Approximations of the rate of growth of switched linear systems

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Abstract. The joint spectral radius of a set of matrices is a measure of the maximal asymptotic growth rate that can be obtained by forming long products of matrices taken from the set. This quantity appears in a number of application contexts, in particular it characterizes the growth rate of switched linear systems. The joint spectral radius is notoriously difficult to compute and to approximate. We introduce in this paper the first polynomial time approximations of guaranteed precision. We provide an approximation $\hat{\rho}$ that is based on ellipsoid norms, that can be computed by convex optimization, and that is such that the joint spectral radius belongs to the interval $[\hat{\rho}/\sqrt{n}, \hat{\rho}]$, where n is the dimension of the matrices. We also provide a simple approximation for the special case where the entries of all the matrices are non-negative; in this case the approximation is proved to be within a factor at most m (m is the number of matrices) of the exact value.

1 Introduction

Let M be a set of square real matrices. The *trajectories* associated to the discrete-time *switched linear system* generated by the set M are given by the vector sequences (x_i) defined by the discrete linear inclusion:

$$x_{k+1} \in \{A_i x_k : A_i \in M\} \quad \forall k.$$

A switched linear system is said to be *stable* if all its trajectories converge to the origin. This condition is equivalent to the condition that all infinite products of matrices taken from the set M converge to zero. Stability can be equivalently expressed by requiring the joint spectral radius of the set M to be less than one. The *joint spectral radius* of a set of matrices is a

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quantity, introduced by Rota and Strang in the early 60's, that measures the maximal asymptotic growth rate that can be obtained by forming long products of matrices taken from the set; see [16]. More formally, the joint spectral radius of the set M is defined by:

$$\rho(M) := \limsup_{k\to\infty} \rho_k(M),$$

$$\text{where} \quad \rho_k(M) = \sup_{A_1,\ldots,A_k\in M} \|A_k \cdots A_1\|^{1/k}.$$

The values of $\rho_k(M)$ do in general depend on the chosen norm but one can show that the limit value $\rho(M)$ does not. When the set M consists of only one matrix A, the joint spectral radius coincides with the usual notion of spectral radius of a single matrix, which is equal to the maximum magnitude of the eigenvalues of the matrix. In the previous definition, if we had used the spectral radius instead of the norm, we would have obtained the *generalized spectral radius*:

$$\rho'(M) = \limsup_{k\to\infty} \rho'_k(M),$$

$$\text{where} \quad \rho'_k(M) = \sup_{A_1,\ldots,A_k\in M} \rho(A_k \cdots A_1)^{1/k}.$$

This quantity appears for the first time in [5], where it is also conjectured that in the case of bounded sets of matrices (and in particular for finite sets of matrices), the joint and generalized spectral radii are equal. This conjecture is proved to be correct in [2].

Questions related to the computability of the joint spectral radius of sets of matrices have been posed in [19] and [13]. The joint spectral radius can easily be approximated to any desired accuracy. Indeed, the following bounds, proved in [13],

$$\rho'_k(M) \le \rho(M) \le \rho_k(M)$$

can be evaluated for increasing values of k and lead to arbitrary close approximations of ρ. These are however expensive calculations. It is proved in [20] that, unless $P = NP$, there is in fact no polynomial-time approximation algorithm for the joint spectral radius of two matrices.

In this paper, we provide two easily computable approximations of the joint spectral radius for finite sets of matrices. The first approximation that we provide, $\hat{\rho}$, is based on the computation of a common quadratic Lyapunov function, or, equivalently, on the computation of an ellipsoid norm. This approximation has the advantage that it can be expressed as a convex optimization problem for which efficient algorithms exist. This first approximation satisfies

$$\frac{1}{\sqrt{n}}\,\hat{\rho} \leq \rho \leq \hat{\rho}$$

where n is the dimension of the matrices. For the special case of symmetric matrices, triangular matrices, or for sets of matrices that have a solvable Lie algebra, we prove equality between the joint spectral radius and its approximation, $\rho = \hat{\rho}$.

We then prove a result of independent interest: the largest spectral radius of the matrices in the convex hull of $M = \{A_1, \ldots, A_m\}$ is a lower bound on the joint spectral radius of M:

$$\max_{0 \leq \lambda_i \leq 1, \sum \lambda_i = 1} \rho(\sum_i \lambda_i A_i) \leq \rho(M).$$

By using this inequality, we prove a simple bound for the joint spectral radius of sets of matrices that have *non-negative entries*. The spectral radius of the matrix S whose entries are the componentwise maximum of the entries of the matrices in M satisfies

$$\frac{\rho(S)}{m} \leq \rho(M) \leq \rho(S)$$

where m is the number of matrices in the set. In this expression, M is a set of matrices, whereas S is a single matrix.

