial\Omega) \tag{28}$$

and u_i represents displacements of the system Ω with elastic properties and mass density defined by the elasticity tensor $C_{ijkl}(x)$ and $\rho=\rho(\mathbf{x})$; the vector $\hat{t}_i$ denotes the stress boundary conditions imposed on $\partial\Omega_\sigma \subset \partial\Omega$. Analogous equation for the homogenized medium has the following form:

$$\int_\Omega \rho^{(eff)} \ddot{u}_i \delta u_i d\Omega + \int_\Omega C_{ijkl}^{(eff)} \varepsilon_{ij} \delta\varepsilon_{kl} d\Omega = \int_\Omega \rho^{(eff)} f_i \delta u_i d\Omega + \int_{\partial\Omega_\sigma} \hat{t}_i \delta u_i d(\partial\Omega) \tag{29}$$

where all material properties of the real system are replaced with the effective parameters. As it is known [9], classical FEM discretization returns the following equations for real heterogeneous and homogenized systems are obtained:

$$M_{\alpha\beta}\ddot{q}_\beta + C_{\alpha\beta}\dot{q}_\beta + K_{\alpha\beta}q_\beta = Q_\alpha, \quad M_{\alpha\beta}^{(eff)}\ddot{\overline{q}}_\beta + C_{\alpha\beta}^{(eff)}\dot{\overline{q}}_\beta + K_{\alpha\beta}^{(eff)}\overline{q}_\beta = Q_\alpha. \tag{30}$$

The R.H.S. vector equals to 0 for free vibrations and then an eigenvalue problem is solved using the following matrix equations:

$$\left(K_{\alpha\beta} - \omega_{(\alpha)} M_{\alpha\beta}\right)\Phi_{\beta\gamma} = 0, \quad \left(K_{\alpha\beta}^{(eff)} - \overline{\omega}_{(\alpha)} M_{\alpha\beta}^{(eff)}\right)\overline{\Phi}_{\beta\gamma} = 0. \tag{31}$$

5 Computational Illustration

First, simply supported periodic composite beam is analyzed, where Young modulus E(x) and mass density in the periodicity cell are given by the following wavelets:

$$h(x) = \begin{cases} 20.0E9; 0 \le x \le 0.5 \\ 2.0E9\ 0.5 < x \le 1 \end{cases}, \quad m(x) = 2 + \frac{1}{\sqrt{2\pi}\sigma^3}\frac{x^2}{\sigma^2 - 1}\exp\left(\frac{-x^2}{2\sigma^2}\right), \sigma\text{=-0.4}; \tag{32}$$

$$E(x) = 10.0 \cdot h(x) + 2.0E9 \cdot m(x). \tag{33}$$

$$\tilde{h}(x) = \begin{cases} 200; 0 \le x \le 0.5 \\ 20; 0.5 < x \le 1 \end{cases}, \quad \rho(x) = 0.5 \cdot \tilde{h}(x) + 0.5 \cdot m(x). \tag{34}$$

The composite specimen is discretized using each time 128 2-noded linear finite elements with unitary inertia moments. The comparison starts from a collection of the eigenvalues reflecting different homogenization techniques given in Tab. 1. Further, the eigenvalues for heterogeneous beams are given for 1st order wavelet projection in Tab. 2, for 2nd order projection – in Tab. 3, 3rd order - in Tab. 4.

The eigenvalues computed for various homogenization models approximate the values computed for the real composite models with different accuracy - the weakest efficiency is detected in case of spatially averaged composite and the difference in relation to the real structure results increase together with the eigenvalue number and the projections order. The results obtained thanks to MRA projection are closer to those relevant to MRA homogenization for a single RVE in composite; classical homogenization is more effective for increasing number of the cells in this model.

Table 1. Eigenvalues for the simply supported homogenized composite beams

Eigenvalue	Spatial averaging	Classical approach	Multiresolutional model
1	1,184 E12	3,665 E11	6,228 E11
2	1,895 E13	5,864 E12	9,965 E12
3	9,592 E13	2,969 E13	5,045 E13
4	3,032 E14	9,383 E13	1,594 E14
5	7,401E14	2,291 E14	3,893 E14
6	1,535 E15	4,750 E14	8,072 E14

Table 2. Eigenvalues for the simply supported composite beam, 1st order wavelet projection

64 RVEs	32 RVEs	16 RVEs	8 RVEs	4 RVEs	2 RVEs	1 RVE
3,534 E11	3,535 E11	3,537 E11	3,550 E11	3,599 E11	3,829 E11	4,529 E11
5,656 E12	5,660 E12	5,679 E12	5,760 E12	6,137 E12	7,887 E12	2,593 E13
2,864 E13	2,870 E13	2,892 E13	2,991 E13	3,512 E13	4,973 E13	7,317 E13
9,056 E13	9,087 E13	9,216 E13	9,828 E13	1,315 E14	4,867 E14	3,512 E14
2,212 E14	2,224 E14	2,275 E14	2,536 E14	3,758 E14	6,743 E14	6,241 E14
4,591 E14	4,627 E14	4,786 E14	5,655 E14	8,448 E14	1,347 E15	1,678 E15

Table 3. Eigenvalues for the simply supported composite beam, 2nd order wavelet projection

