Symbolic Calculation for Frölicher-Nijenhuis $\mathbb{R}$-Algebra for Exploring in Electromagnetic Field Theory* **

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Abstract. The principal aim of this work is the presentation of a symbolic calculation computer analysis for exploring electromagnetic fields for not inertial observer. Based on Frölicher-Nijenhuis super-Lie $\mathbb{R}$-algebra, we developed a learning environment for axiomatic classical electromagnetics and electrodynamics. A collection of programs developed on Mathematical programming environment has been builded for the $\mathbb{N}$-graded Graßmann Algebra, $\mathbb{Z}$-graded endomorphisms and graded commutators.

1 Introduction

In axiomatic classical electromagnetic is postulated that the density pseudo differential form J, $\operatorname{grade} J = d - 1 \in \mathbb{N}$ and the strength or 'magnetic flux' F, $\operatorname{grade} F = 2$ are absolute, are observer-free.

– Charge current conservation

$$\int_{\partial \mathrm{V}} J = 0 \quad \in \mathbb{R} \tag{1}$$

– Conservative force field interaction

$$\int_{\partial C} F = 0 \quad \in \mathbb{R}. \tag{2}$$

* Supported by el Consejo Nacional de Ciencia y Tecnología (CONACyT) de México, proyecto # U41214-F, and by UNAM, DGAPA, Programa de Apoyo a Proyectos de Investigacion e Innovacion Tecnologica, proyecto # IN 105402.

** Submitted to Lecture Notes in Computer Science: International Conference on Computational Science ICCS'04, June 2004, Krakow, Poland. Not yet in final form

*** Zbigniew Oziewicz is a member of Sistema Nacional de Investigadores, México, No de expediente 15337.

M. Bubak et al. (Eds.): ICCS 2004, LNCS 3037, pp. 552–556, 2004.

The first axioms is the conservation of the density pseudo-form of the Charge-Current. The second axiom is interpretd as the conservation of the work (on small test charge-current) in the space-time. See for complementary references and formal approach Cruz & Oziewicz [1, 2003].

Starting with such axiomatic approach we use the Frölicher and Nijenhuis[2, 3] Lie $\mathbb{R}$-operation to derive observer-dependent form of the four Maxwell equations.

In Section 2 we present some necessary definitions. Section 3 is devoted to present the principal lines of the algorithm. Finally in Section 4 we present conclusions and objectives for future work.

2 Basic Definitions

Let $\mathcal{F}$ be an $\mathbb{R}$-algebra of scalar fields on Space-Time. The $\mathcal{F}$-modul of the differential 1-forms is denoted by M.

Definition 1 (Graßmann algebra of differential forms). *The* $\mathbb{N}$*-graded Graßmann* $\mathcal{F}$*-algebra of differential forms is denoted by:*

$$M^{\wedge} \equiv \oplus M^{\wedge i} = M^{\wedge 0} \oplus M^{\wedge 1} \oplus M^{\wedge 2} + \cdots, \quad M^{\wedge 0} \equiv \mathcal{F}, \quad M^{\wedge 1} \equiv M. \tag{3}$$

Definition 2 (Observer field). *Is a not necessarily integrable product structure that splits* $\mathcal{F}$*-modules of the differential one-forms* M *and one-vector fields,* M^**, into 'time-' and 'space-'* $\mathcal{F}$*-submodules. An idempotent* $a^2 = a \in \mathrm{der}(M^{\wedge})$ *is said to be observer if tr* $a = 1$*,and dim im* $a =$ *dim of time* $= 1$.

In the present paper we represent an observer by κ. A 1-dimensional $\mathcal{F}$-modul imκ, is an ideal in the Graßmann $\mathcal{F}$-algebra, and the projector $(\mathrm{id} - \kappa)$, is an $\mathcal{F}$-algebra map

$$(\mathrm{id} - \kappa) \in \mathrm{alg}_{\mathcal{F}} \left(M^{\wedge}, M^{\wedge}/\mathrm{im}\kappa \right). \tag{4}$$