The problem of computing approximations of the joint spectral radius is raised and analyzed in a number of recent contributions. In [15], the exponential number of products that appear in the naive computation of ρ'_k is reduced by avoiding duplicate computation of cyclic permutations; the total number of product to consider remains however exponential. In [7], an algorithm based on the above idea is presented. The algorithm gives arbitrarily small intervals for the joint spectral radius, but no rate of convergence is proved.

This paper gives the first polynomial-time approximations of guaranteed precision. The paper is organized as follows. In the next section, we define the joint spectral radius approximation based on ellipsoid norms. In Section 3, we describe situations for which this approximation is exact, and situations for which it is not. In Section 4, we prove the inequality $\hat{\rho}(M)/\sqrt{n} \leq \rho$ by using a geometrical property of ellipsoids known as John's ellipsoid theorem. Finally, in Section 5 we provide an under-approximation of the joint spectral radius based on the spectral radius of all convex combinations of the matrices in the set M and use this result to prove an approximation for sets of non-negative matrices.

2 The ellipsoid norm approximation

The joint spectral radius can be defined by an extremal norm property. The statement of the following theorem is compiled from results in [12] and also [1].

Theorem 1. *Let $\rho(M)$ be the joint spectral radius of the finite set of matrices M. Then:*

1. *There exists a vector norm $\|.\|_*$ for which $\|A_i x\|_* \leq \rho(M)\ \|x\|_*$, $\forall x$ and $\forall A_i \in M$;*
2. *$\rho(M) = \inf_{\|.\|} \left(\max_{A_i \in M} \|A_i\|\right)$.*

The joint spectral radius is thus given by the infimum over all possible matrix norms of the largest norm of the matrices in the set. A norm achieving this infimum is said to be *extremal* for the set (not every set of matrices possesses an extremal norm, see [21] for a discussion of this issue). In [12], Kozyakin describes the theoretical construction of such an extremal norm. This method is not explicit and partly relies on the a-priori knowledge of the numerical value of the joint spectral radius. One can of course not hope enumerating all possible matrix norms for computing the joint spectral radius, but we can enumerate particular sets of norms. Our first approximation of the joint spectral radius is obtained by finding, among all ellipsoid norms $\|\cdot\|_P$, one that minimizes $\max_i \|A_i\|_P$.

Let us briefly recall the definition of the ellipsoid norm. Let P be a positive definite matrix[1]; the vector P-norm is defined as $\|x\|_P = \sqrt{x^T P x}$. Associated to this vector norm, there is an induced matrix norm:

$$|||A_i|||_P = \sup_x \frac{\|A_i x\|_P}{\|x\|_P} = \sup_x \frac{\sqrt{x^T A_i^T P A_i x}}{\sqrt{x^T P x}} \quad . \tag{1}$$

Further on, we will use the notation $\|.\|_P$ for both the vector and matrix norms. Let us now define the *ellipsoid norm approximation* of the joint spectral radius by:

$$\hat{\rho}(M) = \inf_{P \succ 0} \max_{A_i \in M} \|A_i\|_P.$$

The infimum on all quadratic norms cannot be lower than the infimum on all possible norms and so it immediately follows from Theorem 1 that

[1] Positive definiteness is denoted $\succ 0$ and positive semi-definiteness is denoted $\succeq 0$.

$\rho(M) \leq \hat{\rho}(M)$. The ellipsoid norm approximation can be computed as follows. Notice first that the definition implies that

$$\begin{aligned} \forall x,\ \sqrt{x^T A_i^T P A_i x} &\leq \|A_i\|_P \sqrt{x^T P x} \\ \forall x,\ x^T (A_i^T P A_i - \|A_i\|_P^2 P) x &\leq 0 \\ A_i^T P A_i - \|A_i\|_P^2 P &\preceq 0\ . \end{aligned}$$

One can therefore think of $\|A_i\|_P$ as the smallest scalar value γ for which $A_i^T P A_i \preceq \gamma^2 P$ for some $P \succ 0$. The ellipsoid norm approximation of a set $M = \{A_1, \ldots, A_m\}$ is thus equal to the smallest scalar γ for which there is a solution $P \succ 0$ to $A_i^T P A_i \preceq \gamma^2 P$, $\forall i$; a problem that can be solved efficiently by convex optimization.

