Parallel Chip Firing Game Associated with *n-cube* Edges Orientations

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Abstract. We study the cycles generated by the chip firing game associated with *n-cube* orientations. We consider a particular class of partitions of vertices of *n-cubes* called *left cyclic partitions* that induce *parallel periodic evolutions*. Using this combinatorical model, we show that cycles generated by *parallel evolutions* are of even lengths from 2 to 2^n on H_n $(n \geq 1)$, and of odd lengths different from 3 and ranging from 1 to $2^{n-1}-1$ on H_n $(n \geq 4)$. However, the question weather there exist *parallel evolutions* with period greater that 2^n remains opened.

1 Introduction

A state in the *parallel chip firing game* played on a directed graph $G = (X, A)$ is a mapping $x : V \rightarrow N$ which can be viewed as a distribution of chips onto the vertices of G. In a transition of the game, a state x is transformed into a new state by activating all nodes with more chips that its out-neighbors. The evolution is ultimately *periodic* because the total number of chips remains constant. More precisely, if $x^t, t \geq 0$, denotes the state of the system at time t, then there exists an integer q called *transient length* and another integer p called *period* or *cycle length* such that

$$x^{t+p} = x^t \text{ for } t \geq q, \text{ and } x^{t+p'} \neq x^t \text{ for } p' < p. \tag{1}$$

In this paper, we investigate the dynamics generated by the chip firing game associated with *n-cube* orientations and we provide a model to study possible periods generated in this particular case.

2 Recurrent Construction of Parallel Cycles

Definition 1. *A partition $S_0 \cup S_1 \cup ... \cup S_{k-1}$ of the vertices of an n*-cube *is called a* left cyclic *partition if the two following statements hold.*

• *For all i from 0 to $k-1$, every vertex of S_i has a neighbor in S_{i-1}, where index operations are performed modulo k.*

• *For all i from 0 to $k-1$, there is no edge between two vertices of S_i.*

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The model of *left cyclic partition* clearly gives a characteristic of *parallel evolutions* with unique firing within a cycle. In this paper, we investigate on possible configurations. We first present the construction of *left cyclic partitions* of even lengths.

Lemma 1. *An n-cube admits left cyclic partitions of all even lengths from* 2 *to* 2^n.

Proof. Let $H_n = (V, E)$ be an *n-cube* an let p be an even integer between 2 and 2^n. It is well known that, since p is even, there is a cycle $[x_0, x_1, ..., x_{p-1}, x_0]$ of length p in H_n. Now, for every vertex u, let $\Gamma(u)$ denote the set of all neighbors of u in H_n. This notation is naturally extended to a set of vertices. A *left cyclic partition* of order p is obtained as follows.

For $i = 0, ..., p-1$ **do**
  $S_i \leftarrow \{x_i\}$
endfor
$S = V - \{x_0, x_1, ... x_{p-1}\}$
while $(S \neq \emptyset)$ **do**
  For $i \leftarrow 0$ **to** $p-1$ **do**
    $S_{i+1} \leftarrow S_{i+1} \cup (\Gamma(S_i) \cap S)$
    $S \leftarrow S - (\Gamma(S_i) \cap S)$
  endfor
endwhile

It is obvious that $S_0, ..., S_{p-1}$ is a partition of V and that every vertex in S_i has at least one neighbor in S_{i-1}. So we just need to show that two vertices of the same subset S_i cannot be neighbors. Let a and b be two vertices of S_i.

- There is a path from a to x_0 of length ℓ_1 such that $\ell_1 = i$ **mod** p,
- There is a path from b to x_0 of length ℓ_2 such that $\ell_2 = i$ **mod** p,

Since p is even, it follows that $\ell_1 = \ell_2$ **mod** 2. Hence, if a and b were neighbors, there would exist a cyclic path of odd length $\ell_1 + \ell_2 + 1$ joining a and b in H_n, which is not possible since H_n is a bipartite graph. This shows that two vertices of the same subset cannot be neighbors.

We now turn to the construction of *left cyclic partitions* of odd lengths.

