# **Induction and Co-induction in Sequent Calculus**

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**Abstract.** Proof search has been used to specify a wide range of computation systems. In order to build a framework for reasoning about such specifications, we make use of a sequent calculus involving induction and co-induction. These proof principles are based on a proof theoretic (rather than set-theoretic) notion of *definition* [13, 20, 25, 51]. Definitions are akin to (stratified) logic programs, where the left and right rules for defined atoms allow one to view theories as "closed" or defining fixed points. The use of definitions makes it possible to reason intensionally about syntax, in particular enforcing free equality via unification. We add in a consistent way rules for pre and post fixed points, thus allowing the user to reason inductively and co-inductively about properties of computational system making full use of higher-order abstract syntax. Consistency is guaranteed via cut-elimination, where we give the first, to our knowledge, cut-elimination procedure in the presence of general inductive and co-inductive definitions.

## 1 Introduction

A common approach to specifying computation systems is via deductive systems. Those are used to specify and reason about various logics, as well as aspects of programming languages such as operational semantics, type theories, abstract machines *etc*. Such specifications can be represented as logical theories in a suitably expressive formal logic where *proof-search* can then be used to model the computation. A logic used as a specification language is known as a *logical frameworks* [39], which comes equipped with a representation methodology. The encoding of the syntax of deductive systems inside formal logic can benefit from the use of *higher-order abstract syntax* (HOAS) [40], a high-level and declarative treatment of object-level bound variables and substitution. At the same time, we want to use such a logic in order to reason over the *meta-theoretical* properties of object languages, for example type preservation in operational semantics [26], soundness and completeness of compilation [32] or congruence of bisimulation in transition systems [27]. Typically this involves reasoning by (structural) induction and, when dealing with infinite behavior, co-induction [23].

The need to support both inductive and co-inductive reasoning and some form of HOAS requires some careful design decisions, since the two are prima facie notoriously incompatible. While any meta-language based on a λ-calculus can be used to specify and animate HOAS encodings, meta-reasoning has traditionally involved (co)inductive specifications both at the level of the syntax and of the judgements — which are of course unified at the type-theoretic level. The first provides crucial freeness properties for datatypes constructors, while the second offers principle of case analysis and (co)induction. This is well-known to be problematic, since HOAS specifications lead to non-monotone (co)inductive operators, which by cardinality and consistency reasons are not permitted in inductive logical frameworks. Moreover, even when HOAS is weakened so as to be made compatible with standard proof assistants [12] such as HOL or Coq, the latter suffer the fate of allowing the existence of too many functions and yielding the so called *exotic* terms. Those are canonical terms in the signature of an HOAS encoding that do not correspond to any term in the deductive system under study. This causes a loss of adequacy in HOAS specifications, which is one of the pillar of formal verification, and it undermines the trust in formal derivations. On the other hand, logics such as LF [21] that are weak by design [10] in order to support this style of syntax are not directly endowed with (co)induction principles.

The contribution of this paper lies in the design of a new logic, called Linc<sup>-</sup> (for a logic with  $\lambda$ -terms, induction and co-induction),<sup>3</sup> which carefully adds principles of induction and co-induction to a higher-order intuitionistic logic based on a proof theoretic notion of *definition*, following on work (among others) Lars Hallnäs [20], Eriksson [13], Schroeder-Heister [51] and McDowell and Miller [25]. Definitions are akin to logic programs, but allow us to view theories as "closed" or defining fixed points. This alone allows us to perform case analysis independently from induction principles. Our approach to formalizing induction and co-induction is via the least and greatest solutions of the

<sup>&</sup>lt;sup>3</sup> The "minus" in the terminology refers to the lack of the  $\nabla$  quantifier w.r.t. the eponymous logic in Tiu's thesis [56].

fixed point equations specified by the definitions. Such least and greatest solutions are guaranteed to exist by imposing a stratification condition on definitions (which basically ensures monotonicity). The proof rules for induction and co-induction makes use of the notion of *pre-fixed points* and *post-fixed points* respectively. In the inductive case, this corresponds to the induction invariant, while in the co-inductive one to the so-called simulation.

The simply typed language underlying Linc<sup>-</sup> and the notion of definition make it possible to reason *intensionally* about syntax, in particular enforcing *free* equality via unification, which can be used on first-order terms or higher-order  $\lambda$ -terms. In fact, we can support HOAS encodings of constants without requiring them to be the constructors of a (recursive) datatype, which could not exist for cardinality reasons. In particular we can *prove* the freeness properties of those constructors, namely injectivity, distinctness and case exhaustion. Judgements are encoded as definitions accordingly to their informal semantics, either inductive or co-inductive. Definitions that are true in every fixed point will not be given here special consideration.

Linc<sup>-</sup> can be proved to be a conservative extension of  $FO\lambda^{\Delta IN}$  [25] and a generalization with a higher-order language of Martin-Löf [24] first-order theory of iterated inductive definitions. Moreover, to the best of our knowledge, it is the first sequent calculus with a syntactical cut-elimination theorem for co-inductive definitions. In recent years, several logical systems have been designed that build on the core features of Linc<sup>-</sup>. In particular, one interesting, and orthogonal, extension is the addition of the  $\nabla$ -quantifier [14, 31, 56, 57], which allows one to reason about the intentional aspects of *names and bindings* in object syntax specifications (see, e.g., [15, 58, 59]). The cut elimination proof presented in this paper can be used as a springboard towards cut elimination procedures for more expressive (conservative) extensions of Linc<sup>-</sup> such as the ones with  $\nabla$ . Here lies the added value of the present paper, which extends and revises a conference paper published in the proceedings of TYPES 2003 [33]. In the conference version, the co-inductive rule had a technical side condition that is restrictive and unnatural. The restriction was essentially imposed by the particular cut elimination proof technique outlined in that paper. This restriction has been removed in the present version, and as such the cut elimination proof itself has consequently been significantly revised.

The rest of the paper is organized as follows. Section 2 introduces the sequent calculus for the logic Linc<sup>-</sup>. Section 3 shows some examples of using induction and co-induction to prove properties of list-related predicates and the lazy  $\lambda$ -calculus. Section 4 studies several properties of derivations in Linc<sup>-</sup> that will be used extensively in the cut-elimination proof (Section 5). Section 6 surveys the related work and Section 7 concludes this paper.

## **2** The Logic Linc

The logic Linc<sup>-</sup> shares the core fragment of  $FO\lambda^{\Delta IN}$ , which is an intuitionistic version of Church's Simple Theory of Types. Formulae in the logic are built from predicate symbols and the usual logical connectives  $\bot$ ,  $\top$ ,  $\wedge$ ,  $\vee$ ,  $\supset$ ,  $\forall_{\tau}$  and  $\exists_{\tau}$ . Following Church, formulae will be given type o. The quantification type  $\tau$  (omitted in the rest of the paper) can have base or higher types, but those are restricted not to contain o. Thus the logic has a first-order proof theory but allows the encoding of higher-order abstract syntax.

We assume the usual notion of capture-avoiding substitutions. Substitutions are ranged over by lower-case Greek letters, e.g.,  $\theta$ ,  $\rho$  and  $\sigma$ . Application of substitution is written in postfix notation, *e.g.*  $t\theta$  denotes the term resulting from an application of substitution  $\theta$  to t. Composition of substitutions, denoted by  $\circ$ , is defined as  $t(\theta \circ \rho) = (t\theta)\rho$ .

The whole logic is presented in the sequent calculus in Figure 1. A sequent is denoted by  $\Gamma \longrightarrow C$  where C is a formula and  $\Gamma$  is a multiset of formulae. Notice that in the presentation of the rule schemes, we make use of HOAS, e.g., in the application Bx it is implicit that B has no free occurrence of x. In particular we work modulo  $\alpha$ -conversion without further notice. In the  $\forall \mathcal{R}$  and  $\exists \mathcal{L}$  rules, y is an eigenvariable that is not free in the lower sequent of the rule. Whenever we write a sequent, it is assumed implicitly that the formulae are well-typed and in  $\beta\eta$ -long normal forms: the type context, i.e., the types of the constants and the eigenvariables used in the sequent, is left implicit as well. The mc rule is a generalization of the cut rule that simplifies the presentation of the cut-elimination proof.

We extend the core fragment with a proof theoretic notion of equality and fixed points. Each of these extensions are discussed below.

## 2.1 Equality

The right introduction rule for equality is the standard one, that is, it recognizes that two terms are syntactically equal. The left introduction rule is more interesting. The substitution  $\rho$  in eq $\mathcal{L}$  is a *unifier* of s and t. Note that we specify the premise of eq $\mathcal{L}$  as a set, with the intention that every sequent in the set is a premise of the rule. This set is of

Equality rules:

$$\frac{\{\Gamma\rho \longrightarrow C\rho \mid s\rho =_{\beta\eta} t\rho\}}{s = t, \Gamma \longrightarrow C} \operatorname{eq} \mathcal{L} \qquad \frac{}{\Gamma \longrightarrow t = t} \operatorname{eq} \mathcal{R}$$

Induction rules:

$$\frac{BS\vec{y} \longrightarrow S\vec{y} \quad \Gamma, S\vec{t} \longrightarrow C}{\Gamma, p\vec{t} \longrightarrow C} \quad IL, p\vec{x} \stackrel{\mu}{=} Bp\vec{x} \qquad \frac{\Gamma \longrightarrow Bp\vec{t}}{\Gamma \longrightarrow p\vec{t}} \quad IR, p\vec{x} \stackrel{\mu}{=} Bp\vec{x}$$

Co-induction rules:

$$\frac{B\,p\vec{t},\Gamma\longrightarrow C}{p\vec{t},\Gamma\longrightarrow C}\,\operatorname{CIL},p\vec{x}\stackrel{\vee}{=}B\,p\vec{x}\quad\frac{\Gamma\longrightarrow S\vec{t}\quad S\vec{y}\longrightarrow BS\vec{y}}{\Gamma\longrightarrow p\vec{t}}\,\operatorname{CIR},p\vec{x}\stackrel{\vee}{=}B\,p\vec{x}$$

Fig. 1. The inference rules of Linc<sup>-</sup>

course infinite, since for every unifier of (s,t), we can extend it to another unifier (e.g., by adding substitution pairs for variables not in the terms). However, in many cases, it is sufficient to consider a particular set of unifiers, which is often called a *complete set of unifiers* (CSU) [4], from which any unifier can be obtained by composing a member of the CSU set with a substitution. In the case where the terms are first-order terms, or higher-order terms with the pattern restriction [30], the set CSU is a singleton, i.e., there exists a most general unifier (MGU) for the terms.

In examples and applications, we shall use a more restricted version of eq $\mathcal{L}$  using CSU:

$$\frac{\{\Gamma\rho \longrightarrow C\rho \mid s\rho =_{\beta\eta} t\rho, \rho \in CSU(s,t)\}}{s = t, \Gamma \longrightarrow C} \text{ eq} \mathcal{L}_{CSU}$$

Replacing eq $\mathcal{L}$  with eq $\mathcal{L}_{CSU}$  does not change the class of provable formulae, as shown in [56]. Note that in applying eq $\mathcal{L}$  and eq $\mathcal{L}_{CSU}$ , eigenvariables can be instantiated as a result. Note also that if the premise set of eq $\mathcal{L}$  and eq $\mathcal{L}_{CSU}$  are empty, then the sequent in the conclusion is considered proved.

Our treatment of equality implicitly assumes the notion of *free equality* as commonly found in logic programming. More specifically, the axioms of free equality [9], that is, injectivity of function symbols, inequality between distinct function symbols, and the "occur-check" are enforced via unification in the eq $\mathcal{L}$ -rule. For instance, given a base type nt (for natural numbers) and the constants z:nt (zero) and  $s:nt \to nt$  (successor), we can derive  $\forall x.\ z=(s\ x) \supset \bot$  as follows:

Since z and s x are not unifiable, the eq $\mathcal{L}$  rule above has empty premise, thus concluding the derivation. We can also prove the injectivity of the successor function, i.e.  $\forall x \forall y. (s \ x) = (s \ y) \supset x = y$ .

This proof theoretic notion of equality has been considered in several previous work e.g. by by Schroeder-Heister [51], and McDowell and Miller [25].

#### 2.2 Induction and co-induction

One way of adding induction and co-induction is to introduce fixed point expressions and their associated introduction rules, *i.e.* using the  $\mu$  and  $\nu$  operators of the (first-order)  $\mu$ -calculus. This is essentially what we shall follow here, but with a different notation. Instead of using a "nameless" notation using  $\mu$  and  $\nu$  to express fixed points, we associate a fixed point equation with an atomic formula. That is, we associate certain designated predicates with a *definition*. This notation is clearer and more convenient as far as our examples and applications are concerned. For the proof system using nameless notation for inductive and co-inductive predicates, the interested reader is referred to a recent work by Baelde and Miller [5].

**Definition 1.** An inductive definition clause is written  $\forall \vec{x}.p\vec{x} \stackrel{\mu}{=} B\vec{x}$ , where p is a predicate constant and  $\vec{x}$  is a sequence of variables. The atomic formula  $p\vec{x}$  is called the head of the clause, and the formula  $B\vec{x}$ , where B is a closed term, is called the body. Similarly, a co-inductive definition clause is written  $\forall \vec{x}.p\vec{x} \stackrel{\vee}{=} B\vec{x}$ . The symbols  $\stackrel{\mu}{=}$  and  $\stackrel{\vee}{=}$  are used simply to indicate a definition clause: they are not a logical connective. A definition is a set of definition clauses.

It is technically convenient to bundle up all the definitional clause for a given predicate in a single clause, so that a predicate may occur only at most once in the heads of the clauses of a definition, following the same principles of the *iff-completion* in logic programming [50]. Further, in order to simplify the presentation of some rules that involve predicate substitutions, we sometimes denote a definition using an abstraction over predicates, that is

$$\forall \vec{x}. \ p\vec{x} \stackrel{\mu}{=} B \, p\vec{x}$$

where B is an abstraction with no free occurrence of predicate symbol p and variables  $\vec{x}$ . Substitution of p in the body of the clause with a formula S can then be written simply as  $BS\vec{x}$ . When writing definition clauses, we often omit the outermost universal quantifiers, with the assumption that free variables in a clause are universally quantified (such variables will often be denoted with capital letters). We shall write  $\forall \vec{x}$ .  $p\vec{x} \stackrel{\triangle}{=} Bp\vec{x}$  to denote a definition clause generally, i.e., when we are not interested in the details of whether it is an inductive or a co-inductive definition.

The introduction rules for (co-)inductively defined atoms are given at the bottom of Figure 1. The abstraction S is an invariant of the (co-)induction rule, which is of the same type as p. The variables  $\vec{y}$  are new eigenvariables. For the induction rule IL, S denotes a pre-fixed point of the underlying fixed point operator. Similarly, for the co-induction rule CIL, S can be seen as denoting a post-fixed point of the same operator. Here, we use a characterization of induction and co-induction proof rules as, respectively, the least and the greatest solutions to a fixed point equation. To guarantee soundness of these rules, we shall restrict the (co)inductive definitions to ones which are monotone. In this case, the Knaster-Tarski fixed point theorems tell us that the existence of a pre-fixed point (respectively, post-fixed point) implies the existence of a least (resp., greatest) fixed point. Monotonicity is enforced by a syntactic condition on definitions, as it is used for the logic  $FO\lambda^{\Delta IN}$  [25]: we rule out definitions with circular calling through implications (negations) that can lead to inconsistency [49]. The notion of *level* of a formula allows us to define a proper stratification on definitions.

**Definition 2.** To each predicate p we associate a natural number lvl(p), the level of p. Given a formula B, its level lvl(B) is defined as follows:

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1. \operatorname{lvl}(p\vec{t}) = \operatorname{lvl}(p),
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- 2.  $lvl(\bot) = lvl(\top) = 0$ ,
- 3.  $lvl(B \land C) = lvl(B \lor C) = max(lvl(B), lvl(C))$
- 4.  $lvl(B \supset C) = max(lvl(B) + 1, lvl(C))$
- 5.  $lvl(\forall x. Bx) = lvl(\exists x. Bx) = lvl(Bt)$ , for any term t.