A natural question to ask is how good this approximation is in the general case. In the next section, we describe situations for which the approximation is equal to the joint spectral radius, and we provide an example for which the approximation is larger than the joint spectral radius.

3 The joint spectral radius and its approximation

We prove in this section that the joint spectral radius and the ellipsoid norm approximation are equal (and are equal to the maximal spectral radius) in the following situations: all matrices are symmetric, all matrices are triangular or, more generally, the Lie algebra associated to the matrices is solvable. We close the section with an example for which the joint spectral radius and its approximation are different. We start with the case of symmetric matrices:

Proposition 1. *For a set of symmetric matrices, the joint spectral radius and its ellipsoid norm approximation are equal and are equal to the largest spectral radius of the matrices in the set.*

Proof. Using the identity I as matrix P, we get $A_i^2 \preceq \|A_i\|_I^2 I$, so that $\rho(A_i) = \|A_i\|_I$. Knowing that $\rho(A_i) \leq \inf_{P \succ 0} \|A_i\|_P$, we have actually $\rho(A_i) = \inf_{P \succ 0} \|A_i\|_P$, which finally yields $\max_i \rho(A_i) = \hat{\rho}(M)$. □

In order to derive our result for triangular matrices, we first establish a discrete-time analog to a continuous-time result established in [14] on the existence of a common quadratic Lyapunov function for switched linear systems.

Lemma 1. *Let M be the set $\{A_1, \ldots, A_m\}$ and consider the discrete-time switched linear system*

$$x_{k+1} = A_{i_k}\, x_k \quad A_{i_k} \in M.$$

If the switched system is stable and the matrices are upper-triangular, then there exists a common quadratic Lyapunov function in the form of a diagonal matrix.

Proof. Let $\{A_i, \ldots, A_m\}$ be a set of upper-triangular (possibly complex) matrices and P the candidate Lyapunov function (diagonal, real):

$$A_i = \begin{pmatrix} a^i_{11} & a^i_{12} & \ldots & a^i_{1n} \\ 0 & a^i_{22} & \ldots & a^i_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a^i_{nn} \end{pmatrix}, P = \begin{pmatrix} p_1 & 0 & \ldots & 0 \\ 0 & p_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_n \end{pmatrix}, p_k > 0, \forall k \ .$$

For P to be a Lyapunov function of $x_{k+1} = A_i x_k$ (fixed A_i), the following relation has to hold:

$$P - A_i^* P A_i \succ 0 \ .$$

Developing $P - A_i^* P A_i$, we get:

$$\begin{aligned} &\begin{pmatrix} p_1 & 0 & \ldots & 0 \\ 0 & p_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_n \end{pmatrix} - \begin{pmatrix} {a^i_{11}}^* & 0 & \ldots & 0 \\ {a^i_{12}}^* & {a^i_{22}}^* & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ {a^i_{1n}}^* & {a^i_{2n}}^* & \ldots & {a^i_{nn}}^* \end{pmatrix} \begin{pmatrix} p_1 & 0 & \ldots & 0 \\ 0 & p_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_n \end{pmatrix} \begin{pmatrix} a^i_{11} & a^i_{12} & \ldots & a^i_{1n} \\ 0 & a^i_{22} & \ldots & a^i_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a^i_{nn} \end{pmatrix} \\ &= \begin{pmatrix} p_1 & 0 & \ldots & 0 \\ 0 & p_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_n \end{pmatrix} - \begin{pmatrix} {a^i_{11}}^* p_1 & 0 & \ldots & 0 \\ {a^i_{12}}^* p_1 & {a^i_{22}}^* p_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ {a^i_{1n}}^* p_1 & {a^i_{2n}}^* p_2 & \ldots & {a^i_{nn}}^* p_n \end{pmatrix} \begin{pmatrix} a^i_{11} & a^i_{12} & \ldots & a^i_{1n} \\ 0 & a^i_{22} & \ldots & a^i_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a^i_{nn} \end{pmatrix} \end{aligned}$$

which yields

$$\begin{pmatrix} (1 - |{a^i}_{11}|^2) p_1 & -{a^i_{11}}^* a^i_{12} p_1 & \ldots \\ -a^i_{11} {a^i_{12}}^* p_1 & -|{a^i}_{12}|^2 p_1 + (1 - |{a^i}_{22}|^2) p_2 & \ldots \\ \vdots & \vdots & \ddots \\ -a^i_{11} {a^i_{1n}}^* p_1 & -a^i_{12} {a^i_{1n}}^* p_1 - a^i_{22} {a^i_{2n}}^* p_2 & \ldots \end{pmatrix} \succ 0 \ . \tag{2}$$

The first thing to note is that this matrix is Hermitian, and so its leading principal minors are real (see [9]).