Lemma 2. *If S_0, S_1, S_2 is a* left cyclic partition *of $H_n, n \geq 2$, then every vertex of S_i has at least two neighbors in S_{i-1} for $i = 0, 1, 2$.*

Proof. Because of symmetry considerations, we can assume that $i = 2$. So let x be a vertex of S_2. From the definition of *left cyclic partitions*,

- x has a neighbor $x \oplus e_j$ in S_1, where $\oplus$ is the XOR operator and e_j is a vector of the canonical basis.
- similarly, $x \oplus e_j$ has a neighbor $x \oplus e_j \oplus e_k$ in S_0.

Now consider the vertex $x \oplus e_k$.

- It is a neighbor of x, hence it does not belong to S_2.
- It is a neighbor of $x \oplus e_j \oplus e_k$, hence it does not belong to S_0.

It then follows that $x \oplus e_k$ belongs to S_1, hence x admits two neighbors $x \oplus e_j$ and $x \oplus e_k$ which are both in S_1.

Lemma 3. *If H_n, $n \geq 3$ admits a* left cyclic partition *of order* 3*, then H_{n-1} admits a* left cyclic partition *of order* 3.

Proof. Obvious.

Proposition 1. *n-cubes do not admit left cyclic partitions of order 3.*

Proof. From lemma 2, if a hypercube H admits a *left-cyclic partition* of order 3, then $|H| \geq 6$, which is not the case for H_2. By application of lemma 3, we deduce that no *n-cube*, $n \geq 3$ admits a *left cyclic partition* of order 3.

Proposition 2. *If $S_0, ..., S_{p-1}$ is a left cyclic partition of odd order p of H_n, then $p \leq 2^{n-1} - 1$.*

Proof. We just have to show that in such a case, $|S_i| \geq 2$ for $i = 0, ..., p-1$. Indeed, starting from a vertex $a_{p-1} \in S_{p-1}$, we construct a chain $[a_{p-1}$,a_{p-2},... , a_0, b_{p-1}, b_{p-2}, ..., $b_0]$ such that $a_i, b_i \in S_i$ for $i = 0, ..., p{-}1$. It is clear that $a_i \neq b_i, i = 0, ..., p-1$, otherwise we would have displayed a closed path of odd length in H_n which is not possible.

Lemma 4. *If H_n admits a* left cyclic partition *of order p, then H_{n+1} admits left cyclic partition of order p.*

Proof. If $S_0, ..., S_{p-1}$ is a left cyclic partition of order p in H_n, then $1S_i \cup 0S_{i-1}, i = 0, ..., p-1$ is a also *left cyclic partition* of order p in H_{n+1}.

Lemma 5. *If H_n admits a* left cyclic partition *of odd order p, $p \geq 5$ then H_{n+1} admits a* left cyclic partition *of order $2p-1$. Moreover, if $p \geq 7$, then H_{n+1} admits a left cyclic partition of order $2p-3$.*

Proof. Let $S_0, S_1, ..., S_{p-1}$ be a *left cyclic partition* of odd order p.
- Case $p \geq 5$

The following sequence is a *left cyclic partition* of order $2p{-}1$ in H_{n+1}.
$0S_0$, $1S_0 \cup 0S_1$, $1S_1$, $1S_2$, $0S_2$, $0S_3$, $1S_3$ ..., $1S_{2i}$, $0S_{2i}$, $0S_{2i+1}$, $1S_{2i+1}$,..., $1S_{p-3}$, $0S_{p-3}$, $0S_{p-2}$, $1S_{p-2}$, $1S_{p-1}$, $0S_{p-1}$.
- Case $p \geq 7$

A *left cyclic partition* of order $2p{-}3$ in H_{n+1} is obtained from the *left cyclic partition* exhibited in the case $p \geq 5$ by replacing the subsequence $1S_2$, $0S_2$, $0S_3$, $1S_3$, $1S_4$, $0S_4$, $0S_5$, $1S_5$ by $1S_2$, $0S_2 \cup 1S_3$, $0S_3$, $0S_4$, $1S_4 \cup 0S_5$, $1S_5$.