The level of a sequent  $\Gamma \longrightarrow C$  is the level of C. A formula B is said to be dominated by a predicate symbol p, if  $lvl(B) \le lvl(p)$  and  $lvl(B[\lambda \vec{x}. \top/p]) < lvl(p)$ , where  $\lambda \vec{x}. \top$  is of the same type as p. A definition clause  $\forall \vec{x}. \ p\vec{x} \stackrel{\triangle}{=} B\vec{x}$  is stratified if  $B\vec{x}$  is dominated by p.

Note that when p is vacuous in B and p dominates B, we obviously have lvl(B) < lvl(p).

From now on, we shall be concerned only with stratified definitions. An occurrence of a formula A in a formula C is *strictly positive* if that particular occurrence of A is not to the left of any implication in C. Stratification then implies that all occurrences of the head in the body are strictly positive, and that there is no mutual recursion between different definition clauses. This restriction to non-mutual recursion is just for the sake of simplicity in the presentation of the underlying idea of the cut elimination proof. This proof (Section 5) can be extended to handle mutually recursive definitions with some straightforward, albeit tedious, modifications. In the first-order case, the restriction to non-mutual recursion is immaterial, since one can easily encode mutually recursive predicates as a single predicate with an extra argument. For example, consider the following mutual recursive definitions for even and odd numbers.

even 
$$X \stackrel{\mu}{=} X = z \lor \exists y.y = (s X) \land odd y.$$
  
odd  $X \stackrel{\mu}{=} \exists y.y = (s X) \land even y.$ 

We can collapse these two definition clauses into a single one, with a parameter that takes a constant e (for 'even') or o (for 'odd'):

$$evod\ W\ X \stackrel{\mu}{=} [W = e \land (X = z \lor \exists y.\ y = (s\ X) \land evod\ o\ y)] \lor [W = o \land (\exists y.\ y = (s\ X) \land evod\ e\ y)].$$

We then define even and odd as follows:

$$even X \stackrel{\mu}{=} evod \ e \ X.$$
 $odd X \stackrel{\mu}{=} evod \ o \ X.$ 

This definition can be stratified by assigning levels to the predicate symbols such that

$$lvl(evod) < lvl(even) < lvl(odd)$$
.

## 3 Examples

We now give some examples, starting with some that make essential use of HOAS.

## 3.1 Lazy λ-Calculus

We consider an untyped version of the pure  $\lambda$ -calculus with lazy evaluation, following the usual HOAS style, i.e., object-level  $\lambda$ -operator and application are encoded as constants lam:  $(tm \to tm) \to tm$  and @:  $tm \to tm \to tm$ , where tm is the syntactic category of object-level  $\lambda$ -terms. The evaluation relation is encoded as the following inductive definition

$$M \Downarrow N \stackrel{\mu}{=} [\exists M'. (M = \operatorname{lam} M') \land (M = N)] \lor [\exists M_1 \exists M_2 \exists P. (M = M_1 @ M_2) \land M_1 \Downarrow \operatorname{lam} P \land (PM_2) \Downarrow N]$$

Notice that object-level substitution is realized via  $\beta$ -reduction in the meta-logic.

The notion of applicative simulation of  $\lambda$ -expressions [1] can be encoded as the (stratified) co-inductive definition

$$sim R S \stackrel{\vee}{=} \forall T. R \Downarrow lam T \supset \exists U. S \Downarrow lam U \land \forall P. sim (T P) (U P).$$

Given this encoding, we can prove the reflexivity property of simulation, i.e.,  $\forall s. \ sim \ s. \ s.$  This is proved co-inductively by using the simulation  $\lambda x \lambda y. \ x = y.$  After applying  $\forall \mathcal{R}$  and  $\text{CI}\mathcal{R}$ , it remains to prove the sequents  $\longrightarrow s = s$ , and

$$x = y \longrightarrow \forall x_1. \ x \Downarrow \text{lam} \ x_1 \supset (\exists x_2. \ y \Downarrow \text{lam} \ x_2 \land \forall x_3. (x_1 x_3) = (x_2 x_3))$$

The first sequent is provable by an application of eq $\mathcal{R}$  rule. The second sequent is proved as follows.

The transitivity property is expressed as  $\forall r \forall s \forall t.sim \ r \ s \land sim \ s \ t \supset sim \ r \ t$ . Its proof involves co-induction on  $sim \ r \ t$  with the simulation  $\lambda u \lambda v. \exists w.sim \ u \ w \land sim \ w \ v$ , followed by case analysis (i.e.,  $def \mathcal{L}$  and  $eq \mathcal{L}$  rules) on  $sim \ r \ s$  and  $sim \ s \ t$ . The rest of the proof is purely logical.

We can also show the existence of divergent terms. Divergence is encoded as follows.

divrg 
$$T \stackrel{\vee}{=} [\exists T_1 \exists T_2. \ T = (T_1 @ T_2) \land \text{divrg } T_1] \lor [\exists T_1 \exists T_2. \ T = (T_1 @ T_2) \land \exists E. \ T_1 \Downarrow \text{lam } E \land \text{divrg } (E T_2)].$$

Let  $\Omega$  be the term  $(\operatorname{lam} x.(x@x))$  @  $(\operatorname{lam} x.(x@x))$ . We show that divrg  $\Omega$  holds. The proof is straightforward by coinduction using the simulation  $S := \lambda s.$   $s = \Omega$ . Applying the CIR produces the sequents  $\longrightarrow \Omega = \Omega$  and  $T = \Omega \longrightarrow S_1 \vee S_2$  where

$$S_1 := \exists T_1 \exists T_2. \ T = (T_1 @ T_2) \land (ST_1), \text{ and}$$
 
$$S_2 := \exists T_1 \exists T_2. \ T = (T_1 @ T_2) \land \exists E. \ T_1 \Downarrow lam E \land S(ET_2).$$

Clearly, only the second disjunct is provable, i.e., by instantiating  $T_1$  and  $T_2$  with the same term  $\lim x.(x \otimes x)$ , and E with the function  $\lambda x.(x \otimes x)$ .

#### 3.2 Lists

Lists over some fixed type  $\alpha$  are encoded as the type *lst*, with the usual constructor nil: *lst* for empty list and :: of type  $\alpha \to lst \to lst$ . We consider here the append predicate for both the finite and infinite case.

Finite lists The usual append predicate on finite lists can be encoded as the inductive definition

$$\begin{array}{c} \operatorname{app} L_1 \ L_2 \ L_3 \stackrel{\mu}{=} \left[ (L_1 = \operatorname{nil}) \wedge (L_2 = L_3) \right] \vee \\ \left[ \exists x \exists L_1' \exists L_3' . \ (L_1 = x :: L_1') \wedge (L_3 = x :: L_3') \wedge \operatorname{app} L_1' \ L_2 \ L_3' \right]. \end{array}$$

Associativity of append is stated formally as

$$\forall l_1 \forall l_2 \forall l_{12} \forall l_3 \forall l_4$$
.(app  $l_1 \ l_2 \ l_{12} \land app \ l_{12} \ l_3 \ l_4$ )  $\supset \forall l_{23}$ .app  $l_2 \ l_3 \ l_{23} \supset app \ l_1 \ l_{23} \ l_4$ .

Proving this formula requires us to prove first that the definition of append is functional, that is,

$$\forall l_1 \forall l_2 \forall l_3 \forall l_4$$
.app  $l_1 \ l_2 \ l_3 \land \text{app} \ l_1 \ l_2 \ l_4 \supset l_3 = l_4$ .

This is done by induction on  $l_1$ , i.e., we apply the IL rule on app  $l_1$   $l_2$   $l_3$ , after the introduction rules for  $\forall$  and  $\supset$ , of course. The invariant in this case is

$$S := \lambda r_1 \lambda r_2 \lambda r_3 . \forall r. app r_1 r_2 r \supset r = r_3.$$

It is a simple case analysis to check that this is the right invariant. Back to our original problem: after applying the introduction rules for the logical connectives in the formula, the problem of associativity is reduced to the following sequent

app 
$$l_1 l_2 l_{12}$$
, app  $l_{12} l_3 l_4$ , app  $l_2 l_3 l_{23} \longrightarrow \text{app } l_1 l_{23} l_4$ . (1)

We then proceed by induction on the list  $l_1$ , that is, we apply the IL rule to the hypothesis app  $l_1$   $l_2$   $l_{12}$ . The invariant is simply

$$S := \lambda l_1 \lambda l_2 \lambda l_{12} . \forall l_3 \forall l_4. \text{app } l_{12} \ l_3 \ l_4 \supset \forall l_{23}. \text{app } l_2 \ l_3 \ l_{23} \supset \text{app } l_1 \ l_{23} \ l_4.$$

Applying the IL rule, followed by  $\vee L$ , to sequent (1) reduces the sequent to the following sub-goals

- (i)  $S l_1 l_2 l_{12}$ , app  $l_{12} l_3 l_4$ , app  $l_2 l_3 l_{23} \longrightarrow \text{app } l_1 l_{23} l_4$ ,
- (ii)  $(l_1 = \text{nil} \land l_2 = l_3) \longrightarrow S \ l_1 \ l_2 \ l_3,$ (iii)  $\exists x, l'_1, l'_3, l_1 = x :: l'_1 \land l_3 = x :: l'_3 \land S \ l'_1 \ l_2 \ l'_3 \longrightarrow S \ l_1 \ l_2 \ l_3$

The proof for the second sequent is straightforward. The first sequent reduces to

app 
$$l_{12}$$
  $l_3$   $l_4$ , app  $l_{12}$   $l_3$   $l_{23}$   $\longrightarrow$  app nil  $l_{23}$   $l_4$ .

This follows from the functionality of append and IR. The third sequent follows by case analysis. Of course, these proofs could have been simplified by using a derived principle of structural induction. While this is easy to do, we have preferred here to use the primitive IL rule.

Infinite lists The append predicate on infinite lists is defined via co-recursion, that is, we define the behavior of destructor operations on lists (i.e., taking the head and the tail of the list). In this case we never construct explicitly the result of appending two lists, rather the head and the tail of the resulting lists are computed as needed. The co-recursive append requires case analysis on all arguments.

$$\begin{array}{c} \operatorname{coapp} L_1 \ L_2 \ L_3 \overset{\vee}{=} \left[ (L_1 = \operatorname{nil}) \wedge (L_2 = \operatorname{nil}) \wedge (L_3 = \operatorname{nil}) \right] \vee \\ \left[ (L_1 = \operatorname{nil}) \wedge \exists x \exists L_2' \exists L_3'. \ (L_2 = x :: L_2') \wedge (L3 = x :: L_3') \ \wedge \ \operatorname{coapp} \ \operatorname{nil} \ L_2' \ L_3' \right] \vee \\ \left[ \exists x \exists L_1' \exists L_3'. \ (L_1 = x :: L_1') \wedge (L_3 = x :: L_3') \ \wedge \ \operatorname{coapp} \ L_1' \ L_2 \ L_3' \right]. \end{array}$$

The corresponding associativity property is stated analogously to the inductive one and the main statement reduces to proving the sequent

coapp 
$$l_1$$
  $l_2$   $l_{12}$ , coapp  $l_{12}$   $l_3$   $l_4$ , coapp  $l_2$   $l_3$   $l_{23}$   $\longrightarrow$  coapp  $l_1$   $l_{23}$   $l_4$ .

We apply the CIR rule to coapp  $l_1$   $l_{23}$   $l_4$ , using the simulation

$$S := \lambda l_1 \lambda l_2 \lambda l_{12} \cdot \exists l_{23} \exists l_3 \exists l_4 \cdot \text{coapp } l_{12} \ l_3 \ l_4 \wedge \text{ coapp } l_2 \ l_3 \ l_{23} \wedge \text{ coapp } l_1 \ l_{23} \ l_4.$$

Subsequent steps of the proof involve mainly case analysis on coapp  $l_{12}$   $l_3$   $l_4$ . As in the inductive case, we have to prove the sub-cases when  $l_{12}$  is nil. However, unlike in the former case, case analysis on the arguments of coapp suffices.

## 4 Properties of derivations

We discuss several properties of derivations in Linc<sup>-</sup>. Some of them involve transformations on derivations which will be used extensively in the cut-elimination proof in Section 5. Before we proceed, some remarks on the use of eigenvariables in derivations are useful. In proof search involving  $\forall \mathcal{R}, \exists \mathcal{L} \text{ IL}, \text{ CIR}$  or eq $\mathcal{L}$ , new eigenvariables can be introduced in the premises of the rules. Let us refer to such variables as internal eigenvariables, since they occur only in the premise derivations. We view the choice of such eigenvariables as arbitrary and therefore we identify derivations that differ only in the choice of the eigenvariables introduced by those rules. Another way to look at it is to consider eigenvariables as proof-level binders. Hence when we work with a derivation, we actually work with an equivalence class of derivations modulo renaming of internal eigenvariables.

## 4.1 Instantiating derivations

The following definition extends substitutions to apply to derivations. Since we identify derivations that differ only in the choice of variables that are not free in the end-sequent, we will assume that such variables are chosen to be distinct from the variables in the domain of the substitution and from the free variables of the range of the substitution. Thus applying a substitution to a derivation will only affect the variables free in the end-sequent.

**Definition 3.** If  $\Pi$  is a derivation of  $\Gamma \longrightarrow C$  and  $\theta$  is a substitution, then we define the derivation  $\Pi\theta$  of  $\Gamma\theta \longrightarrow C\theta$  as follows:

1. Suppose  $\Pi$  ends with the eq $\mathcal{L}$  rule

$$\frac{\left\{\Gamma'\rho \xrightarrow{\Pi^{\rho}} C\rho\right\}_{\rho}}{s = t, \Gamma' \xrightarrow{} C} \text{ eq} \mathcal{L}$$

where  $s\rho =_{\beta\eta} t\rho$ . Observe that any unifier for the pair  $(s\theta,t\theta)$  can be transformed to another unifier for (s,t), by composing the unifier with  $\theta$ . Thus  $\Pi\theta$  is

$$\frac{\left\{ \Gamma'\theta\rho' \longrightarrow C\theta\rho' \right\}_{\rho'}}{s\theta = t\theta.\Gamma'\theta \longrightarrow C\theta} \text{ eq} \mathcal{L} .$$

where  $s\theta \rho' =_{\beta \eta} t\theta \rho'$ .

2. If  $\Pi$  ends with any other rule and has premise derivations  $\Pi_1, \dots, \Pi_n$ , then  $\Pi\theta$  also ends with the same rule and has premise derivations  $\Pi_1\theta, \dots, \Pi_n\theta$ .

Among the premises of the inference rules of Linc<sup>-</sup>, certain premises share the same right-hand side formula with the sequent in the conclusion. We refer to such premises as major premises. This notion of major premise will be useful in proving cut-elimination, as certain proof transformations involve only major premises.

**Definition 4.** Given an inference rule R with one or more premise sequents, we define its major premise sequents as follows.

- 1. If R is either  $\supset \mathcal{L}$ , mc or I $\mathcal{L}$ , then its rightmost premise is the major premise
- 2. If R is CIR, then its left premise is the major premise.
- 3. Otherwise, all the premises of R are major premises.

A minor premise of a rule R is a premise of R which is not a major premise. The definition extends to derivations by replacing premise sequents with premise derivations.

The following two measures on derivations will be useful later in proving many properties of the logic. Given a set of measures S, we denote with lub(S) the least upper bound of S.

**Definition 5.** Given a derivation  $\Pi$  with premise derivations  $\{\Pi_i\}_i$ , the measure  $\operatorname{ht}(\Pi)$  is  $\operatorname{lub}(\{\operatorname{ht}(\Pi_i)\}_i)+1$ .