As A_i is assumed to be stable, $a^i_{jj} < 1, \forall j$. The first diagonal element in (2) is therefore positive, for any value of p_1. Let it be chosen as 1. Moreover, the value of p_2 can be chosen in such a way that the (2×2) leading principal minor is positive. Indeed, p_2 only appears in its last diagonal element, and its coefficient $(1 - |a^i{}_{22}|^2)$ is positive, as $a^i{}_{22} < 1$. So, taking p_2 such that

$$\begin{vmatrix} (1-|a^i{}_{11}|^2) & -{a^i_{11}}^* a^i_{12} \\ -a^i_{11}{a^i_{12}}^* & -|a^i{}_{12}|^2 + (1-|a^i{}_{22}|^2)p_2 \end{vmatrix} > 0$$

is possible, and simple developments give the following condition:

$$p_2 > \frac{1}{(1-|a^i{}_{22}|^2)} \left[\frac{\left(|a^i_{11}||a^i_{12}|\right)^2}{(1-|a^i{}_{11}|^2)} + |a^i{}_{12}|^2 \right] \ .$$

We can define in this way a p_2 that satisfies this for all matrices A_i of the set by choosing

$$p_2 > \max_i \frac{1}{(1-|a^i{}_{22}|^2)} \left[\frac{(|a^i_{11}||a^i_{12}|)^2}{(1-|a^i{}_{11}|^2)} + |a^i{}_{12}|^2 \right] \ .$$

The same argument shows that we can successively choose the values of $p_3, \ldots, p_n$ in a way such that the leading principal minors of (2) are all positive, and this for any matrix A_i of the set. Indeed, let the leading principal minor of order k be > 0. Then, the leading principal minor of order $k+1$ can be made > 0 too, because p_{k+1} only appears in its last diagonal term, with a strictly positive coefficient. So, taking p_{k+1} large enough is sufficient. The finiteness of the elements of A_i guarantees us that such a value p_{k+1} exists and is finite.

A Hermitian matrix H is positive definite if and only if all its leading principal minors are positive ([9]), and so we can deduce that the Hermitian matrix appearing in (2) is indeed positive definite, for any i. So, the P matrix built in this way is a common quadratic Lyapunov function for the set M. □

Corollary 1. *For a set of triangular matrices, the joint spectral radius and its ellipsoid norm approximation are equal. Their value is the largest spectral radius of the matrices in the set.*

Proof. From lemma 1, it turns out that, for a set of stable upper-triangular matrices A_i, there exists a positive definite P_* such that $\|A_i\|_{P_*} < 1, \forall i$. This is equivalent to expressing

$$\max_i \rho(A_i) < 1 \Rightarrow \exists P_* \succ 0 : \max_i \|A_i\|_{P_*} < 1 \ .$$

By linearity, this implies that $\max_i \rho(A_i) \geq \max_i \|A_i\|_{P_*}$. Indeed, let us pose $\max_i \rho(A_i) = r$, so that $\forall y > r,\ \max_i \rho\left(\frac{A_i}{y}\right) < 1$. This implies that $\forall y > r, \exists P_* : \max_i \frac{\|A_i\|_{P_*}}{y} < 1$ or again, $\forall y > \max_i \rho(A_i), \exists P_* : \max_i \|A_i\|_{P_*} < y$. So, $\max_i \|A_i\|_{P_*}$ is arbitrarily close (from above) to $\max_i \rho(A_i)$ and the announced inequality $\max_i \rho(A_i) \geq \max_i \|A_i\|_{P_*}$ holds.