Definition 3 (Derivations of Graßmann algebra). *Let* $|\mathcal{A}| \equiv \mathrm{grade}\, \mathcal{A} \in \mathbb{Z}$. *A map* $\mathcal{A} : M^{\wedge} \to M^{\wedge}$ *is said to be a* graded $\mathcal{F}$*-derivation,* $\mathcal{A} \in$ *der*$(M^{\wedge})$*, if Leibniz axiom holds:*

$$\mathcal{A}(\alpha \wedge \beta) = (\mathcal{A}\alpha) \wedge \beta + (-)^{|\mathcal{A}||\alpha|} \alpha \wedge \mathcal{A}\beta. \quad \forall\, \alpha, \beta \in M^{\wedge} \tag{5}$$

Example: every observer is $\mathcal{F}$-derivation, $a \in \mathrm{der}_{\mathcal{F}}(M^{\wedge})$, grade $a = 0 \in \mathbb{N}$.

Definition 4 (Lie Super ($\equiv \mathbb{Z}_2$-graded) $\mathcal{F}$-algebra of derivations). The graded commutator (bracket) *is:*

$$\begin{aligned}
(End\, M^{\wedge}) \otimes (End\, M^{\wedge}) &\xrightarrow[\textit{super-bracket}]{\{\ ,\ \}} (End\, M^{\wedge}),\\
\{\mathcal{A}, \mathcal{B}\} &\equiv \mathcal{A} \circ \mathcal{B} - (-)^{|\mathcal{A}||\mathcal{B}|} \mathcal{B} \circ \mathcal{A} \in \mathrm{der}(M^{\wedge}),\\
\mathrm{grade}\{\mathcal{A}, \mathcal{B}\} &= \mathrm{grade}\, \mathcal{A} + \mathrm{grade}\, \mathcal{B}.
\end{aligned}$$

For the graded commutator the graded version of the Jacobi identity holds,

$$\{A, \{B, C\}\} = \{\{A, B\}, C\} + (-)^{AB}\{B, \{A, C\}\}. \tag{6}$$

if $\mathcal{A}$ and $\mathcal{B}$ be graded derivations; $\mathcal{A}, \mathcal{B} \in \mathrm{der}(M^\wedge)$. Then $\{\mathcal{A}, \mathcal{B}\} \in \mathrm{der}(M^\wedge)$.

Definition 5 (Ślebodziński Lie derivation). *Let* $\mathcal{A} \in \mathrm{der}(M^\wedge)$ *be any derivation. The* $\mathbb{R}$*-derivation introduced in 1931 [4] is:*

$$\mathcal{L}_{\mathcal{A}} \equiv \{\mathcal{A}, d\} \in \mathrm{der}_{\mathbb{R}}(M^\wedge). \tag{7}$$

Definition 6 (Frölicher and Nijenhuis $\mathbb{R}$-algebra). *A derivation* $[\kappa, \rho]_{FN} \in \mathrm{der}_{\mathcal{F}}(M^\wedge)$ *exit, such that*

$$\begin{aligned} \mathcal{L}_{[\kappa,\rho]_{FN}} &\equiv \{\mathcal{L}_\kappa, \mathcal{L}_\rho\} \in \mathrm{der}_{\mathbb{R}}(M^\wedge), \\ [\kappa, \rho]_{FN} &= (-1)^{\kappa+\rho+\kappa\rho} \cdot [\rho, \kappa]_{FN}. \end{aligned} \tag{8}$$

The Frölicher-Nijenhuis [1956] binary operation on the Lie $M^\wedge$*-module* $\mathrm{der}_{\mathcal{F}}(M^\wedge)$, *denoted by* $[\cdot, \cdot]_{FN} \in \mathrm{der}_{\mathcal{F}}(M^\wedge)$, *with* $\mathrm{grade}[\cdot, \cdot] = +1$, *is an example of the Gerstenhaber* $\mathbb{R}$*-algebra.*

An arbitrary derivation $D \in \mathrm{der}_{\mathbb{R}}(M^\wedge)$ possess the following unique decomposition

$$D = (\mathcal{L} \circ i + i \circ \mathcal{L})D = \{i_D, d\} + i_{\{D,d\}}. \tag{9}$$

The generalization of the Frölicher and Nijenhuis decomposition for an extension of DGA, was given in [Oziewicz 1991].