**Definition 6.** Given a derivation  $\Pi$  with premise derivations  $\{\Pi_i\}_i$ , the measure  $\operatorname{indm}(\Pi)$  is defined as follows

$$indm(\Pi) = \begin{cases} lub(\{indm(\Pi_i)\}_i) + 1, & \text{if } \Pi \text{ ends with } IL, \\ lub(\{indm(\Pi_i)\}_i), & \text{otherwise.} \end{cases}$$

Note that given the possible infinite branching of eq $\mathcal{L}$  rule, these measures in general can be ordinals. Therefore in proofs involving induction on those measures, transfinite induction is needed. However, in most of the inductive proofs to follow, we often do case analysis on the last rule of a derivation. In such a situation, the inductive cases for both successor ordinals and limit ordinals are basically covered by the case analysis on the inference figures involved, and we shall not make explicit use of transfinite induction.

**Lemma 1.** For any substitution  $\theta$  and derivation  $\Pi$  of  $\Gamma \longrightarrow C$ ,  $\Pi\theta$  is a derivation of  $\Gamma\theta \longrightarrow C\theta$ .

*Proof.* This lemma states that Definition 3 is well-constructed, and follows by induction on  $ht(\Pi)$ .

**Lemma 2.** For any derivation  $\Pi$  and substitution  $\theta$ ,  $ht(\Pi) \ge ht(\Pi\theta)$  and  $ht(\Pi) \ge ht(\Pi\theta)$ .

*Proof.* By induction on  $ht(\Pi)$ . The measures may not be equal because in the case where the derivation ends with the eq $\mathcal{L}$  rule, some of the premise derivations of  $\Pi$  may not be needed to construct the premise derivations of  $\Pi\theta$ .

**Lemma 3.** For any derivation  $\Pi$  and substitutions  $\theta$  and  $\rho$ , the derivations  $(\Pi\theta)\rho$  and  $\Pi(\theta\circ\rho)$  are the same derivation.

*Proof.* By induction on the measure  $ht(\Pi)$ .

## 4.2 Atomic initial rule

It is a common property of most logics that the initial rule can be restricted to atomic form, that is, the rule

$$\overline{p\vec{t} \longrightarrow p\vec{t}}$$
 init

where p is a predicate symbol. The more general rule is derived as follows.

**Definition 7.** We construct a derivation  $Id_C$  of the sequent  $C \longrightarrow C$  inductively as follows. The induction is on the size of C. If C is an atomic formula we simply apply the atomic initial rule. Otherwise, we apply the left and right introduction rules for the topmost logical constant in C, probably with some instances of the contraction and the weakening rule.

The proof of the following lemma is straightforward by induction on  $ht(Id_C)$ .

**Lemma 4.** For any formula C, it holds that  $indm(Id_C) = 0$ .

Restricting the initial rule to atomic form will simplify some technical definitions to follow. We shall use Id instead of Id<sub>C</sub> to denote identity derivations since the formula C is always known from context.

## 4.3 Unfolding of derivations

**Definition 8.** Inductive unfolding. Let  $p\vec{x} \stackrel{\mu}{=} B p\vec{x}$  be an inductive definition. Let  $\Pi$  be a derivation of  $\Gamma \longrightarrow C$  where p dominates C. Let S be a closed term of the same type as p and let  $\Pi_S$  be a derivation of the sequent

$$BS\vec{x} \longrightarrow S\vec{x}$$

where  $\vec{x}$  are new eigenvariables not free in  $\Gamma$  and C. We define the derivation  $\mu_C^p(\Pi, \Pi_S)$  of  $\Gamma \longrightarrow C[S/p]$  as follows. If p is vacuous in C, then  $\mu_C^p(\Pi, \Pi_S) = \Pi$ . Otherwise, we define  $\mu_C^p(\Pi, \Pi_S)$  according to the last rule of  $\Pi$ .

1. Suppose  $\Pi$  ends with init

$$\overline{p\vec{t} \longrightarrow p\vec{t}}$$
 init.

Then  $\mu_C^p(\Pi, \Pi_S)$  is the derivation

$$\begin{array}{ccc}
\Pi_{S} & \text{Id} \\
\underline{BS\vec{x} \longrightarrow S\vec{x}} & S\vec{t} \longrightarrow S\vec{t} \\
p\vec{t} \longrightarrow S\vec{t}
\end{array}$$
 IL

2. Suppose  $\Pi$  ends with  $\supset \mathcal{L}$ 

$$\frac{\prod_1 \qquad \prod_2}{\Gamma' \longrightarrow D_1 \quad D_2, \Gamma' \longrightarrow C} \supset \mathcal{L}$$

Then  $\mu_C^p(\Pi, \Pi_S)$  is the derivation

$$\frac{ \prod_1 \qquad \mu_C^p(\Pi_2, \Pi_S)}{ D_1 \longrightarrow D_1 \quad D_2, \Gamma' \longrightarrow C[S/p]} \supset \mathcal{L}$$

$$\frac{\Gamma' \longrightarrow D_1 \quad D_2, \Gamma' \longrightarrow C[S/p]}{ D_1 \supset D_2, \Gamma' \longrightarrow C[S/p]} \supset \mathcal{L}$$

*3.* Suppose  $\Pi$  ends with  $\supset \mathcal{R}$ 

$$\frac{\Gamma, C_1 \xrightarrow{\Pi'} C_2}{\Gamma \xrightarrow{} C_1 \supset C_2} \supset \mathcal{R}$$

Note that since p dominates C, it must be the case that p does not occur in  $C_1$ . The derivation  $\mu(\Pi,\Pi_S)$  is then defined as follows.

$$\frac{\mu_{C_2}^p(\Pi',\Pi_S)}{\Gamma,C_1\longrightarrow C_2[S/p]}\supset \mathcal{R}.$$

4. Suppose  $\Pi$  ends with mc

$$\frac{\Delta_1 \longrightarrow B_1 \quad \dots \quad \Delta_m \longrightarrow B_m \quad B_1, \dots, B_m, \Gamma' \longrightarrow C}{\Delta_1, \dots, \Delta_m, \Gamma' \longrightarrow C} \quad mc$$

Then  $\mu_C^p(\Pi,\Pi_S)$  is

$$\frac{\Delta_1 \longrightarrow B_1 \quad \dots \quad \Delta_m \longrightarrow B_m \quad \mu_C^p(\Pi', \Pi_S)}{\Delta_1, \dots, \Delta_m, \Gamma' \longrightarrow C[S/p]} \ mc$$

5. Suppose  $\Pi$  ends with IL on some predicate q given a definition clause  $q\vec{z} \stackrel{\mu}{=} Dq\vec{z}$ .

$$\frac{DI\vec{z} \xrightarrow{} I\vec{z} \quad I\vec{t}, \Gamma' \xrightarrow{} C}{a\vec{t}, \Gamma' \xrightarrow{} C} \text{ I} \mathcal{L}$$

Then  $\mu_C^p(\Pi, \Pi_S)$  is the derivation

$$\frac{\Psi}{DI\vec{z} \longrightarrow I\vec{z}} \frac{\mu_C^p(\Pi', \Pi_S)}{I\vec{t}, \Gamma' \longrightarrow C[S/p]} \text{ I.f.}$$

6. Suppose  $\Pi$  ends with IR

$$\frac{\Gamma'}{\Gamma \longrightarrow B \, p \, \vec{t}} \, I \mathcal{R}.$$

Then  $\mu_C^p(\Pi, \Pi_S)$  is the derivation

$$\begin{array}{ccc} \mu_{Bp}^p(\Pi',\Pi_S) & \Pi_S[\vec{t}/\vec{x}] \\ \underline{\Gamma \longrightarrow BS\vec{t}} & BS\vec{t} \longrightarrow S\vec{t} \\ \Gamma \longrightarrow S\vec{t} & mc. \end{array}$$

7. If  $\Pi$  ends with any other rules, and has premise derivations

$$\left\{ \Gamma_i \xrightarrow{\Pi_i} C_i \right\}_{i \in I}$$

for some index set I, then  $\mu_C^p(\Pi,\Pi_S)$  also ends with the same rule and has premise derivations  $\{\mu_{C_i}^p(\Pi_i,\Pi_S)\}_{i\in I}$ .

**Definition 9.** Co-inductive unfolding. Let  $p\vec{x} \stackrel{\vee}{=} B p\vec{x}$  be a co-inductive definition. Let S be a closed term of the same type as p and let  $\Pi_S$  be a derivation of

$$S\vec{x} \longrightarrow BS\vec{x}$$
.

Let C be a formula dominated by p, and let  $\Pi$  be a derivation of  $\Gamma \longrightarrow C[S/p]$ . We define the derivation  $V_C^p(\Pi, \Pi_S)$  of  $\Gamma \longrightarrow C$  as follows.

If p is vacuous in C, then  $v_C^p(\Pi, \Pi_S) = \Pi$ . If  $C = p\vec{t}$  then  $C[S/p] = S\vec{t}$  and  $v_C^p(\Pi, \Pi_S)$  is the derivation

$$\frac{\Gamma \longrightarrow S\vec{t} \quad S\vec{x} \longrightarrow BS\vec{x}}{\Gamma \longrightarrow p\vec{t}} \text{ CIR}$$

Otherwise, we define  $V_C^p(\Pi,\Pi_S)$  based on the last rule in  $\Pi$ .

1. Suppose  $\Pi$  ends with  $\supset \mathcal{L}$ 

$$\frac{\Gamma' \longrightarrow D_1 \quad D_2, \Gamma' \longrightarrow C[S/p]}{D_1 \supset D_2, \Gamma' \longrightarrow C[S/p]} \supset \mathcal{L}$$

Then  $v_C^p(\Pi, \Pi_S)$  is the derivation

$$\frac{\Pi_1 \qquad \mathsf{V}^p_C(\Pi_2,\Pi_S)}{\Gamma' \longrightarrow D_1 \quad D_2,\Gamma' \longrightarrow C} \supset \mathcal{L}$$

2. Suppose  $\Pi$  ends with  $\supset \mathcal{R}$ 

$$\frac{\Gamma, C_1 \longrightarrow C_2[S/p]}{\Gamma \longrightarrow (C_1 \supset C_2)[S/p]} \supset \mathcal{R}$$

Note that since p dominates C, it must be the case that p is vacuous in  $C_1$ . Therefore we construct the derivation  $V_C^p(\Pi,\Pi_S)$  as follows.

$$\begin{array}{l}
\nu_{C_2}^p(\Pi',\Pi_S) \\
\underline{\Gamma,C_1 \longrightarrow C_2} \\
\underline{\Gamma \longrightarrow C_1 \supset C_2}
\end{array} \supset \mathcal{R}$$

3. Suppose  $\Pi$  ends with mc

$$\frac{\Delta_1 \xrightarrow{\Pi_1} B_1 \dots \Delta_m \xrightarrow{\Pi_m} B_m \quad B_1, \dots, B_m, \Gamma' \longrightarrow C[S/p]}{\Delta_1, \dots, \Delta_m, \Gamma' \longrightarrow C[S/p]} \text{ mc}$$

Then  $v_C^p(\Pi,\Pi_S)$  is

$$\underline{\Delta_1 \longrightarrow B_1} \quad \dots \quad \underline{\Lambda_m \longrightarrow B_m} \quad B_1, \dots, B_m, \Gamma' \longrightarrow C$$

$$\underline{\Delta_1, \dots, \Delta_m, \Gamma' \longrightarrow C} \quad mc$$

4. Suppose  $\Pi$  ends with IL on a predicate  $q\vec{t}$ , given an inductive definition  $q\vec{z} \stackrel{\mu}{=} Dq\vec{z}$ .

$$\frac{DI\vec{z} \xrightarrow{\Psi} I\vec{z} \quad I\vec{t}, \Gamma' \xrightarrow{\Pi'} C[S/p]}{g\vec{t}, \Gamma' \xrightarrow{} C[S/p]} \text{ I} \mathcal{L}$$

Then  $v_C^p(\Pi, \Pi_S)$  is the derivation

$$\frac{\Psi}{DI\vec{z} \longrightarrow I\vec{z}} \frac{V_C^p(\Pi', \Pi_S)}{I\vec{t}, \Gamma' \longrightarrow C} \text{ I} \mathcal{L}$$

5. If  $\Pi$  ends with any other rules, and has premise derivations

$$\left\{ \begin{array}{c} \Pi_i \\ \Gamma_i \longrightarrow C_i[S/p] \end{array} \right\}_{i \in I}$$

for some index set I, then  $v_C^p(\Pi,\Pi_S)$  also ends with the same rule and has premise derivations  $\{v_C^p(\Pi_i,\Pi_S)\}_{i\in I}$ .

The following two lemmas state that substitutions commute with unfolding of derivations. Their proofs follow straightforwardly from the fact that the definitions of (co-)inductive unfolding depend only on the logical structures of conclusions of sequents, hence is orthogonal to substitutions of eigenvariables. In these lemmas, we assume that the formulas C, p and derivations  $\Pi$  and  $\Pi_S$  satisfy the conditions of Definition 8 and Definition 9.

**Lemma 5.** The derivations  $\mu_C^p(\Pi, \Pi_S)\theta$  and  $\mu_C^p(\Pi\theta, \Pi_S)$  are the same derivation.

**Lemma 6.** The derivations  $\mathbf{v}_C^p(\Pi, \Pi_S)\theta$  and  $\mathbf{v}_C^p(\Pi\theta, \Pi_S)$  are the same derivation.

## 5 Cut elimination for Linc-

A central result of our work is cut-elimination, from which consistency of the logic follows. Gentzen's classic proof of cut-elimination for first-order logic uses an induction on the size of the cut formula, i.e., the number of logical connectives in the formula. The cut-elimination procedure consists of a set of reduction rules that reduce a cut of a compound formula to cuts on its sub-formulae of smaller size. In the case of Linc<sup>-</sup>, the use of induction/co-induction complicates the reduction of cuts. Consider for example a cut involving the induction rules

$$\frac{\Delta \xrightarrow{Bpt} IR \xrightarrow{BSy \xrightarrow{Sy} St, \Gamma \xrightarrow{C} C} \Pi}{\Delta \xrightarrow{pt, \Gamma \xrightarrow{C}} mc} IL$$

There are at least two problems in reducing this cut. First, any permutation upwards of the cut will necessarily involve a cut with S that can be of larger size than p, and hence a simple induction on the size of cut formula will not work. Second, the invariant S does not appear in the conclusion of the left premise of the cut. The latter means that we need to transform the left premise so that its end sequent will agree with the right premise. Any such transformation will most likely be global, and hence simple induction on the height of derivations will not work either.

We shall use the *reducibility* techniques to prove cut elimination. More specifically, we shall build on the notion of reducibility introduced by Martin-Löf to prove normalization of an intuitionistic logic with iterative inductive definition [24]. Martin-Löf's proof has been adapted to sequent calculus by McDowell and Miller [25], but in a restricted setting where only natural number induction is allowed. Since our logic involves arbitrary stratified inductive definitions, which also includes iterative inductive definitions, we shall need a more general cut reductions. But the real difficulty in our case is really in establishing cut elimination in the presence of co-inductive definitions, for which there is no known cut elimination proof for the sequent calculus formulation.

The main part of the reducibility technique is a definition of the family of reducible sets of derivations. In Martin-Löf's theory of iterative inductive definition, this family of sets is defined inductively by the level of the derivations they contain. Extending this definition of reducibility to Linc<sup>-</sup> is not obvious. In particular, in establishing the reducibility of a derivation  $\Xi$  ending with a CI $\Re$  rule:

$$\frac{\Gamma \xrightarrow{\prod S\vec{t} \quad S\vec{x} \xrightarrow{\longrightarrow} BS\vec{x}}}{\Gamma \xrightarrow{\longrightarrow} p\vec{t}} \text{ CI}\mathcal{R}, p\vec{x} \stackrel{\vee}{=} Bp\vec{x}$$

one must first establish the reducibility of its premise derivations. But a naive definition of reducibility for  $\Xi$ , i.e., a definition that postulates the reducibility of  $\Xi$  from the reducibility of its premises, is not a monotone definition, since the premise derivations of  $\Xi$  may be derivations that have a higher level than  $\Xi$ .