On the other hand, we know that the joint spectral radius is greater or equal to the largest spectral radius of the matrices in the set, that is $\rho(M) \geq \max_i \rho(A_i)$. So, summing up, we have

$$\rho(M) \geq \max_i \rho(A_i) \geq \max_i \|A_i\|_P \geq \hat{\rho}(M) \ .$$

As $\hat{\rho}$ is an over-approximation of $\rho(M)$, this yields $\rho(M) = \max_i \rho(A_i)$ and $\hat{\rho}(M) = \rho(M)$. □

We now generalize the previous result to a more general class of sets of matrices. This development is very similar to the one presented in [14]. Let us recall the following notations and definitions. The Lie algebra $\{A_0, A_1\}_{LA}$ is the linear span of

$$\{A_0, A_1, [A_0, A_1], [A_0, [A_0, A_1]], [A_1, [A_0, A_1]], \ldots\}$$

and all possible combinations of commutators.

The *commutator series* of a Lie algebra $\mathfrak{g}$ is the sequence of subalgebras recursively defined by $\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k]$, and $\mathfrak{g}^0 = \mathfrak{g}$. Noting $[a, b]$ the linear span of elements of the form $[A, B]$, where $A \in a, B \in b$, we have

$$\begin{aligned}
&\mathfrak{g}^1 = [\mathfrak{g}^0, \mathfrak{g}^0] = \mathrm{span}\{[A_0, A_1], [A_0, [A_0, A_1]], [A_1, [A_0, A_1]], \ldots\} \ , \\
&\mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}^1] = \mathrm{span}\{[[A_0, A_1], [A_0, [A_0, A_1]]], [[A_0, A_1], [A_1, [A_0, A_1]]], \ldots\}, \\
&\mathfrak{g}^3 = [\mathfrak{g}^2, \mathfrak{g}^2] = \ldots \ .
\end{aligned}$$

These sets are such that $\mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \ldots$, and if $\mathfrak{g}^k = \mathfrak{g}^{k+1}$ then all subsequent $\mathfrak{g}^{k+p}$ ($p \in \mathbb{N}$) are also equal to $\mathfrak{g}^k$. A Lie Algebra is *solvable* if its commutator series $\mathfrak{g}^k$ vanishes for some k.

An often used example of solvable Lie algebra is the vector space of upper-triangular matrices. It is easy to check that the sequence of subalgebras $\mathfrak{g}^k$ is the set of upper-triangular matrices whose elements on the diagonal at distance less than k from the main diagonal are all zero.

We make use of the following result (cited in [14], referring to [17]):

Lemma 2. *Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field, and let ρ be a representation of $\mathfrak{g}$ on a vector space V of finite* *dimension n. Then there exists a basis $\{v_1, \dots, v_n\}$ of V such that for each $X \in \mathfrak{g}$ the matrix of $\rho(X)$ in that basis takes the upper-triangular form*

$$\begin{pmatrix} \lambda_1(X) & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n(X) \end{pmatrix} ,$$

where the $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of the matrix $\rho(X)$.

Theorem 2. *Let $M = \{A_1, \dots, A_m\}$ and consider the switched linear system*

$$x_{k+1} = A_{i_k} x_k, \; A_{i_k} \in M.$$

If all matrices in M have a spectral radius less than 1 and the Lie algebra associated to M is solvable, then the system has a common quadratic Lyapunov function.

Proof. So, if $\{A_i : A_i \in M\}_{LA}$ is solvable, then there exists a (possibly complex) invertible matrix T such that

$$A_i = T^{-1} \tilde{A}_i T, \text{ with } \tilde{A}_i \text{ upper-triangular}, \forall i \;\; .$$

This introduction of complex values does not change the main argument.

Lemma 1 shows that there exists a real common quadratic Lyapunov function $\tilde{P}$ in diagonal form for such a set of matrices $\tilde{M} = \{\tilde{A}_1, \dots, \tilde{A}_n\}$. ¿From this $\tilde{P}$, we can deduce the form of the corresponding P for the non-upper-triangular set $M = \{A_1, \dots, A_n\}$:

$$\begin{aligned} \tilde{A}_i^{\,*} \tilde{P} \tilde{A}_i - \tilde{P} &\prec 0 \\ (T A_i T^{-1})^* \tilde{P} T A_i T^{-1} - \tilde{P} &\prec 0 \\ T^{*-1} A_i^* T^* \tilde{P} T A_i T^{-1} - \tilde{P} &\prec 0 \\ A_i^* (T^* \tilde{P} T) A_i - (T^* \tilde{P} T) &\prec 0 \;\; . \end{aligned}$$