3 Outline of Algorithm

The builded functions contains:

- Appropriate data types designed for: $\mathbb{N}$-graded $\mathcal{F}$-algebra of the differential forms $M^\wedge$. $\mathbb{Z}$-homogeneous graded endomorphism $\mathrm{End}\,(M^\wedge)$, Poisson graded commutator (bracket) $[\ ,\ ]_{FN}$, etc.
- Constructors for graded $\mathcal{F}$- and $\mathbb{R}$-derivations, Lie $\mathcal{F}$- and $\mathbb{R}$-derivation, observers $\in$ der, and creations and annihilation operators $\in$ End. Every annihilation opertator is a $\mathcal{F}$-derivation.
- Procedures for Graßmann multiplication $\wedge$, composition $\circ$ of graded endomorphisms and Jacobi identity for Lie and Poisson $\mathbb{R}$-brackets.

A grammar definition is necessary in the way that the abstract representation be in accordance with mathematical formal definition of objets. By example: for the $\mathbb{N}$-graded Graßmann $\mathcal{F}$-algebra of differential forms $M^\wedge$we have:

- $M^\wedge$ is generated by $\mathcal{F}$ and M ie $M^\wedge = \mathrm{gen}\{\mathcal{F}, M\}$.
- If $\alpha, \beta \in M^\wedge$ then $\alpha + \beta \in M^\wedge$ and $\alpha \wedge \beta \in M^\wedge$.

- $\operatorname{grade}(\alpha \wedge \beta) = \operatorname{grade}\alpha + \operatorname{grade}\beta + \operatorname{grade}\wedge, \operatorname{grade}\wedge = 0 \in \mathbb{N}$.
- $\forall \alpha \in M \Rightarrow \operatorname{grade}\alpha = 1$ and $\alpha \wedge \alpha = 0$.
- The creation Graßmann operators is defined by: $e_\alpha \beta \equiv \alpha \wedge \beta$, $e_\alpha \in \operatorname{End}(M^\wedge)$.
- The 'inner' or 'interior' action 'i' of $M^\wedge$ on $M^{*\wedge}$ is said to be the annihilation operator $i_M \in \operatorname{der}_{\mathcal{F}}(M^\wedge)$.

In practical computation operators e and i play an important role, e is a data type constructor and i is a data type selector:

$$M^\wedge \otimes M^\wedge \xrightarrow{\ e\ } M^\wedge$$

$$M^* \otimes M^\wedge \xrightarrow{\ i\ } M^\wedge.$$

For derivations operator we need the set of symbols: $\{\text{name}, \text{grade}, +\}$. With those and the Leibniz axiom we build the necessary expressions for the derivation operation, and for graded commutator we need $\{\text{name}, \text{grade}, +, \circ\}$.

For Frölicher-Nijenhuis binary operation we use the explicit form of the bracket: Let $\rho, \kappa \in \operatorname{der}$ and $\delta_{\kappa\circ\rho} \in \operatorname{der}$

$$[\rho, \kappa]_{FN} \equiv -\{\kappa, \mathcal{L}_\rho\} - \{\delta_{\kappa\circ\rho}, d\} \quad \in \operatorname{der}_{\mathbb{R}} \tag{10}$$

Here we use a $\mathcal{F}$-modul map $p \in \hom_{\mathcal{F}}(M, M^\wedge)$ lifted to the unique ($\mathbb{Z}_2$-graded) $\mathcal{F}$-derivation

$$\hom_{\mathcal{F}}(M, M^\wedge) \simeq M^\wedge \otimes_{\mathcal{F}} M^* \ni p \mapsto \delta_p \in \operatorname{der}_{\mathcal{F}}(M^\wedge),$$

with $\operatorname{grade}(\delta) = 0$, such that $\delta_p|\mathcal{F} = 0$ and $\delta_p|M = p$.

4 Conclusions and Future Work

The axiomatic approach could possess pedagogical advantage in teaching the fundamentals of electromagnetic laws as explicitly observer-dependent and gives a power tool in the exploration of physical laws. Symbolic calculus offer an appropriate tool in the exploration of abstract algebra and its applications. Future work will be devoted to

- Get procedures for splitting differential forms and differential equation in Space Time,
- automatize algorithmically all expressions of four Maxwell equations for non inertial observers
- build higher order words of Lie Poisson brackets for Lie derivation of observer operators $\{k, d\}$ and find all identities inside the Lie algebra $\operatorname{gen}\{d, k_1, k_2\}$.

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