Lemma 6. *H_4 admits left cyclic partitions of orders* 5 *and* 7*.*

Proof.
- A *left cyclic partition* of order 5 in H_4 is the following :

$\{0000, 1101\}$, $\{0001, 1100, 0010, 1111\}$, $\{0110, 1011\}$, $\{0100, 0111, 1001, 1010\}$, $\{0011, 0101, 1000, 1110\}$.
- A *left cyclic partition* of order 7 in H_4 is the following :

$\{0000, 1101\}$, $\{0001, 1100\}$, $\{0011, 1110\}$, $\{0010, 1111\}$, $\{0110, 1011\}$, $\{0100, 0111, 1001, 1010\}$, $\{0101, 1000\}$.

Lemma 7. *H_n, $n \geq 4$, admits a* left cyclic *partition of order* $2^{n-1} - 1$.

Proof. Consider the sequence $\{u_i; 0 \leq i \leq 2^{n-1}-1\}$, defined by $u_i = bin(i) \oplus bin(i/2)$, where $bin(x)$ is the n-position binary representation of the integer x, and symbol / denotes integer division. It can be easilly checked that this sequence corresponds to a *hamiltonian cycle* in H_{n-1}. Now, let us denote $v_i = u_i \oplus 1 \oplus 2^{n-2}$ (i.e. v_i is obtained from u_i by changing the first and last bits) and $N = 2^n$. It is also easy to check that $\{v_i; 0 \leq i \leq 2^{n-1}-1\}$ is a *hamiltonian cycle* of H_{n-1}.Now, observe that $0u_i \oplus 1v_i = 2^{n-1} \oplus (u_i \oplus v_i) = 2^{n-1} \oplus 1 \oplus 2^{n-2}$. Hence, $0u_i$ and $1v_i$ are not neighbors in the hypercube H_n . On the other hand, $u_{N-4} = 100...010$, $u_{N-2} = 10...01$, $u_{N-1} = 10...0$ and $v_0 = u_0 \oplus 1 \oplus 2^{n-2} = 10...01 = u_{N-2}$. Hence, $0u_{N-4}, 0u_{N-1}, 0u_{N-2}, 1v_0$ is a chain of H_n. Moreover, $v_{N-4} = 0...011$, $v_{N-2} = 0...0 = u_0$ and $v_{N-1} = 0...01 = u_1$. Hence $1v_{N-4}, 1v_{N-1}, 1v_{N-2}, 0u_0$ is a chain of H_n. Hence, by considering the two chains and the two previous *hamiltonian cycles*, we see that the partition $\{0u_0, 1v_0\}, \{0u_{N-4}, 1v_{N-4}\}$, $\{0u_{N-3}, 0u_{N-1}, 1v_{N-3}, 1v_{N-1}\}$, $\{0u_{N-2}, 1v_{N-2}\}$ is a *left cyclic partition.*

Proposition 3. *H_n, $n \geq 4$, admits left cyclic partitions of all odd orders from* 5 *to* $2^{n-1} - 1$.

Proof. We proceed by induction. For $n = 4$ the result follows from lemma 6. Assuming that the result holds for $n \geq 4$, let us consider an $(n+1)$-*cube* together with an odd integer $p \in [5, 2^n - 1]$.

- Case 1 : $5 \leq p \leq 2^{n-1} - 1$. The result follows from the induction hypothesis and lemma 4.
- Case 2 : $2^{n-1}-1 < p < 2^n - 1$. There is an odd integer q, $7 < q < 2^{n-1} - 1$, such that $p = 2q - 1$ or $p = 2q - 3$. The result follows from the induction hypothesis and lemma 5.
- Case 3 : $p = 2^n - 1$. The result follows from lemma 7.

Theorem 1. *The cycles generated by the parallel chip firing game associated with n-cube orientations, $n \geq 4$, are of even lengths from* 2 *to* 2^n*, and of odd lengths different from 3 and ranging from* 1 *to* $2^{n-1}-1$.

3 Conclusion

In this paper, we have given, in the particular case of *parallel evolutions*, a range of possible periods which can be generated by the chip firing game associated with *n-cube* orientations. The question of the existence of cycles with length greater than 2^n remains to be clarified. We suggest, for the case of n-cubes, to consider a recurrent approach based on the sub-evolutions induced in every face of the n-cube by a *block sequential evolution.*

References

1. M. A. Kiwi, R. Ndoundam, M. Tchuente and E. Goles *No polynomial bound for the period of the parallel chip firing game on graphs*, Theoretical Computer Science, 136, pp. 527-532, 1994.