And we get $P = T^* \tilde{P} T$. As $\tilde{P}$ is positive definite, so is P. Moreover, $\tilde{P}$ being diagonal, $T^* \tilde{P} T$ is actually Hermitian, but is not guaranteed to be real. Let us then denote

$$-R := A_i^* (T^* \tilde{P} T) A_i - (T^* \tilde{P} T)$$

where R is, by construction, Hermitian positive definite. We can write, by separating the real and imaginary parts,

$$P = \mathbb{R}(P) + i\mathbb{I}(P) \;\; \text{and} \;\; R = \mathbb{R}(R) + i\mathbb{I}(R) \;\; .$$

As P and R are Hermitian, $\mathbb{R}(P)$, $\mathbb{R}(R)$ are symmetric positive definite and $\mathbb{I}(P)$, $\mathbb{I}(R)$ are skew-symmetric. We can rewrite

$$A_i^*(\mathbb{R}(P) + i\mathbb{I}(P))A_i - (\mathbb{R}(P) + i\mathbb{I}(P)) = -(\mathbb{R}(R) + i\mathbb{I}(R))$$

and taking the real part,

$$A_i^*\mathbb{R}(P)A_i - \mathbb{R}(P) = -\mathbb{R}(R) \ .$$

As a consequence, $\mathbb{R}(P)$ is a real common quadratic Lyapunov function for the solvable Lie algebra $\{A_i : A_i \in M\}_{LA}$. □

Corollary 2. *If the Lie algebra associated to the set $M = \{A_1, \ldots, A_m\}$ is solvable, then the joint spectral radius of M is equal to $\max_i \rho(A_i)$ and also to its ellipsoid norm approximation.*

Proof. Indeed, Theorem 2 allows us to deduce that $\rho(M) < 1 \Rightarrow \hat{\rho}(M) < 1$, which yields $\rho(M) \geq \hat{\rho}(M)$, allowing to deduce the strict equality, thanks to the already known $\rho(M) \leq \hat{\rho}(M)$. Here again, we already know that $\rho(M) \geq \max_i \rho(A_i)$, and Theorem 2 teaches us that $\max_i \rho(A_i) < 1 \Rightarrow \rho(M) < 1$, so $\max_i \rho(A_i) \geq \rho(M)$. And we deduce $\rho(M) = \max_i \rho(A_i)$. □

We have equality between the joint spectral radius and its ellipsoid norm approximation when the Lie algebra is solvable. One could wonder whether the solvability of the Lie algebra is necessary for this equality to hold. This is not the case. In order to exhibit a counter-example, we first prove a property of independent interest.

Proposition 2. *The joint spectral radius of $\{A, A^T\}$ is equal to its ellipsoid norm approximation and to the largest singular value of A.*

Proof. In such a particular case, we use the inequalities $\rho(A, A^T) \leq \hat{\rho}(A, A^T) \leq \sigma(A)$, which can be seen by using $P = I$ in the definition of $\hat{\rho}$, so that we get $A^T I A \leq \gamma^2 I$, holding for $\gamma \geq \sigma(A)$. And finally,

$$\begin{aligned} & \rho(A, A^T) \geq \rho(AA^T)^{1/2} = \sigma(A) \\ \text{yields} \quad & \rho(A, A^T) = \hat{\rho}(A, A^T) = \sigma(A) \ . \end{aligned}$$

□

Consider now the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ A^T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

It is easy to check that the Lie algebra associated to these matrices is not solvable. On the other hand it follows from the above proposition that for this pair of matrices the joint spectral radius and its ellipsoid norm approximation are equal (and are equal to $\sigma(A) = \frac{1+\sqrt{5}}{2} \simeq 1.618$).

We close this section with a numerical example of two matrices for which we do not have equality between the joint spectral radius and its ellipsoid norm approximation. Let us consider the following matrices (inspired by [6]):

$$A_1 = \begin{pmatrix} 1 & 2\ a_1 \\ -2/a_1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 2\ a_2 \\ -2/a_2 & 1 \end{pmatrix} \quad .$$

Assuming that $a_2 \geq a_1 \geq 1$, extensive calculations that are not reproduced here (see the Technical Report [18] for more details) show that the approximation $\hat{\rho}$ is such that

$$\hat{\rho}(A_1, A_2) \geq \sqrt{1 + 4\ a_2/a_1} \quad .$$

For $a_1 = 1, a_2 = 2$, the joint spectral radius can be shown to be strictly less than 2.8584 by using an exhaustive calculation of all the products of 5 matrices (2.783 if 16 matrices). This is strictly less than $\sqrt{1 + 4 \times 2/1} = 3$, so here $\rho < \hat{\rho}$. The gap between the joint spectral radius and its approximation can be seen on figure 1 for $a_1 = 1$ and varying a_2.