To define a proper notion of reducibility for the co-inductive cases, we use a notion of parametric reducibility, similar to that used in the strong normalisation proof of System F [19]. The notion of a parameter in our case is essentially a coinductive predicate. As with strong normalisation of System F, these parameters are substituted with some "reducibility candidates", which in our case are certain sets of derivations of a co-inductive invariant which we call saturated sets. Let us say that a derivation  $\Psi$  has type B if its end sequent is of the form  $\Gamma \longrightarrow B$ , for some  $\Gamma$ . Roughly, a parametric reducibility set of type C, under a parameter substitution [S/p], where p is a co-inductive predicate and S is an invariant of the same type as S, is a certain set of derivations of type C[S/p] satisfying some closure conditions which are very similar to the definition of reducibility sets, but without the co-inductive part. The definition of reducibility in the case involving co-induction rules, e.g., as in the derivation  $\Xi$  above, can then be defined in terms parametric reducibility sets, under appropriate parameter substitutions. Details of the definition will be given later in this section.

#### 5.1 Cut reduction

We follow the idea of Martin-Löf in using derivations directly as a measure by defining a well-founded ordering on them. The basis for the latter relation is a set of reduction rules (called the contraction rules in [24]) that are used to eliminate the applications of the cut rule. For the cases involving logical connectives, the cut-reduction rules used to prove the cut-elimination for Linc<sup>-</sup> are the same to those of  $FO\lambda^{\Delta IN}$ . The crucial differences are in the reduction rules involving induction and co-induction rules.

**Definition 10.** We define a reduction relation between derivations. The redex is always a derivation  $\Xi$  ending with the multicut rule

$$\frac{\Delta_1 \xrightarrow{\Pi_1} B_1 \cdots \Delta_n \xrightarrow{\Pi_n} B_n \quad B_1, \dots, B_n, \Gamma \longrightarrow C}{\Delta_1, \dots, \Delta_n, \Gamma \longrightarrow C} mc$$

We refer to the formulas  $B_1, \ldots, B_n$  produced by the mc as cut formulas.

*If* n = 0,  $\Xi$  *reduces to the premise derivation*  $\Pi$ .

For n>0 we specify the reduction relation based on the last rule of the premise derivations. If the rightmost premise derivation  $\Pi$  ends with a left rule acting on a cut formula  $B_i$ , then the last rule of  $\Pi_i$  and the last rule of  $\Pi$  together determine the reduction rules that apply. We classify these rules according to the following criteria: we call the rule an essential case when  $\Pi_i$  ends with a right rule; if it ends with a left rule, it is a left-commutative case; if  $\Pi_i$  ends with the init rule, then we have an axiom case; a multicut case arises when it ends with the mc rule. When  $\Pi$  does not end with a left rule acting on a cut formula, then its last rule is alone sufficient to determine the reduction rules that apply. If  $\Pi$  ends in a rule acting on a formula other than a cut formula, then we call this a right-commutative case. A structural case results when  $\Pi$  ends with a contraction or weakening on a cut formula. If  $\Pi$  ends with the init rule, this is also an axiom case; similarly a multicut case arises if  $\Pi$  ends in the mc rule.

For simplicity of presentation, we always show i = 1.

Essential cases:

 $\wedge \mathcal{R} / \wedge \mathcal{L}$ : If  $\Pi_1$  and  $\Pi$  are

$$\frac{\Delta_1 \xrightarrow{\Pi_1'} \qquad \Pi_1''}{\Delta_1 \xrightarrow{B_1'} \Delta_1 \xrightarrow{B_1''} \wedge \mathcal{R}} \qquad \frac{B_1', B_2, \dots, B_n, \Gamma \xrightarrow{C} C}{B_1' \wedge B_1'', B_2, \dots, B_n, \Gamma \xrightarrow{C} \Lambda \mathcal{L}},$$

then  $\Xi$  reduces to

$$\frac{\Pi'_1}{\Delta_1 \longrightarrow B'_1} \quad \frac{\Pi_2}{\Delta_2 \longrightarrow B_2} \quad \cdots \quad \frac{\Pi_n}{\Delta_n \longrightarrow B_n} \quad B'_1, B_2, \dots, B_n, \Gamma \longrightarrow C$$

$$\frac{\Delta_1, \dots, \Delta_n, \Gamma \longrightarrow C}{\Delta_1, \dots, \Delta_n, \Gamma \longrightarrow C} \quad mc$$

*The case for the other*  $\land \bot$  *rule is symmetric.* 

 $\vee \mathcal{R} / \vee \mathcal{L}$ : If  $\Pi_1$  and  $\Pi$  are

$$\frac{\Pi'_1}{\Delta_1 \longrightarrow B'_1} \vee \mathcal{R} \qquad \frac{\Pi' \qquad \Pi''}{B'_1, B_2, \dots, B_n, \Gamma \longrightarrow C \qquad B''_1, B_2, \dots, B_n, \Gamma \longrightarrow C}}{B'_1 \vee B''_1, B_2, \dots, B_n, \Gamma \longrightarrow C} \vee \mathcal{L}$$

then  $\Xi$  reduces to

$$\frac{\Delta_1 \longrightarrow B_1' \qquad \Gamma_2}{\Delta_1 \longrightarrow B_1' \qquad \Delta_2 \longrightarrow B_2 \qquad \cdots \qquad \Delta_n \longrightarrow B_n \qquad B_1', B_2, \dots, B_n, \Gamma \longrightarrow C}{\Delta_1, \dots, \Delta_n, \Gamma \longrightarrow C} mc$$

*The case for the other*  $\vee \mathcal{R}$  *rule is symmetric.* 

 $\supset \mathcal{R}/\supset \mathcal{L}$ : Suppose  $\Pi_1$  and  $\Pi$  are

$$\frac{\Pi_1'}{B_1',\Delta_1 \longrightarrow B_1''} \supset \mathcal{R} \qquad \qquad \frac{\Pi'}{B_2,\ldots,B_n,\Gamma \longrightarrow B_1'} \frac{\Pi''}{B_1',B_2,\ldots,B_n,\Gamma \longrightarrow C} \supset \mathcal{L}$$

Let  $\Xi_1$  be

$$\frac{\left\{ \Delta_{i} \stackrel{\Pi_{i}}{\longrightarrow} B_{i} \right\}_{i \in \left\{2...n\right\}} \quad B_{2}, \dots, B_{n}, \Gamma \stackrel{\Pi'}{\longrightarrow} B'_{1}}{\Delta_{2}, \dots, \Delta_{n}, \Gamma \stackrel{}{\longrightarrow} B'_{1}} \quad mc \quad \prod'_{1} \quad B'_{1}, \Delta_{1} \stackrel{}{\longrightarrow} B''_{1}} \quad mc \quad \Delta_{1}, \dots, \Delta_{n}, \Gamma \stackrel{}{\longrightarrow} B''_{1}} \quad mc$$

Then  $\Xi$  reduces to

$$\frac{\Xi_{1}}{\cdots \longrightarrow B_{1}''} \quad \begin{cases} \Pi_{i} \\ \Delta_{i} \longrightarrow B_{i} \end{cases} \prod_{i \in \{2..n\}} \prod_{\substack{B_{1}'', \{B_{i}\}_{i \in \{2..n\}}, \Gamma \longrightarrow C \\ \Delta_{1}, \dots, \Delta_{n}, \Gamma, \Delta_{2}, \dots, \Delta_{n}, \Gamma \longrightarrow C}} mc$$

We use the double horizontal lines to indicate that the relevant inference rule (in this case, cL) may need to be applied zero or more times.

 $\forall \mathcal{R}/\forall \mathcal{L}$ : If  $\Pi_1$  and  $\Pi$  are

$$\frac{\Pi'_1}{\Delta_1 \longrightarrow B'_1[y/x]} \xrightarrow{} \forall \mathcal{R} \qquad \frac{B'_1[t/x], B_2, \dots, B_n, \Gamma \longrightarrow C}{\forall x. B'_1, B_2, \dots, B_n, \Gamma \longrightarrow C} \forall \mathcal{L}$$

then  $\Xi$  reduces to

$$\frac{ \prod_{1}^{\prime}[t/y] }{ \Delta_{1} \longrightarrow B_{1}^{\prime}[t/x] } \begin{cases} \Delta_{i} \xrightarrow{\Pi_{i}} B_{i} \\ \delta_{i} \longrightarrow B_{i} \end{cases}_{i \in \{2..n\}} \xrightarrow{\Pi^{\prime}} C \xrightarrow{mc}$$

 $\exists \mathcal{R}/\exists \mathcal{L}$ : If  $\Pi_1$  and  $\Pi$  are

$$\frac{\Pi'_1}{\Delta_1 \longrightarrow B'_1[t/x]} \xrightarrow{\exists \mathcal{R}} \frac{\Pi'}{\exists x.B'_1, B_2, \dots, B_n, \Gamma \longrightarrow C} \xrightarrow{\exists \mathcal{L}},$$

then  $\Xi$  reduces to

$$\frac{\Pi'_1}{\Delta_1 \longrightarrow B'_1[t/x]} \frac{\Pi'[t/y]}{\dots B'_1[t/x], B_2, \dots, \Gamma \longrightarrow C} mc$$

\*/IL: Suppose  $\Pi$  is the derivation

$$\frac{DS\vec{x} \longrightarrow S\vec{x} \quad S\vec{t}, B_2, \dots, B_n, \Gamma \longrightarrow C}{p\vec{t}, B_2, \dots, B_n, \Gamma \longrightarrow C} \text{ I} \mathcal{L}$$

where  $p\vec{x} \stackrel{\mu}{=} B p\vec{x}$ . Then  $\Xi$  reduces to

$$\mu_{p\vec{t}}^{p}(\Pi_{1},\Pi_{S}) \qquad \Pi' \\
\underline{\Delta_{1} \longrightarrow S\vec{t} \quad \dots \quad S\vec{t},\dots,B_{n},\Gamma \longrightarrow C} \quad mc$$

CIR/CIL: Suppose  $\Pi_1$  and  $\Pi$  are

$$\frac{\Pi'_1}{\Delta_1 \longrightarrow S\vec{t}} \frac{\Pi_S}{S\vec{x} \longrightarrow DS\vec{x}} \text{ CIR} \qquad \frac{\Pi'}{p\vec{t}, \dots, \Gamma \longrightarrow C} \text{ CIL}$$

Let  $\Xi_1$  be the derivation

$$\frac{\Pi'_1}{\Delta_1 \longrightarrow S\vec{t}} \frac{\Pi_S[\vec{t}/\vec{x}]}{S\vec{t} \longrightarrow DS\vec{t}} mc$$

Then  $\Xi$  reduces to

$$\frac{\mathsf{v}^p_{Dp}(\Xi_1,\Pi_S)}{\Delta_1 \longrightarrow D\,p\vec{t}} \; \left\{ \begin{matrix} \Pi_j \\ \Delta_j \longrightarrow B_j \end{matrix} \right\}_{j \in \{2,...,n\}} \; \; D\,p\vec{t}, \ldots, \Gamma \longrightarrow C \\ \Delta_1, \ldots, \Delta_n, \Gamma \longrightarrow C \end{cases} \; mc$$

 $eq \mathcal{R}/eq \mathcal{L}$ : Suppose  $\Pi_1$  and  $\Pi$  are

$$\frac{\left\{\begin{matrix}\Pi^{\rho}\\B_{2}\rho,\ldots,B_{n}\rho,\Gamma\rho\longrightarrow C\rho\right\}_{\rho}}{s=t,B_{2},\ldots,B_{n},\Gamma\longrightarrow C}\text{ eq}\mathcal{L}$$

Then by the definition of eq $\mathcal{R}$  rule, s and t are equal terms (modulo  $\lambda$ -conversion), and hence are unifiable by the empty substitution. Note that in this case  $\Pi^{\epsilon} \in \{\Pi^{\rho}\}_{0}$ . Therefore  $\Xi$  reduces to

$$\frac{\left\{ \Delta_{i} \stackrel{\prod_{i}}{\longrightarrow} B_{i} \right\}_{i \in \left\{2..n\right\}} \stackrel{B_{2}, \dots, B_{n}, \Gamma}{\longrightarrow} C}{\Delta_{2}, \dots, \Delta_{n}, \Gamma \stackrel{\longrightarrow}{\longrightarrow} C} \ mc} \frac{\Delta_{2}, \dots, \Delta_{n}, \Gamma \stackrel{\longrightarrow}{\longrightarrow} C}{\boxtimes} \ wL$$

Left-commutative cases: In the following cases, we suppose that  $\Pi$  ends with a left rule, other than  $\{cL, wL, IL\}$ , acting on  $B_1$ .

• $L/\circ L$ : Suppose  $\Pi_1$  is

$$\frac{\left\{\begin{array}{c}\Pi_1^i\\\Delta_1^i\longrightarrow B_1\end{array}\right\}}{\Delta_1\longrightarrow B_1}\bullet L$$

where  $\bullet L$  is any left rule except  $\supset L$ , eqL, or IL. Then  $\Xi$  reduces to

$$\frac{\left\{ \begin{array}{ccc} \Pi_{1}^{i} & \left\{ \Delta_{j} \stackrel{\Pi_{j}}{\longrightarrow} B_{j} \right\}_{j \in \{2..n\}} & \Pi \\ \Delta_{1}^{i} \stackrel{\Pi}{\longrightarrow} B_{1} & \left\{ \Delta_{j} \stackrel{\Pi_{j}}{\longrightarrow} B_{j} \right\}_{j \in \{2..n\}} & B_{1}, \dots, B_{n}, \Gamma \stackrel{\Pi}{\longrightarrow} C \end{array} \right\}}{\Delta_{1}, \Delta_{2}, \dots, \Delta_{n}, \Gamma \stackrel{\Pi}{\longrightarrow} C}$$

$$\frac{\Delta_{1}, \Delta_{2}, \dots, \Delta_{n}, \Gamma \stackrel{\Pi}{\longrightarrow} C}{\Delta_{1}, \Delta_{2}, \dots, \Delta_{n}, \Gamma \stackrel{\Pi}{\longrightarrow} C}$$

 $\supset \mathcal{L}/\circ \mathcal{L}$ : Suppose  $\Pi_1$  is

$$\frac{D_1' \qquad D_1'' \qquad D_1''}{D_1' \supset D_1'', \Delta_1' \longrightarrow B_1} \supset \mathcal{L}$$

Let  $\Xi_1$  be

$$\frac{D_1'', \Delta_1' \longrightarrow B_1 \quad D_2 \qquad \prod_n \qquad \prod_n \qquad \prod_{b_1, \dots, B_n, \Gamma \longrightarrow C}}{D_1'', \Delta_1', \Delta_2, \dots, \Delta_n, \Gamma \longrightarrow C} mc$$

Then  $\Xi$  reduces to

$$\frac{ \begin{array}{c} \Pi_1' \\ \Delta_1' \longrightarrow D_1' \\ \hline \Delta_1', \Delta_2, \dots, \Delta_n, \Gamma \longrightarrow D_1' \end{array} \stackrel{WL}{\longrightarrow} \begin{array}{c} \Xi_1 \\ D_1'', \Delta_1', \Delta_2, \dots, \Delta_n, \Gamma \longrightarrow C \\ \hline D_1' \supset D_1'', \Delta_1', \Delta_2, \dots, \Delta_n, \Gamma \longrightarrow C \end{array} \supset \mathcal{L}$$

 $I\mathcal{L}/\circ\mathcal{L}$ : Suppose  $\Pi_1$  is

$$\frac{DS\vec{x} \longrightarrow S\vec{x} \quad S\vec{t}, \Delta'_1 \longrightarrow B_1}{p\vec{t}, \Delta'_1 \longrightarrow B_1} \text{ IL}$$

where  $p\vec{x} \stackrel{\mu}{=} Dp\vec{x}$ . Let  $\Xi_1$  be

$$\frac{\Pi'_1}{S\vec{t}, \Delta'_1 \longrightarrow B_1 \quad \dots \quad \Delta_n \longrightarrow B_n \quad B_1, \dots, B_n, \Gamma \longrightarrow C} M_1 \quad \text{mc}$$