32 RVEs	16 RVEs	8 RVEs	4 RVEs	2 RVEs	1 RVE
3,636 E11	3,639 E11	3,652 E11	3,703 E11	3,936 E11	4,604 E11
5,823 E12	5,842 E12	5,925 E12	6,309 E12	8,006 E12	2,603 E13
2,952 E13	2,975 E13	3,075 E13	3,605 E13	5,090 E13	7,420 E13
9,348 E13	9,480 E13	1,010 E14	1,334 E14	4,875 E14	3,531 E14
2,288 E14	2,340 E14	2,605 E14	3,846 E14	6,803 E14	6,292 E14
4,760 E14	4,921 E14	5,805 E14	8,641 E14	1,362 E15	1,690 E15

Table 4. Eigenvalues for the simply supported composite beam, 3rd order wavelet projection

16 RVEs	8 RVEs	4 RVEs	2 RVEs	1 RVE
3,662 E11	3,674 E11	3,726 E11	3,964 E11	4,664 E11
5,879 E12	5,962 E12	6,354 E12	8,121 E12	2,600 E13
2,993 E13	3,096 E13	3,637 E13	5,174 E13	7,479 E13
9,540 E13	1,017 E14	1,354 E14	4,876 E14	3,529 E14
2,355 E14	2,626 E14	3,903 E14	6,839 E14	6,341 E14
4,954 E14	5,857 E14	8,796 E14	1,373 E15	1,691 E15

Free vibrations for 2 and 3-bays periodic beams are solved using classical and homogenization-based FEM implementation. The unitary inertia momentum is taken in all computational cases, ten periodicity cells compose each bay, while material properties inserted in the numerical model are calculated by spatial averaging, classical and multiresolutional homogenization schemes and compared against the real structure response. First 10 eigenvalues changes for all these beams are contained in Figs. 1,2 – the resulting values are marked on the vertical axes, while the number of eigenvalue being computed – on the horizontal ones.

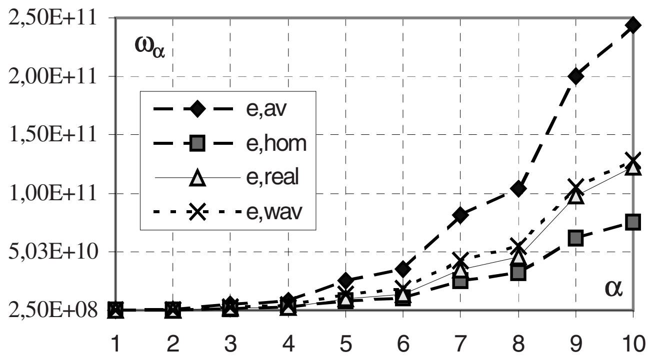

Fig. 1. Eigenvalues progress for various two-bays composite structures

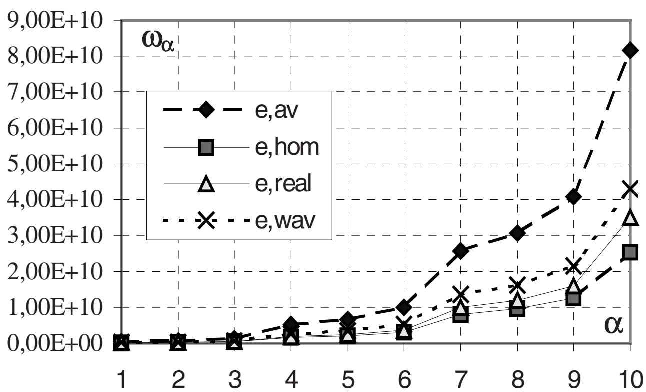

Fig. 2. Eigenvalues progress for various three-bays composite structures

Eigenvalues obtained for various homogenization models approximate the values computed for the real composite with different accuracy - the worst efficiency in eigenvalues modeling is detected in case of spatially averaged composite and the difference in relation to the real structure results increase together with the eigenvalue number. Wavelet-based and classical homogenization methods give more accurate results – the first method is better for smaller number of the bays, and classical homogenization approach is recommended in case of increasing number of the bays and the RVEs. The justification of this observation comes from the fact, that the wavelet function is less important for the increasing number of the periodicity cells in the structure. Another interesting result is that the efficiency of approximation of the maximum deflections for a multi-bay periodic composite beam by the deflections encountered for homogenized systems increase together with an increase of the total number of the bays.

6 Conclusions

The most important result of the homogenization-based Finite Element modeling of the periodic unidirectional composites is that the real composite behavior is rather well approximated by the homogenized model response. MRA homogenization technique giving more accurate approximation of the real structure behavior is decisively more complicated in numerical implementation since necessity of usage of the combined symbolic-FEM approach. The technique introduces new opportunities to calculate effective parameters for the composites with material properties approximated by various wavelet functions. A satisfactory agreement between the real and homogenized structures models enables the application to other transient problems with deterministic as well as stochastic material parameters.

Multiresolutional homogenization procedure has been established here using the Haar basis to determine complete mathematical equations for homogenized coefficients and to make implementation of the FEM-based homogenization analysis. As it was documented above, the Haar basis approximation gives sufficient approximation of various mathematical functions describing most of possible spatial distributions of composites physical properties.

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