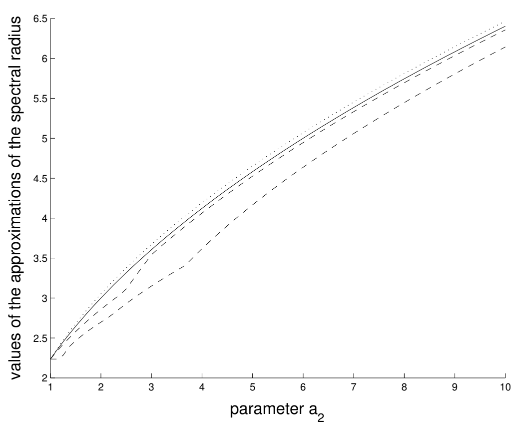

Fig. 1. The spectral radius and its ellipsoid norm approximation as functions of the real parameter a_2 (a_1 is fixed to 1). The two lowest curves (dashed) represent upper and lower bounds on the exact value of the spectral radius (computed with words of length 6). The middle curve (solid) represents $\sqrt{1 + 4a_2}$. The two highest curves (dotted) represent upper and lower bounds on the approximation.

4 Guaranteed precision of the approximation

The ellipsoid norm approximation of the joint spectral radius can be shown to have a guaranteed precision. The argument for this is simple: Let ρ be the joint spectral radius of the set $\{A_1, \ldots, A_m\}$; we know by Theorem 1 that there exists a vector norm $\|.\|_*$ for which $\|A_i x\|_* \leq \rho \|x\|_*$ for all x and i. The level curves of this norm define closed convex set that can be approximated by ellipsoids, the quality of these approximations can be measured and provides a guaranteed precision for the approximation.

We start by describing the quality of best possible ellipsoids (the result below is known as John's theorem; it is stated in [10], referring to [11]).

Theorem 3. *Let $K \subset \mathbb{R}^n$ be a compact convex set with nonempty interior. Then there is an ellipsoid E with center c such that the inclusions $E \subseteq K \subseteq n(E - c)$ hold. If K is symmetric about the origin ($K = -K$), the constant n can be changed into $\sqrt{n}$.*

Knowing this, we can now prove:

Theorem 4. *Let ρ be the joint spectral radius of a finite set of matrices of dimension n. Let $\hat{\rho}$ be the ellipsoid norm approximation of the joint spectral radius. Then $\hat{\rho}/\sqrt{n} \leq \rho \leq \hat{\rho}$.*

Proof. The norm mentioned above is symmetric about the center, as $\|x\|_* = \|-x\|_*, \forall x$. So, the above theorem guarantees us that, whatever the norm $\|.\|_*$ is, there exists a quadratic norm $\|x\|_P = x^T P x$ (of which level curves are ellipsoids) such that

$$\|x\|_P \leq \|x\|_* \leq \sqrt{n}\|x\|_P \ .$$

As, $\forall q > \rho(M)$, the norm $\|.\|_*$ satisfies $\|A_i x\|_* \leq q\|x\|_*, \forall x, \forall i$, we can now write

$$\begin{gathered}
\forall x, \forall i, \|A_i x\|_P \leq \|A_i x\|_* \leq q\|x\|_* \leq q\|x\|_P\sqrt{n} \\
\forall x, \forall i, \ \|A_i x\|_P \leq q\|x\|_P\sqrt{n} \\
\forall x, \forall i, \ x^T A_i^T P A_i x \leq q^2 n \ x^T P x \\
\forall i, A_i^T P A_i - q^2 n P \preceq 0 \ .
\end{gathered}$$

Thus, the approximation $\hat{\rho}$ defined by $\hat{\rho}(M) = \inf_{P \succ 0} \max_{A_i \in M} \|A_i\|_P$ is $\leq q\sqrt{n}$. So, at worst, the approximation will result in the value $\rho(M)\sqrt{n}$. Summing up, this gives:

$$\rho(M) \leq \hat{\rho}(M) \leq \rho(M)\sqrt{n} \ .$$

□

5 Matrices with non-negative entries

In this section, we introduce an approximation of the joint spectral radius for matrices with non-negative entries. We first provide a result for general matrices that is of independent interest.