Then  $\Xi$  reduces to

$$\frac{DS\vec{x} \longrightarrow S\vec{x} \quad S\vec{t}, \Delta'_1, \dots, \Delta_n, \Gamma \longrightarrow C}{p\vec{t}, \Delta'_1, \dots, \Delta_n \longrightarrow C} \text{ I.C.}$$

 $eq \mathcal{L} / \circ \mathcal{L}$ : Suppose  $\Pi_1$  is

$$\frac{\left\{\begin{array}{c}\Pi_{1}^{\rho}\\\Delta_{1}'\rho\longrightarrow B_{1}\rho\end{array}\right\}}{s=t,\Delta_{1}'\longrightarrow B_{1}}\operatorname{eq}\mathcal{L}$$

then  $\Xi$  reduces to

$$\frac{\left\{ \frac{\Pi_{1}^{\rho}}{\Delta'_{1}\rho \longrightarrow B_{1}\rho} \left\{ \begin{array}{c} \Pi_{1}\rho \\ \Delta_{i}\rho \longrightarrow B_{i}\rho \end{array} \right\}_{i\in\{2..n\}} \quad \Pi\rho \\ \frac{\Delta'_{1}\rho, \Delta_{2}\rho, \dots, \Delta_{n}\rho, \Gamma\rho \longrightarrow C\rho}{s = t, \Delta'_{1}, \Delta_{2}, \dots, \Delta_{n}, \Gamma \longrightarrow C} \text{ eq} \right\}}{s = t, \Delta'_{1}, \Delta_{2}, \dots, \Delta_{n}, \Gamma \longrightarrow C}$$

Right-commutative cases:

 $-/\circ \mathcal{L}$ : Suppose  $\Pi$  is

$$egin{array}{c} \left\{ egin{array}{c} \Pi^i \ B_1, \dots, B_n, \Gamma^i \longrightarrow C \end{array} 
ight\} \ B_1, \dots, B_n, \Gamma \longrightarrow C \end{array} \circ \mathcal{L}$$

where  $\circ \mathcal{L}$  is any left rule other than  $\supset \mathcal{L}$ , eq $\mathcal{L}$ , or  $I\mathcal{L}$  acting on a formula other than  $B_1, \dots, B_n$ . The derivation  $\Xi$  reduces to

$$\left\{ \frac{\Delta_1 \xrightarrow{\Pi_1} B_1 \cdots \Delta_n \xrightarrow{\Pi_n} B'_n B_1, \dots, B_n, \Gamma^i \longrightarrow C}{\Delta_1, \dots, \Delta_n, \Gamma^i \longrightarrow C} mc \right\}$$

$$\frac{\Delta_1 \xrightarrow{\Pi_1} B_1 \cdots \Delta_n \xrightarrow{\Pi_n} B_1, \dots, B_n, \Gamma^i \longrightarrow C}{\Delta_1, \dots, \Delta_n, \Gamma \longrightarrow C} \circ L$$

$$-/\supset \mathcal{L}$$
: Suppose  $\Pi$  is

$$\frac{B_1,\ldots,B_n,\Gamma'\longrightarrow D'\quad B_1,\ldots,B_n,D'',\Gamma'\longrightarrow C}{B_1,\ldots,B_n,D'\supset D'',\Gamma'\longrightarrow C}\supset \mathcal{L}$$

Let  $\Xi_1$  be

$$\frac{\Delta_1 \longrightarrow B_1 \quad \cdots \quad \Delta_n \longrightarrow B_n \quad B_1, \dots, B_n, \Gamma' \longrightarrow D'}{\Delta_1, \dots, \Delta_n, \Gamma' \longrightarrow D'} \quad mc$$

and  $\Xi_2$  be

$$\frac{\Delta_1 \xrightarrow{\Pi_1} B_1 \cdots \Delta_n \xrightarrow{\Pi_n} B_n B_1, \dots, B_n, D'', \Gamma' \longrightarrow C}{\Delta_1, \dots, \Delta_n, D'', \Gamma' \longrightarrow C} mc$$

Then  $\Xi$  reduces to

$$\frac{\Xi_1}{\Delta_1,\ldots,\Delta_n,\Gamma'\longrightarrow D'} \frac{\Xi_2}{\Delta_1,\ldots,\Delta_n,D'',\Gamma'\longrightarrow C}\supset \mathcal{L}$$

 $-/I\mathcal{L}$ : Suppose  $\Pi$  is

$$\frac{DS\vec{x} \longrightarrow S\vec{x} \quad B_1, \dots, B_n, S\vec{t}, \Gamma' \longrightarrow C}{B_1, \dots, B_n, p\vec{t}, \Gamma' \longrightarrow C} \text{ I} \mathcal{L}$$

where  $p\vec{x} \stackrel{\mu}{=} Dp\vec{x}$ . Let  $\Xi_1$  be

$$\frac{\Delta_1 \xrightarrow{\Pi_1} B_1 \cdots \Delta_n \xrightarrow{\Pi_n} B_n B_1, \dots, B_n, S\vec{t}, \Gamma' \longrightarrow C}{\Delta_1, \dots, \Delta_n, S\vec{t}, \Gamma' \longrightarrow C} mc$$

Then  $\Xi$  reduces to

$$\frac{DS\vec{x} \xrightarrow{\Pi_S} S\vec{x} \quad \Delta_1, \dots, \Delta_n, S\vec{t}, \Gamma' \longrightarrow C}{\Delta_1, \dots, \Delta_n, p\vec{t}, \Gamma' \longrightarrow C} \text{ I} \mathcal{L}$$

 $-/eq\mathcal{L}$ : If  $\Pi$  is

$$\frac{\left\{\begin{matrix}\Pi^{\rho}\\B_{1}\rho,\ldots,B_{n}\rho,\Gamma'\rho\longrightarrow C\rho\end{matrix}\right\}}{B_{1},\ldots,B_{n},s=t,\Gamma'\longrightarrow C} \text{ eq}\mathcal{L}$$

then  $\Xi$  reduces to

$$\frac{\left\{ \begin{cases} \Pi_{i}\rho \\ \Delta_{i}\rho \xrightarrow{\longrightarrow} B_{i}\rho \end{cases} \prod_{i\in\{1..n\}} \prod_{\substack{B_{i}\rho,\ldots,\Gamma'\rho \xrightarrow{\longrightarrow} C\rho \\ \Delta_{1}\rho,\ldots,\Delta_{n}\rho,\Gamma'\rho \xrightarrow{\longrightarrow} C\rho} mc \right\}}{\Delta_{1}\rho,\ldots,\Delta_{n}\rho,\Gamma'\rho \xrightarrow{\longrightarrow} C\rho} mc \right\}$$

$$\frac{\left\{ \begin{cases} \Pi_{i}\rho \\ \Delta_{1}\rho \xrightarrow{\longrightarrow} B_{i}\rho \end{cases} \prod_{i\in\{1..n\}} \prod_{\substack{B_{i}\rho,\ldots,\Gamma'\rho \xrightarrow{\longrightarrow} C\rho \\ \Delta_{1}\rho,\ldots,\Delta_{n}\rho,\Gamma'\rho \xrightarrow{\longrightarrow} C\rho} mc \right\}}{\Delta_{1}\rho,\ldots,\Delta_{n}\rho,\Gamma'\rho \xrightarrow{\longrightarrow} C\rho} mc \right\}$$

 $-/\circ \mathcal{R}$ : If  $\Pi$  is

$$\frac{\left\{\begin{array}{c}\Pi^{i}\\B_{1},\ldots,B_{n},\Gamma^{i}\longrightarrow C^{i}\end{array}\right\}}{B_{1},\ldots,B_{n},\Gamma\longrightarrow C}\circ\mathcal{R},$$

where  $\circ \mathcal{R}$  is any right rule except CI $\mathcal{R}$ , then  $\Xi$  reduces to

$$\frac{\left\{\begin{array}{cccc}
\Pi_{1} & \Pi_{n} & \Pi^{i} \\
\Delta_{1} \longrightarrow B_{1} & \cdots & \Delta_{n} \longrightarrow B_{n} & B_{1}, \dots, B_{n}, \Gamma^{i} \longrightarrow C^{i} \\
& \Delta_{1}, \dots, \Delta_{n}, \Gamma^{i} \longrightarrow C^{i}
\end{array}\right\}}{\Delta_{1}, \dots, \Delta_{n}, \Gamma \longrightarrow C} \circ \mathcal{R}$$

 $-/\text{CI}\mathcal{R}$ : Suppose  $\Pi$  is

$$\frac{B_1, \dots, B_n, \Gamma \longrightarrow S\vec{t} \quad S\vec{x} \xrightarrow{\Pi_S} DS\vec{x}}{B_1, \dots, B_n, \Gamma \longrightarrow p\vec{t}} \text{ CIR}$$

where  $p\vec{x} \stackrel{\vee}{=} Dp\vec{x}$ . Let  $\Xi_1$  be

$$\frac{\Delta_1 \xrightarrow{\Pi_1} B_1 \cdots \Delta_n \xrightarrow{\Pi_n} B_n \quad B_1, \dots, B_n, \Gamma \xrightarrow{S\vec{t}} S\vec{t}}{\Delta_1, \dots, \Delta_n, \Gamma \xrightarrow{S\vec{t}} S\vec{t}} mc$$

Then  $\Xi$  reduces to

$$\frac{\Xi_1}{\Delta_1, \dots, \Delta_n, \Gamma \longrightarrow S\vec{t}} \frac{\Pi_S}{S\vec{x} \longrightarrow DS\vec{x}} \text{ CIR}$$

$$\frac{\Delta_1, \dots, \Delta_n, \Gamma \longrightarrow p\vec{t}}{\Delta_1, \dots, \Delta_n, \Gamma \longrightarrow p\vec{t}}$$

Multicut cases:

 $mc/\circ L$ : If  $\Pi$  ends with a left rule, other than cL, wL and IL, acting on  $B_1$  and  $\Pi_1$  ends with a multicut and reduces to  $\Pi'_1$ , then  $\Xi$  reduces to

$$\frac{\Delta_1 \longrightarrow B_1 \quad \Delta_2 \longrightarrow B_2 \quad \cdots \quad \Delta_n \longrightarrow B_n \quad B_1, \dots, B_n, \Gamma \longrightarrow C}{\Delta_1, \dots, \Delta_n, \Gamma \longrightarrow C} \quad mc$$

-/mc: Suppose  $\Pi$  is

$$\frac{\left\{\{B_i\}_{i\in I^j},\Gamma^j\longrightarrow D^j\right\}_{j\in\{1..m\}}}{\{B_i\}_{i\in I^j},\Gamma^j\longrightarrow D}\frac{\Pi'}{\{B_i\}_{j\in\{1..m\}},\{B_i\}_{i\in I'},\Gamma'\longrightarrow C}$$

$$B_1,\ldots,B_n,\Gamma^1,\ldots,\Gamma^m,\Gamma'\longrightarrow C$$

$$mc$$

where  $I^1, \ldots, I^m, I'$  partition the formulas  $\{B_i\}_{i \in \{1..n\}}$  among the premise derivations  $\Pi_1, \ldots, \Pi_m, \Pi'$ . For  $1 \le j \le m$  let  $\Xi^j$  be

$$\frac{\left\{ \Delta_{i} \xrightarrow{\Pi_{i}} B_{i} \right\}_{i \in I^{j}} \prod_{i \in I^{j}} \Pi^{j}}{\left\{ \Delta_{i} \right\}_{i \in I^{j}}, \Gamma^{j} \longrightarrow D^{j}} mc$$

Then  $\Xi$  reduces to

$$\frac{\left\{ \dots \xrightarrow{\Xi^{j}} D^{j} \right\}_{j \in \{1..m\}} \quad \left\{ \begin{array}{c} \Pi_{i} \\ \Delta_{i} \longrightarrow B_{i} \end{array} \right\}_{i \in I'} \quad \dots \xrightarrow{\Pi'} C}{\Delta_{1}, \dots, \Delta_{n}, \Gamma^{1}, \dots \Gamma^{m}, \Gamma' \longrightarrow C} \ mc$$

Structural cases:

 $-/c\mathcal{L}$ : If  $\Pi$  is

$$egin{array}{c} \Pi' \ B_1, B_2, \dots, B_n, \Gamma \longrightarrow C \ B_1, B_2, \dots, B_n, \Gamma \longrightarrow C \end{array}$$
 c.l.

then  $\Xi$  reduces to

$$\frac{\Delta_{1} \xrightarrow{\Pi_{1}} \left\{ \Delta_{i} \xrightarrow{\Pi_{i}} B_{i} \right\}_{i \in \{1..n\}} \xrightarrow{B_{1}, B_{1}, B_{2}, \dots, B_{n}, \Gamma \xrightarrow{} C} mc}{\frac{\Delta_{1}, \Delta_{1}, \Delta_{2}, \dots, \Delta_{n}, \Delta_{n}, \Gamma \xrightarrow{} C}{\Delta_{1}, \Delta_{2}, \dots, \Delta_{n}, \Gamma \xrightarrow{} C} cL} mc$$

$$-/w\mathcal{L}$$
: If  $\Pi$  is

$$\frac{B_2,\ldots,B_n,\Gamma\longrightarrow C}{B_1,B_2,\ldots,B_n,\Gamma\longrightarrow C} \text{ WL} ,$$

then  $\Xi$  reduces to

$$\frac{\Delta_2 \xrightarrow{\Pi_2} B_2 \quad \dots \quad \Delta_n \xrightarrow{\Pi_n} B_n \quad B_2, \dots, B_n, \Gamma \longrightarrow C}{\Delta_2, \dots, \Delta_n, \Gamma \longrightarrow C} \quad mc$$

$$\frac{\Delta_2, \dots, \Delta_n, \Gamma \longrightarrow C}{\Delta_1, \Delta_2, \dots, \Delta_n, \Gamma \longrightarrow C} \quad wL$$

Axiom cases:

init/ $\circ \mathcal{L}$ : Suppose  $\Pi$  ends with a left-rule acting on  $B_1$  and  $\Pi_1$  ends with the init rule. Then it must be the case that  $\Delta_1 = \{B_1\}$  and  $\Xi$  reduces to

$$\frac{\Delta_2 \longrightarrow B_2 \quad \cdots \quad \Delta_n \longrightarrow B_n \quad B_1, B_2, \dots, B_n, \Gamma \longrightarrow C}{B_1, \Delta_2, \dots, \Delta_n, \Gamma \longrightarrow C} \quad mc$$

-/init: If  $\Pi$  ends with the init rule, then n = 1,  $\Gamma$  is the empty multiset, and C must be a cut formula, i.e.,  $C = B_1$ . Therefore  $\Xi$  reduces to  $\Pi_1$ .

Notice that the reductions in the essential case for induction and co-induction are not symmetric. This is because we use an asymmetric measure to show the termination of cut-reduction, that is, the complexity of cut is always reduced on the right premise. The difficulty in getting a symmetric measure, in the presence of contraction and implication (in the body of definition), is already observed in logics with definitions but without (co-)induction [49].

It is clear from an inspection of the rules of the logic and the definition of cut reduction that every derivation ending with a multicut has a reduct. But because we use multisets in sequents, there may be some ambiguity as to whether a formula occurring on the left side of the rightmost premise of a multicut rule is in fact a cut formula, and if so, which of the left premises corresponds to it. As a result, several of the reduction rules may apply, and so a derivation may have multiple redexes.

The following lemmas show that the reduction relation is preserved by some of the transformations of derivations defined previously.

**Lemma 7.** Let  $\Pi$  be a derivation of  $\Gamma \longrightarrow C$  ending with a mc and let  $\theta$  be a substitution. If  $\Pi \theta$  reduces to  $\Xi$  then there exists a derivation  $\Pi'$  such that  $\Xi = \Pi' \theta$  and  $\Pi$  reduces to  $\Pi'$ .