Proposition 3. *Let $M = \{A_1, \ldots, A_m\}$. Then*

$$\max_{\sum_{i=0}^{m} \alpha_i = 1,\ \alpha_i \geq 0} \rho\left(\sum \alpha_i A_i\right) \leq \rho(M).$$

Proof. We have, using the inequality $\rho(.) \leq \|.\|$ (for any valid matrix norm $\|.\|$) and the subadditivity of the norm,

$$\begin{aligned} \tilde{\rho}(M) := \rho\left(\sum_i \alpha_i A_i\right) &\leq \left\|\sum_i \alpha_i A_i\right\| \\ &\leq \sum_i \|\alpha_i A_i\| = \sum_i \alpha_i \|A_i\| \\ &\leq \max_i (\|A_i\|) \quad , \text{as } \textstyle\sum_i \alpha_i = 1 \ . \end{aligned}$$

Now, if the system converges to the origin, that is $\rho(M) < 1$, we know there exists a norm $\|.\|_*$ such that $\forall i, \|A_i\|_* < 1$ (see [12]). We can then deduce from the previous inequality that $\tilde{\rho}(M) < 1$. So,

$$\rho(M) < 1 \Rightarrow \tilde{\rho}(M) < 1 \ ,$$

and we deduce, using linearity, that $\tilde{\rho}(M) \leq \rho(M)$. □

For deriving the approximation of this section we need one more property.

Lemma 3. *Let M_1 and M_2 be matrices with non-negative entries. If $(M_2)_{ij} \geq (M_1)_{ij}$, then $\rho(M_2) \geq \rho(M_1)$.*

We are now ready to prove:

Theorem 5. *Let $M = \{A_1, \ldots, A_m\}$ be a set of matrices with non-negative entries and define $S_{ij} = \max_{1 \leq k \leq m}(A_k)_{ij}$. We have*

$$\frac{\rho(S)}{m} \leq \rho(M) \leq \rho(S). \tag{3}$$

Proof. For the first inequality of (3), non-negativity, Lemma 3 and Proposition 3 give

$$S \leq \sum_{k=1}^{m} A_k \Rightarrow \rho(S) \leq \rho\left(\sum_{k=1}^{m} A_k\right) = m\rho\left(\frac{\sum_{k=1}^{m} A_k}{m}\right) \leq m\rho(M) \ .$$

To prove the second inequality of (3), we may note that, as the elements are non-negative, for any sequence ω of k indices, the product A_ω satisfies $(S^k)_{ij} \geq (A_\omega)_{ij}$. Lemma 3 allows us to deduce, $\forall \omega : |\omega| = k$,

$$S^k \geq A_\omega \Rightarrow \limsup_{k\to\infty} \|S^k\|^{1/k} \geq \limsup_{k\to\infty} \left(\max_{|\omega|=k} \|A_\omega\|^{1/k} \right) \Rightarrow \rho(S) \geq \rho(M) \ .$$

□

For all set cardinalities m, the equality $\rho(S)/m = \rho(M)$ is achieved for some particular matrices. For $m = 2$ consider the following pair:

$$\left\{ A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

for which $\rho(S) = 2$. On the other hand, we see that $A^2 = A$, $B^2 = B$, $AB = B$ and $BA = A$. Any product generated by $\{A, B\}$ is either A or B, so $\rho(A, B) = 1$ and we indeed have $\hat{\rho}/2 = \rho$. A similar construction is immediate for the cases $m \geq 3$.

6 Conclusion

We introduce in this paper a polynomial-time approximation of the joint spectral radius that is easy to compute and that is guaranteed to be within a factor $\sqrt{n}$ of the exact value, where n is the dimension of the matrices. We describe particular classes of matrices for which our approximation is equal to the joint spectral radius. The problem of characterizing exactly the sets of matrices for which equality holds is a question that remains open. We also provide an easy way of approximating the joint spectral radius of matrices with non-negative entries, and show that this approximation is within a factor at most n of the exact value, where n is the number of matrices in the set. This last result does not depend on the size of the matrices. The question remains open to find better approximations at a reasonable computational cost. In particular, both approximations presented in this paper have relative errors that increase with the size or number of the matrices. It is yet unclear if a polynomial time approximation is possible that gives a fixed guaranteed relative error.

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