*Proof.* Observe that the redexes of a derivation are not affected by substitution, since the cut reduction rules are determined by the last rules of the premise derivations of the derivation, which are not changed by substitution. Therefore, any cut reduction rule that is applied to  $\Pi\theta$  to get  $\Xi$  can also be applied to  $\Pi$ . Suppose that  $\Pi'$  is the reduct of  $\Pi$  obtained this way. In all cases, except for the cases where the reduction rule applied is either \*/IL or CIL/CIR, it is a matter of routine to check that  $\Pi'\theta = \Xi$ . For the reduction rules \*/IL and CIL/CIR, we need Lemma 5 and Lemma 6 which show that substitution commutes with (co-)inductive unfolding.

**Lemma 8.** Let  $p\vec{x} \stackrel{\mu}{=} Dp\vec{x}$  be an inductive definition and let  $\Pi_S$  be a derivation of  $DS\vec{x} \longrightarrow S\vec{x}$  for some invariant S. Let C be a non-atomic formula dominated by p. Let  $\Pi$  and  $\Pi'$  be two derivations of the same sequent  $\Gamma \longrightarrow C$ , and  $\Pi$  ends with an mc-rule. If  $\mu_C^p(\Pi,\Pi_S)$  reduces to  $\Xi$  then there exists a derivation  $\Pi'$  such that  $\Xi = \mu_C^p(\Pi',\Pi_S)$  and  $\Pi$  reduces to  $\Pi'$ .

*Proof.* By case analysis on the reduction rules. The case analysis can be much simplified by the following observations. First, the reduction rules are driven only by outermost connectives in the formulas in the sequent. Second, the unfolding of a derivation affects only the right-hand-side of the sequents appearing in the derivation (or more specifically, only the branches containing major premises). By a quick inspection on the definition of reduction rules in Definition 10, we see that the only non-trivial case to consider is the right-commutative case  $-/ \circ \mathcal{R}$ . Since C is non-atomic (and assuming that it has at least one occurrence of p, otherwise it is trivial since  $\Pi = \mu_C^p(\Pi, \Pi_S)$  in this case), the only cases we need to verify is when its topmost logical connective is either  $\land$ ,  $\lor$ ,  $\supset$ ,  $\forall$  and  $\exists$ . In these cases, the unfolding does not change the topmost connective, therefore any reduction rule that applies to  $\mu(\Pi, \Pi_S)$  also applies to  $\Pi$ . Lemma 5 and Lemma 6 are used when substitutions are involved (right/left commutative cases with eq $\mathcal{L}$ ).

**Lemma 9.** Let  $p\vec{x} \stackrel{\mu}{=} D p\vec{x}$  be an inductive definition and let  $\Pi_S$  be a derivation of  $DS\vec{x} \longrightarrow S\vec{x}$  for some invariant S. Let  $\Pi$  be the derivation

$$\frac{\Delta_1 \longrightarrow B_1 \quad \cdots \quad \Delta_n \longrightarrow B_n \quad B_1, \dots, B_n, \Gamma \longrightarrow p\vec{t}}{\Delta_1, \dots, \Delta_n, \Gamma \longrightarrow p\vec{t}} mc$$

Suppose that  $\Pi'$  ends with a rule other than init and  $I\mathcal{R}$ . If  $\mu_{p\bar{t}}^p(\Pi,\Pi_S)$  reduces to  $\Xi$  then there exists a derivation  $\Pi''$  such that  $\Xi = \mu_{p\bar{t}}^p(\Pi'',\Pi_S)$  and  $\Pi$  reduces to  $\Pi''$ .

*Proof.* The proof is straightforward by inspection on the cut reduction rules and the definition of inductive unfolding.

**Lemma 10.** Let  $p\vec{x} \stackrel{\vee}{=} Dp\vec{x}$  be a co-inductive definition and let  $\Pi_S$  be a derivation of  $S\vec{x} \longrightarrow DS\vec{x}$  for some invariant S. Let C be a non-atomic formula dominated by p. Let  $\Pi$  and  $\Pi'$  be two derivations of the sequent  $\Gamma \longrightarrow C[S/p]$ , where  $\Pi$  ends with a mc rule. If  $\mathbf{v}_C^p(\Pi,\Pi_S)$  reduces to  $\Xi$  then there exists a derivation  $\Pi'$  such that  $\Xi = \mathbf{v}_C^p(\Pi',\Pi_S)$  and  $\Pi$  reduces to  $\Pi'$ .

Proof. Analogous to the proof of Lemma 8.

## 5.2 Normalizability

**Definition 11.** We define the set of normalizable derivations to be the smallest set that satisfies the following conditions:

- 1. If a derivation  $\Pi$  ends with a multicut, then it is normalizable if every reduct of  $\Pi$  is normalizable.
- 2. If a derivation ends with any rule other than a multicut, then it is normalizable if the premise derivations are normalizable.

Following Martin-Löf [24], instead of assigning some ordinal measures to derivations and define an ordering on them, we shall use the derivation figures themselves as a measure. Each clause in the definition of normalizability asserts that a derivation is normalizable if certain (possibly infinitely many) other derivations are normalizable. We call the latter the *predecessors* of the former. Thus a derivation is normalizable if the tree of its successive predecessors is well-founded. We refer to this well-founded tree as its *normalization*.

Since a normalization is well-founded, it has an associated induction principle: for any property P of derivations, if for every derivation  $\Pi$  in the normalization, P holds for every predecessor of  $\Pi$  implies that P holds for  $\Pi$ , then P holds for every derivation in the normalization.

The set of all normalizable derivations is denoted by NM.

**Lemma 11.** If there is a normalizable derivation of a sequent, then there is a cut-free derivation of the sequent.

*Proof.* Let  $\Pi$  be a normalizable derivation of the sequent  $\Gamma \longrightarrow B$ . We show by induction on the normalization of  $\Pi$  that there is a cut-free derivation of  $\Gamma \longrightarrow B$ .

- 1. If  $\Pi$  ends with a multicut, then any of its reducts is one of its predecessors and so is normalizable. But the reduct is also a derivation of  $\Gamma \longrightarrow \mathcal{B}$ , so by the induction hypothesis this sequent has a cut-free derivation.
- 2. Suppose  $\Pi$  ends with a rule other than multicut. Since we are given that  $\Pi$  is normalizable, by definition the premise derivations are normalizable. These premise derivations are the predecessors of  $\Pi$ , so by the induction hypothesis there are cut-free derivations of the premises. Thus there is a cut-free derivation of  $\Gamma \longrightarrow \mathcal{B}$ .

The next lemma states that normalization is closed under substitutions.

**Lemma 12.** If  $\Pi$  is a normalizable derivation, then for any substitution  $\theta$ ,  $\Pi\theta$  is normalizable.

*Proof.* We prove this lemma by induction on the normalization of  $\Pi$ .

- 1. If  $\Pi$  ends with a multicut, then  $\Pi\theta$  also ends with a multicut. By Lemma 7 every reduct of  $\Pi\theta$  corresponds to a reduct of  $\Pi$ , therefore by induction hypothesis every reduct of  $\Pi\theta$  is normalizable, and hence  $\Pi\theta$  is normalizable.
- 2. Suppose  $\Pi$  ends with a rule other than multicut and has premise derivations  $\{\Pi_i\}$ . By Definition 3 each premise derivation in  $\Pi\theta$  is either  $\Pi_i$  or  $\Pi_i\theta$ . Since  $\Pi$  is normalizable,  $\Pi_i$  is normalizable, and so by the induction hypothesis  $\Pi_i\theta$  is also normalizable. Thus  $\Pi\theta$  is normalizable.

#### 5.3 Parametric reducibility

Let us first define some terminology concerning derivations. We say that a derivation  $\Pi$  has type C if the end sequent of  $\Pi$  is of the form  $\Gamma \longrightarrow C$  for some  $\Gamma$ . We say that a set of derivations S has type C, if every derivation  $\Pi \in S$  has type C. A set of derivations R is *closed under substitution* if for every  $\Pi \in R$  and for every substitution  $\theta$ ,  $\Pi \theta \in R$ .

To simplify presentation, we shall use the following notations to denote certain types of derivations. The derivation

$$\underline{\Delta_1 \xrightarrow{\Pi_1} B_1 \cdots \Delta_n \xrightarrow{\Pi_n} B_n \quad \Gamma \xrightarrow{\Pi} C}_{\Delta_1, \dots, \Delta_n, \Gamma \xrightarrow{} C} mc$$

is abbreviated as  $mc(\Pi_1, \dots, \Pi_n, \Pi)$ . The derivation

$$\frac{BS\vec{x} \longrightarrow S\vec{x} \quad \Gamma, S\vec{u} \longrightarrow C}{\Gamma, p\vec{u} \longrightarrow C} \quad \text{I.L.}$$

is abbreviated as  $ind(\Pi_S, \Pi)$ , and the derivation

$$\frac{\Pi}{\Gamma \longrightarrow S\vec{u}} \frac{\Pi_S}{S\vec{x} \longrightarrow BS\vec{x}} \text{ CIR}$$

$$\Gamma \longrightarrow p\vec{u}$$

is abbreviated as  $coind(\Pi, \Pi_S)$ .

**Definition 12.** Let F be a closed term of type  $\alpha_1 \to \cdots \to \alpha_n \to o$ . A set of derivations S is said to be F-indexed if every derivation in S has type  $Ft_1 \dots t_n$  for some  $t_1, \dots, t_n$ .

Given a set S of derivations and a formula C, we denote with  $S \downarrow_C$  the set

$$\{\Pi \in \mathcal{S} \mid \Pi \text{ is of type } C \}.$$

We shall now define a family of sets of derivations, which we call parametric reducibility sets.

**Definition 13.** Parametric Reducibility. Let  $p\vec{x} \stackrel{\vee}{=} B p\vec{x}$  be a co-inductive definition, let I be a closed term of the same type as p, let  $\mathcal{R}$  be a set of derivations, and let S be an I-indexed set of derivations. Let C be a formula dominated by p. We define the parametric reducibility sets  $\mathbf{RED}_C^p[\mathcal{R}, S]$ , consisting of derivations of type C[I/p], by induction on the size of C, as follows. (In the following, we shall refer to C as the type of  $\mathbf{RED}_C^p[\mathcal{R}, S]$ .)

- 1. If p does not appear in C then  $\mathbf{RED}_{C}^{p}[\mathcal{R},\mathcal{S}] = \mathcal{R}\downarrow_{C}$ .
- 2. If  $C = p\vec{u}$ , for some  $\vec{u}$ , then  $\mathbf{RED}_C^p[\mathcal{R}, \mathcal{S}] = \mathcal{S}\downarrow_{I\vec{u}}$ .
- 3. Otherwise, the family of parametric reducibility sets  $\{\mathbf{RED}_{C\theta}^p[\mathcal{R},\mathcal{S}]\}_{\theta}$  is the smallest family that satisfies the following: for every  $\theta$  and for every derivation  $\Pi$  of type  $C\theta[I/p]$ ,  $\Pi \in \mathbf{RED}_{C\theta}^p[\mathcal{R},\mathcal{S}]$  if one of the following holds:
  - (a)  $\Pi$  ends with mc, and all its reducts are in  $\mathbf{RED}_{C\theta}^p[\mathcal{R}, \mathcal{S}]$ .
  - (b)  $\Pi$  ends with  $\supset \mathcal{R}$ , i.e.,

$$\frac{\Gamma, B \longrightarrow D[I/p]}{\Gamma \longrightarrow B \supset D[I/p]} \supset \mathcal{R}$$

 $\Pi' \in \mathbf{RED}_D^p[\mathcal{R}, \mathcal{S}]$ , and for every substitution  $\rho$  and for every derivation  $\Xi$  of  $\Delta \longrightarrow B\rho$  in  $\mathcal{R}$ , we have  $mc(\Xi, \Pi'\rho) \in \mathbf{RED}_{D_0}^p[\mathcal{R}, \mathcal{S}]$ .

(c)  $\Pi$  ends with a rule  $\rho$  other than mc and  $\supset \mathcal{R}$ , the minor premise derivations of  $\Pi$  are normalizable, and its major premise derivations are in the parametric reducibility sets of the appropriate types.

From now on, when we write  $\mathbf{RED}_C^p[\mathcal{R}, \mathcal{S}]$ , it is understood that p is a co-inductive predicate, C is dominated by p,  $\mathcal{R}$  is a set of derivations, and  $\mathcal{S}$  is an I-indexed set of normalizable derivations, for some I.

Note that in Definition 13 (3), we define simultaneously the reducibility sets  $\mathbf{RED}_{C\theta}^{P}[\mathcal{R}, \mathcal{S}]$  for all substitution  $\theta$ . This is because in the case the derivation  $\Pi$  ends with eq $\mathcal{L}$ , reducibility of  $\Pi$  may depend on the reducibility of (possibly infinitely many) derivations which are in  $\mathbf{RED}_{C0}^{P}[\mathcal{R}, \mathcal{S}]$  for some  $\rho$ . Since  $C\rho$  is of the same size as  $C\theta$ , its

parametric reducibility set may not yet be defined by induction on the size. We therefore need to define this and other reducibility sets which are indexed by instances of *C* simultaneously.

As with the definition of normalizability, clause (3) in Definition 13 defines a monotone fixed point operator (assuming the parametric reducibility sets of smaller types have been fixed), and it therefore induces a well-founded tree of derivations in the family  $\{\mathbf{RED}_{C\theta}^p[\mathcal{R},\mathcal{S}]\}_{\theta}$ . It is immediately clear from the definition that a derivation  $\Pi'$  in the family is a predecessor of  $\Pi$  (in the same family) if either

- $\Pi$  ends with a left rule and  $\Pi'$  is a major premise of  $\Pi$ , or
- $\Pi$  ends with mc and  $\Pi'$  is a reduct of  $\Pi$ .

We shall call the well-founded tree of successive predecessors of a derivation  $\Pi$  in the family  $\{\mathbf{RED}_{C\theta}^p[\mathcal{R},\mathcal{S}]\}_{\theta}$  the parametric reduction of  $\Pi$ . As with the normalization of a derivation, it has an associated induction principle. Note that, however, this ordering on derivations is defined only in the case where C satisfies the syntactic condition defined in Definition 13(3), i.e., it contains at least an occurrence of p and is not an atomic formula.

The definition of parametric reducibility can be seen as defining a function on S-indexed sets. In the case where the type of the parametric reducibility set is the body of the co-inductive definition for p, this function corresponds to the underlying fixed point operator for p. We shall now define a class of S-indexed sets which are closed under this fixed point operator. These sets, called saturated sets in the following, can be seen as post-fixed points of the fixed point operator for the co-inductive definition for p. They will be used in defining the reducibility of derivations involving the co-induction rule CIR.

**Definition 14.** Let  $\forall \vec{x}. \ p\vec{x} \stackrel{\vee}{=} B \ p\vec{x}$  be an co-inductive definition. Let S be a closed term of the same type as p. Let  $\Pi_S$  be a derivation of  $S\vec{x} \longrightarrow BS\vec{x}$ . Let  $\mathcal{R}$  be a set of derivations. An S-indexed set S is a  $(\mathcal{R}, \Pi_S)$ -saturated set if the following hold:

- 1. Every derivation in S is normalizable.
- 2. If  $\Pi \in S$  then  $\Pi \theta \in S$  for any  $\theta$ .
- 3. If  $\Pi \in \mathcal{S}$  and  $\Pi$  is of type  $S\vec{u}$  for some  $\vec{u}$ , then  $mc(\Pi, \Pi_S[\vec{u}/\vec{x}]) \in \mathbf{RED}^p_{Bp\vec{u}}[\mathcal{R}, \mathcal{S}]$ .

### 5.4 Reducibility

We now define a family of reducible sets  $\mathbf{RED}_i$  of level i.

**Definition 15.** Reducibility. We define the family  $\{\mathbf{RED}_i\}_i$  of reducible sets of level i by induction on i. In defining the reducible set of level i, we assume that reducible sets of smaller levels have been defined. Each set  $\mathbf{RED}_i$  the smallest set that satisfies the following: For every derivation  $\Pi$  of level i,  $\Pi \in \mathbf{RED}_i$  if one of the following holds:

- 1.  $\Pi$  ends with mc and all its reducts are in **RED**<sub>i</sub>.
- 2. Π *is*

$$\frac{\Gamma, B \xrightarrow{\Pi'} D}{\Gamma \xrightarrow{B \supset D}} \supset \mathcal{R},$$

 $\Pi' \in \mathbf{RED}_{\mathrm{lvl}(D)}$ , and for every substitution  $\theta$  and for every derivation  $\Xi$  of  $\Delta \longrightarrow B\theta$  in  $\mathbf{RED}_{\mathrm{lvl}(B\theta)}$ , we have  $mc(\Xi, \Pi'\theta) \in \mathbf{RED}_{\mathrm{lvl}(D\theta)}$ .

3.  $\Pi$  ends with CIR, i.e.,  $\Pi$  is

$$\frac{\Gamma \xrightarrow{\Pi'} \frac{\Pi_S}{S\vec{x} \xrightarrow{} BS\vec{x}} \text{CIR}}{\Gamma \xrightarrow{} n\vec{t}}$$

where  $p\vec{x} \stackrel{\vee}{=} B p\vec{x}$ ,  $\Pi'$  and  $\Pi_S$  are normalizable, and there exists a  $(\mathcal{R}, \Pi_S)$ -saturated set S, where  $\mathcal{R} = \bigcup \{ \mathbf{RED}_j \mid j < i \}$ , such that  $\Pi' \in S$ .

4.  $\Pi$  ends with a rule  $\rho$  other than mc and  $\supset \mathcal{R}$ , the minor premise derivations of  $\Pi$  are normalizable, and its major premise derivations are in the reducibility sets of the appropriate levels.

As in the definition of normalizability, each clause in the definition of reducibility asserts that a derivation is reducible provided that certain other derivations, called the predecessors of the derivation, are reducible. The definition of reducibility induces a well-founded ordering on derivations in the reducibility sets. We shall refer to this ordering as *reducibility ordering* and the induced well-founded tree as the *reduction* of the derivation. We say that a derivation is *reducible* if it is in  $\mathbf{RED}_i$  for some i.

## **Lemma 13.** Every reducible derivation is normalizable.

*Proof.* Given a reducible derivation  $\Pi$ , it is straightforward to show by induction on its reduction that it is normalizable. In the case where  $\Pi$  ends with CI $\mathcal{R}$ , by the definition of saturated sets (Definition 14) and reducibility (Definition 15), its premise derivations are normalizable, and therefore  $\Pi$  is also normalizable.

## **Lemma 14.** *If* $\Pi$ *is reducible then for every derivation* $\theta$ *,* $\Pi\theta$ *is also reducible.*

*Proof.* The proof is by induction on the reduction of  $\Pi$ . We consider two non-trivial cases here: the case where  $\Pi$  ends with mc and the case where it ends with CIR. For the former, suppose that  $\Pi = mc(\Pi_1, ..., \Pi_n, \Pi')$ . By Lemma 7, every reduct of  $\Pi\theta$ , say  $\Xi$ , is the result of substituting a reduct of  $\Pi$ . By induction hypothesis, every reduct of  $\Pi\theta$  is reducible, hence  $\Pi\theta$  is also reducible.

We now consider the case  $\Pi$  ends with  $CI\mathcal{R}$ , i.e.,  $\Pi$  is

$$\frac{\Gamma' \qquad \Pi_S}{\Gamma \longrightarrow S\vec{t} \quad S\vec{x} \longrightarrow BS\vec{x}} \text{ CIR}$$

where  $p\vec{x} \stackrel{\vee}{=} Bp\vec{x}$ . Let i be the level of p and let  $\mathcal{R} = \bigcup \{ \mathbf{RED}_j \mid j < \mathrm{lvl}(p) \}$ . By the definition of reducibility, we have that  $\Pi'$  and  $\Pi_S$  are both normalizable, and moreover, there exists a  $(\mathcal{R}, \Pi_S)$ -saturated set  $\mathcal{S}$ , such that  $\Pi' \in \mathcal{S}$ . Suppose that  $\vec{u} = (\vec{t})\theta$ . To show that  $\Pi\theta$  is reducible, we must first show that both  $\Pi'\theta$  and  $\Pi_S$  are normalizable. This is straightforward from the fact that both  $\Pi'$  and  $\Pi_S$  are normalizable and that normalisation is closed under substitutions (Lemma 12). It remains to show that there exists a  $(\mathcal{R}, \Pi_S)$ -saturated set  $\mathcal{S}'$  such that  $\Pi'\theta \in \mathcal{S}'$ . Let  $\mathcal{S}' = \mathcal{S}$ . Since saturated sets are closed under substitution and  $\Pi' \in \mathcal{S}'$ , we have  $\Pi'\theta \in \mathcal{S}'$ .

**Lemma 15.** Let p be a co-inductive predicate, let S be a closed term of the same type as p. Let  $\mathcal{R} = \bigcup \{ \mathbf{RED}_j \mid j < \text{lvl}(p) \}$ , let

$$S = \bigcup \{\Xi \mid \Xi \text{ is reducible and has type } S\vec{t} \text{ for some } \vec{t}\}$$

and let C be a formula dominated by p. Then for every reducible derivation  $\Pi$  of type C[S/p],  $\Pi \in \mathbf{RED}_C^p[\mathcal{R},\mathcal{S}]$ .

*Proof.* By induction on the reduction of  $\Pi$ . If p does not occur in C then  $\Pi \in \mathcal{R}$ , since in this case lvl(C) < lvl(p) (recall that C is dominated by p), therefore  $\Pi \in \mathbf{RED}^p_C[\mathcal{R}, \mathcal{S}]$ . If C = p then  $\Pi \in \mathcal{S}$  (since  $\Pi$  is reducible), hence  $\Pi \in \mathbf{RED}^p_C[\mathcal{R}, \mathcal{S}]$ . The other cases follow from straightforwardly from induction hypothesis. We show here the case where  $\Pi$  ends with  $\supset \mathcal{R}$ .

$$\frac{\Gamma, B \longrightarrow D[S/p]}{\Gamma \longrightarrow B \supset D[S/p]} \supset \mathcal{R}$$

Note that in this case  $C = B \supset D$ , and p does not occur in B by the restriction on C (p dominates C). Since  $\Pi$  is reducible, we have that  $\Pi'$  is a reducible predecessor of  $\Pi$ , and for every substitution  $\theta$  and every reducible derivation  $\Xi$  of type  $B\theta$ , we have  $mc(\Xi,\Pi'\theta)$  is also a reducible predecessor of  $\Pi$ . It thus follows from induction hypotheses that  $\Pi' \in \mathbf{RED}_D^p[\mathcal{R}, S]$  and for every  $\Xi \in \mathcal{R}$  of type  $B\theta$  (which is reducible by the definition of  $\mathcal{R}$ ),  $mc(\Xi,\Pi'\theta) \in \mathbf{RED}_{D\theta}^p[\mathcal{R}, S]$ . Therefore, by the definition of parametric reducibility, we have that  $\Pi \in \mathbf{RED}_D^p[\mathcal{R}, S]$ .

## 5.5 Reducibility of unfolded derivations

The following lemmas state that reducibility is preserved by (co)inductive unfolding, under certain assumptions.

**Lemma 16.** Inductive unfolding. Let  $p\vec{x} \stackrel{\mu}{=} Bp\vec{x}$  be an inductive definition. Let  $\Pi_S$  be a reducible derivation of  $BS\vec{x} \longrightarrow S\vec{x}$ . Let  $\Pi$  be a reducible derivation of  $\Gamma \longrightarrow C$  such that p dominates C. Suppose the following statements hold:

- 1. For every derivation  $\Xi$  of  $\Delta \longrightarrow B p \vec{u}$ , if  $\mu(\Xi, \Pi_S)$  is reducible, then the derivation  $mc(\mu(\Xi, \Pi_S), \Pi_S[\vec{u}/\vec{x}])$  is reducible.
- 2. For every reducible derivation  $\Xi$  of  $\Delta \longrightarrow S\vec{u}$  the derivation  $mc(\Xi, Id_{S\vec{u}})$  is reducible.
- 3. The derivation ind  $(\Pi_S, \operatorname{Id}_{S\vec{u}})$  is reducible, for every  $\vec{u}$  of the appropriate types.

*Then the derivation*  $\mu_C^p(\Pi, \Pi_S)$  *of*  $\Gamma \longrightarrow C[S/p]$  *is reducible.* 

*Proof.* By induction on the reduction of  $\Pi$ . We show the non-trivial cases, assuming that p is not vacuous in C. To simplify presentation, we shall write  $\mu(.,.)$  instead of  $\mu_F^p(.,.)$ , since in each of the following cases, it is easy to infer from the context which F we are referring to.

- 1. Suppose  $\Pi$  ends with *init* rule on  $p\vec{u}$ . Then  $\mu(\Pi, \Pi_S) = ind(\Pi_S, \operatorname{Id}_{S\vec{u}})$ , which is reducible by assumption.
- 2. Suppose  $\Pi$  ends with  $\supset \mathcal{R}$ , that is,  $C = C_1 \supset C_2$ .

$$\frac{\Gamma, C_1 \longrightarrow C_2}{\Gamma \longrightarrow C_1 \supset C_2} \supset \mathcal{R}$$

By the restriction on C, we know that p is vacuous in  $C_1$ , hence  $C[S/p] = C_1 \supset C_2[S/p]$ . By the definition of reducibility, the derivation  $\Pi'$  is reducible and for every substitution  $\theta$  and every reducible derivation  $\Psi$  of  $\Delta \longrightarrow C_1\theta$ , the derivation  $\Xi$ 

$$\frac{\Delta \xrightarrow{\Psi} C_1 \theta \qquad \Gamma \theta, C_1 \theta \xrightarrow{\Pi' \theta} C_2 \theta}{\Delta, \Gamma \theta \xrightarrow{} C_2 \theta} mc$$

is reducible. We want to show that the derivation  $\mu(\Pi, \Pi_S)$ 

$$\frac{\mu(\Pi',\Pi_S)}{\Gamma,C_1\,p\longrightarrow C_2\,S} \over \Gamma\longrightarrow C_1\,p\supset C_2[S/p]}\supset \mathcal{R}$$

is reducible. This reduces to showing that  $\mu(\Pi',\Pi_S)$  is reducible and that

$$\frac{\Delta \longrightarrow C_1 \theta \qquad \frac{\mu(\Pi', \Pi_S) \theta}{\Gamma \theta, C_1 \theta \longrightarrow C_2 \theta[S/p]}}{\Delta, \Gamma \theta \longrightarrow C_2 \theta[S/p]} mc$$

is reducible. The first follows from induction hypothesis on  $\Pi'$ . For the second derivation, we know from Lemma 5 that

$$\mu(\Pi',\Pi_S)\theta = \mu(\Pi'\theta,\Pi_S).$$

It follows from this and the definition of inductive unfolding (Definition 8) that

$$mc(\Psi, \mu(\Pi', \Pi_S)\theta) = mc(\Psi, \mu(\Pi'\theta, \Pi_S)) = \mu(mc(\Psi, \Pi'\theta), \Pi_S) = \mu(\Xi, \Pi_S)$$

We can apply induction hypothesis on  $\Xi$ , since it is a predecessor of  $\Pi$ , to establish the reducibility of  $\mu(\Xi, \Pi_S)$ . This, together with reducibility of  $\mu(\Pi', \Pi_S)$  implies that  $\mu(\Pi, \Pi_S)$  is reducible.

3. Suppose  $\Pi$  ends with  $I\mathcal{R}$  rule on  $p\vec{u}$ .

$$\frac{\Gamma \longrightarrow B \, p \, \vec{u}}{\Gamma \longrightarrow p \, \vec{u}} \, \mathbf{I} \mathcal{R}$$

Then  $\mu(\Pi, \Pi_S)$  is the derivation

$$\frac{\mu(\Pi', \Pi_S)}{\Gamma \longrightarrow BS\vec{u}} \frac{\Pi_S[\vec{u}/\vec{x}]}{BS\vec{u} \longrightarrow S\vec{u}} mc$$

The derivation  $\mu(\Pi', \Pi_S)$  is reducible by induction hypothesis. This, together with assumption (1) of the lemma, imply that  $\mu(\Pi, \Pi_S)$  is reducible.

#### 4. Suppose $\Pi$ ends with mc.

$$\frac{\Delta_1 \xrightarrow{\Pi_1} D_1 \cdots \Delta_n \xrightarrow{\Pi_n} D_m D_1, \dots, D_m, \Gamma' \longrightarrow C}{\Delta_1, \dots, \Delta_m, \Gamma' \longrightarrow C} mc$$

Then  $\mu(\Pi, \Pi_S)$  is the derivation

$$\frac{\Delta_1 \xrightarrow{\Pi_1} \qquad \prod_n \qquad \mu(\Pi', \Pi_S)}{\Delta_1 \xrightarrow{D_1} \qquad \Delta_n \xrightarrow{D_m} \qquad D_1, \dots, D_m, \Gamma' \xrightarrow{D_m} C[S/p]} mc$$

By the definition of reducibility, every reduct of  $\Pi$  is reducible. We need to show that every reduct of  $\mu(\Pi,\Pi_S)$  is reducible.

From Lemma 8, we know that for the case where C is not atomic every reduct of  $\mu(\Pi, \Pi_S)$  corresponds to some reduct of  $\Pi$ . Similarly, for the case where  $\Pi'$  ends with a rule other than *init* or  $I\mathcal{R}$ , by Lemma 9, the reducts of  $\mu(\Pi, \Pi_S)$  are in one-to-one correspondence with the reducts of  $\Pi$ . Therefore in these cases, the inductive hypothesis can be applied to show the reducibility of each reduct of  $\mu(\Pi, \Pi_S)$ . This leaves us the following two cases, where  $C = p\vec{u}$  and  $\Pi'$  ends with either  $I\mathcal{R}$  or *init* rules.

- Suppose  $\Pi'$  is the derivation

$$\frac{D_1, \dots, D_m, \Gamma' \longrightarrow B p \vec{u}}{D_1, \dots, D_m, \Gamma' \longrightarrow p \vec{u}} I \mathcal{R}$$

Let  $\Xi_1$  be the derivation

$$\frac{\left\{ \Delta_{j} \xrightarrow{} D_{j} \right\}_{j \in \{1, \dots, m\}} \prod_{j \in \{1, \dots, m\}} \Pi'' \longrightarrow B p \vec{u}}{\Delta_{1}, \dots, \Delta_{m}, \Gamma' \longrightarrow B p \vec{u}} mc$$

then the derivation

$$\frac{\Delta_1, \dots, \Delta_m, \Gamma' \longrightarrow B \, p \, \vec{u}}{\Delta_1, \dots, \Delta_m, \Gamma' \longrightarrow p \, \vec{u}} \, \, \mathbf{I} \mathcal{R}$$

is a reduct of  $\Pi$  (by the reduction rule  $-/I\mathcal{R}$ ), and therefore by the definition of reducibility both this reduct and  $\Xi_1$  are reducible predecessors of  $\Pi$ . Let  $\Psi$  be the derivation

$$\frac{\mu(\Pi'',\Pi_S)}{D_1,\dots,\Gamma'\longrightarrow BS\vec{u}} \frac{\Pi_S'}{BS\vec{u}\longrightarrow S\vec{u}} \ mc$$

Then the derivation  $\mu(\Pi, \Pi_S)$  is the following

$$\frac{\left\{ \Delta_{j} \xrightarrow{\prod_{j} D_{j}} \right\}_{j \in \{1,\dots,m\}} \Psi}{\Delta_{1},\dots,\Delta_{m},\Gamma' \longrightarrow S\vec{u}} mc$$

The only applicable reduction rule to  $\mu(\Pi, \Pi_S)$  is -/mc, which gives us the reduct  $\Xi$ 

$$\frac{\Delta_1, \dots, \Delta_m, \Gamma' \longrightarrow BS\vec{u} \quad BS\vec{u} \longrightarrow S\vec{u}}{\Delta_1, \dots, \Delta_m, \Gamma' \longrightarrow S\vec{u}} mc$$

where  $\Psi'$  is the derivation

$$\frac{\left\{ \begin{array}{l} \Pi_{j} \\ \Delta_{j} \longrightarrow D_{j} \end{array} \right\}_{j \in \{1, \dots, m\}} \quad \begin{array}{l} \mu(\Pi'', \Pi_{S}) \\ D_{1}, \dots, \Gamma' \longrightarrow BS\vec{u} \end{array}}{\Delta_{1}, \dots, \Delta_{m}, \Gamma' \longrightarrow BS\vec{u}} \ mc$$

Notice that  $\Psi'$  is exactly  $\mu(\Xi_1, \Pi_S)$ , and is reducible by inductive hypothesis. Therefore assumption (1) applies, and the reduct  $\Xi$  is reducible, hence  $\mu(\Pi, \Pi_S)$  is also reducible.

- Otherwise, suppose  $\Pi'$  ends with *init*, then  $D_1 = p\vec{u}$  and  $\Pi$  is the derivation

$$\frac{\Delta_1 \longrightarrow p\vec{u} \quad \overline{p\vec{u} \longrightarrow p\vec{u}} \quad \text{init}}{\Delta_1 \longrightarrow p\vec{u}} \quad mc$$

The only reduct of  $\Pi$  is  $\Pi_1$  since the only applicable reduction is -/init. On the other hand, the derivation  $\mu(\Pi, \Pi_S)$  is

$$\underbrace{ \begin{array}{c} \Pi_1 \\ \Delta_1 \longrightarrow p\vec{u} \end{array} \underbrace{ \begin{array}{c} BS\vec{x} \longrightarrow S\vec{x} & S\vec{u} \longrightarrow S\vec{u} \\ p\vec{u} \longrightarrow S\vec{u} \end{array} }_{\Delta_1 \longrightarrow S\vec{u}} \text{ IL} }_{ D\vec{u} \longrightarrow S\vec{u}$$

Its only reduct is (by \*/IL)

$$\frac{\mu(\Pi_1,\Pi_S)}{\Delta_1 \longrightarrow S\vec{u}} \xrightarrow{Id} S\vec{u} \xrightarrow{M} S\vec{u}$$

$$\frac{\Delta_1 \longrightarrow S\vec{u}}{\Delta_1 \longrightarrow S\vec{u}} mc$$

The derivation  $\mu(\Pi_1, \Pi_S)$  is reducible by inductive hypothesis ( $\Pi_1$  is a predecessor of  $\Pi$ ) and assumption (2) applies, and the above reduct is reducible.

Remark 1. Intuitively, condition (1) of Lemma 16 can be seen as asserting that the set of reducible derivations whose types are instances of  $S\vec{x}$  forms a pre-fixed point of the fixed point operator induced by the inductive definition of p.

**Lemma 17.** Co-inductive unfolding. Let  $p\vec{x} \stackrel{\vee}{=} B p\vec{x}$  be a co-inductive definition. Let  $\Pi_S$  be a normalizable derivation of  $S\vec{x} \longrightarrow BS\vec{x}$  for some invariant S. Let  $\mathcal{R} = \{\mathbf{RED}_j \mid j < \text{lvl}(p)\}$ , and let S be a  $(\mathcal{R}, \Pi_S)$ -saturated set. Let  $\Pi$  be a derivation of  $\Gamma \longrightarrow C[S/p]$  for some C dominated by p. If  $\Pi \in \mathbf{RED}_C[\mathcal{R}, S]$  then  $\mathbf{v}_C^p(\Pi, \Pi_S)$  is reducible.

*Proof.* By induction on the size of C, with sub-induction on the parametric reduction of  $\Pi$ . As in the proof of inductive unfolding, we omit the subscript and superscript in the  $\nu$  function to simplify the presentation of the proof.

- 1. If p is not free in C, then  $v(\Pi, \Pi_S) = \Pi$ . Since  $\Pi \in \mathbf{RED}_C[\mathcal{R}, \mathcal{S}]$ , it follows from the definition of parametric reducibility that  $\Pi \in \mathcal{R}$ , hence it is reducible by assumption.
- 2. Suppose  $C = p\vec{u}$ . Then  $C[S/p] = S\vec{u}$  and  $v(\Pi, \Pi_S)$  is the derivation

$$\frac{\prod \qquad \prod_{S}}{\Gamma \longrightarrow S\vec{u} \quad S\vec{x} \longrightarrow BS\vec{x}} \text{ CIR}$$

$$\Gamma \longrightarrow p\vec{u}$$

To show that this derivation is reducible, we first show that there exist a  $(\mathcal{R}, \Pi_S)$ -saturated set  $\mathcal{S}'$  such that  $\Pi \in \mathcal{S}'$ . Since  $\Pi \in \mathbf{RED}^p_{p\bar{u}}[\mathcal{R}, \mathcal{S}]$ , by the definition of parametric reducibility, we have  $\Pi \in \mathcal{S}$ . Let  $\mathcal{S}' = \mathcal{S}$ . Then  $\mathcal{S}'$  is indeed a  $(\mathcal{R}, \Pi_S)$ -saturated set containing  $\Pi$ . It remains to show that both  $\Pi$  and  $\Pi_S$  are normalizable. This follows from the assumption on  $\Pi_S$  and the fact that saturated sets contain only normalizable derivations.

- 3. Suppose p occurs in C but  $C \neq p\vec{u}$  for any  $\vec{u}$ . There are several subcases, depending on the last rule in  $\Pi$ . Then we show by induction on parametric reducibility of  $\Pi$  that it is also reducible.
  - (a) The base cases are those where  $\Pi$  ends with a rule with empty premises and where  $\Pi$  ends with a right-introduction rule. In the former case, its reducibility is immediate from the definition of reducibility (Definition 15). For the latter, in most cases, the reducibility of  $\Pi$  follows from the outer induction hypothesis (since in this case, the premise derivations of  $\Pi$  are in the parametric reducibility sets of smaller types) and Definition 15. We show here a non-trivial case involving implication-right: Suppose  $\Pi$  ends with  $\supset \mathcal{R}$ , i.e.,  $C = C_1 \supset C_2$  for some  $C_1$  and  $C_2$ .

$$\frac{\Gamma, C_1 \longrightarrow C_2[S/p]}{\Gamma \longrightarrow C_1 \supset C_2[S/p]} \supset \mathcal{R}$$

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Note that p is vacuous in  $C_1$  by the restriction on C. The derivation  $v(\Pi, \Pi_S)$  is

$$rac{
u(\Pi',\Pi_S)}{\Gamma \longrightarrow C_1 \supset C_2} \supset \mathcal{R}$$

To show that  $\nu(\Pi, \Pi_S)$  is reducible, we need to show that  $\nu(\Pi', \Pi_S)$  is reducible, and for every  $\theta$  and every  $\Psi \in \mathbf{RED}_{C_1\theta}$ , we have  $mc(\Psi, \nu(\Pi', \Pi_S)\theta) \in \mathbf{RED}_{C_2\theta}$ .

The parametric reducibility of  $\Pi$  implies that  $\Pi' \in \mathbf{RED}_{C_2}[\mathcal{R}, \mathcal{S}]$  and for every  $\theta$  and every derivation  $\Psi' \in \mathcal{R}$ ,  $mc(\Psi', \Pi'\theta) \in \mathbf{RED}_{C_2\theta}[\mathcal{R}, \mathcal{S}]$ . Note that  $\Psi$  is in  $\mathcal{R}$  since  $lvl(C_1\theta) < lvl(p)$ . Therefore we also have  $mc(\Psi, \Pi'\theta) \in \mathbf{RED}_{C_2\theta}[\mathcal{R}, \mathcal{S}]$ . By the outer induction hypothesis, we have that both

$$\nu(\Pi',\Pi_S)$$
 and  $\nu(mc(\Psi,\Pi'\theta),\Pi_S)$ 

are reducible. It remains to show that the  $mc(\Psi, \nu(\Pi', \Pi_S)\theta)$  is reducible. Note that by Lemma 6 this derivation is equivalent to  $mc(\Psi, \nu(\Pi'\theta, \Pi_S))$ . To show that this derivation is reducible, there are two cases to consider. If  $C_2$  is non-atomic then it is easy to see that  $mc(\Psi, \nu(\Pi'\theta, \Pi_S))$  is equivalent to  $\nu(mc(\Psi, \Pi'\theta), \Pi_S)$ , which is reducible by the outer induction hypothesis. If, however,  $C_2 = p\vec{u}$  for some  $\vec{u}$ , then  $mc(\Psi, \nu(\Pi'\theta, \Pi_S))$  is the derivation (supposing that the end sequent of  $\Psi$  is  $\Delta \longrightarrow C_1\theta$ ):

$$\frac{\Psi}{\Delta \longrightarrow C_1 \theta} \frac{C_1 \theta, \Gamma \theta \longrightarrow S \vec{u} \quad S \vec{x} \longrightarrow B S \vec{x}}{C_1 \theta, \Gamma \theta \longrightarrow p \vec{u}} \text{ CIR}$$

$$\frac{\Delta \longrightarrow C_1 \theta}{\Delta, \Gamma \theta \longrightarrow p \vec{u}} mc$$

To show that this derivation is reducible, we must show that all its reducts are reducible. There is only one reduction rule that is applicable in this case, i.e., the -/CIR-case, which leads to the following derivation:

$$\frac{\Delta \xrightarrow{} C_1 \theta \quad C_1 \theta, \Gamma \theta \xrightarrow{} S \vec{u}}{\Delta, \Gamma \theta \xrightarrow{} S \vec{u}} mc \quad \prod_{S \vec{x} \xrightarrow{} BS \vec{x}} CI \mathcal{R}.$$

But notice that this is exactly the derivation  $v(mc(\Psi,\Pi'\theta),\Pi_S)$ , which is reducible by the outer induction hypothesis.

Having shown that  $v(\Pi', \Pi_S)$  and  $mc(\Psi, v(\Pi', \Pi_S)\theta)$  are reducible, we have sufficient conditions to conclude that  $v(\Pi, \Pi_S)$  is indeed reducible.

(b) For the inductive cases,  $\Pi$  ends either with mc or a left-rule. We show the former case here (the other cases are straightforward). Suppose  $\Pi$  is

$$\frac{\Delta_1 \longrightarrow D_1 \quad \cdots \quad \Delta_n \longrightarrow D_m \quad D_1, \dots, D_m, \Gamma' \longrightarrow C[S/p]}{\Delta_1, \dots, \Delta_n, \Gamma' \longrightarrow C[S/p]} mc$$

Then  $v(\Pi, \Pi_S)$  is the derivation

$$\frac{\Delta_1 \longrightarrow D_1 \quad \cdots \quad \Delta_n \longrightarrow D_m \quad \nu(\Pi', \Pi_S)}{\Delta_1, \dots, \Delta_n, \Gamma' \longrightarrow C} \quad mc$$

The derivation  $v(\Pi, \Pi_S)$  is reducible if every reduct of  $v(\Pi, \Pi_S)$  is also reducible. From Lemma 10, it follows that every reduct of  $v(\Pi, \Pi_S)$  is of the form  $v(\Xi, \Pi_S)$  where  $\Xi$  is a reduct of  $\Pi$ . Since all reducts of  $\Pi$  are predecessors of  $\Pi$  in the parametric reducibility ordering, we can apply the inductive hypothesis to show that every reduct of  $v(\Pi, \Pi_S)$  is reducible, hence  $v(\Pi, \Pi_S)$  is also reducible.

#### 5.6 Cut elimination

Most cases in the cut elimination proof for Linc<sup>-</sup> in the following are similar to those of  $FO\lambda^{\Delta IN}$ . The crucial differences are in the handling of the essential cut reductions for inductive and co-inductive rules.<sup>4</sup> In the case of derivations of inductive predicates, a crucial part of the proof is in establishing that the *S*-indexed set of reducible derivations (where *S* is an inductive invariant) satisfies the conditions of Lemma 16 (in effect, demonstrating that the said set forms a pre-fixed point). Dually, in the case for co-inductive proofs, one must show that the *S*-indexed set of reducible derivations, where *S* is a co-inductive invariant, forms a saturated set (i.e., a post fixed point of the co-inductive definition involved).

**Lemma 18.** For any derivation  $\Pi$  of  $B_1, \dots, B_n, \Gamma \longrightarrow C$ , for any reducible derivations

$$\begin{array}{cccc}
\Pi_1 & \Pi_n \\
\Delta_1 \longrightarrow B_1, & \dots & \Delta_n \longrightarrow B_n
\end{array}$$

where  $n \ge 0$ , and for any substitutions  $\delta_1, \dots, \delta_n, \gamma$  such that  $B_i \delta_i = B_i \gamma$  for every  $i \in \{1, \dots, n\}$ , the derivation  $\Xi$ 

is reducible.

*Proof.* The proof is by induction on  $\operatorname{indm}(\Pi)$  with subordinate induction on  $\operatorname{ht}(\Pi)$ , on n and on the reductions of  $\Pi_1, \ldots, \Pi_n$ . The proof does not rely on the order of the inductions on reductions. Thus when we need to distinguish one of the  $\Pi_i$ , we shall refer to it as  $\Pi_1$  without loss of generality. The derivation  $\Xi$  is reducible if all its reducts are reducible.

If n=0, then  $\Xi$  reduces to  $\Pi\gamma$ , thus in this case we show that  $\Pi\gamma$  is reducible. Since reducibility is preserved by substitution (Lemma 14), it is enough to show that  $\Pi$  is reducible. This is proved by a case analysis of the last rule in  $\Pi$ . For each case, the result follows easily from the induction hypothesis on  $ht(\Pi)$  and Definition 15. The  $\supset \mathcal{R}$  case requires that substitution for variables does not increase the measures of a derivation. In the cases for  $\supset \mathcal{L}$  and  $I\mathcal{L}$  we need the additional information that reducibility implies normalizability (Lemma 13). The case for  $CI\mathcal{R}$  requires special attention. Let  $p\vec{x} \stackrel{\vee}{=} Dp\vec{x}$  be a co-inductive definition. Suppose  $\Pi$  is the derivation

$$\frac{\Gamma \xrightarrow{\Pi'} S\vec{t} \quad S\vec{x} \xrightarrow{\Pi_S} DS\vec{x}}{\Gamma \xrightarrow{} p\vec{t}} CI\mathcal{R}$$

for some invariant S. Let  $\mathcal{R} = \bigcup \{ \mathbf{RED}_j \mid j < \mathrm{lvl}(p) \}$ . To show that  $\Pi$  is reducible we must show that its premises are normalizable and that there exists a  $(\mathcal{R}, \Pi_S)$ -saturated set S such that  $\Pi' \in S$ . The former follows from the outer induction hypothesis and Lemma 13. For the latter, the set S is defined as follows:

$$S = \{ \Psi \mid \Psi \text{ is a reducible derivation of type } S\vec{u}, \text{ for some } \vec{u} \}.$$

Since  $\Pi'$  is reducible by induction hypothesis, we have  $\Pi' \in \mathcal{S}$ . It remains to show that  $\mathcal{S}$  is a  $(\mathcal{R}, \Pi_{\mathcal{S}})$ -saturated set. More specifically, we show that  $\mathcal{S}$  has the following properties.

- 1. Every derivation in S is normalizable.
- 2. If  $\Psi \in \mathcal{S}$  then  $\Psi \theta \in \mathcal{S}$  for any  $\theta$ .
- 3. If  $\Psi \in \mathcal{S}$  and  $\Psi$  is of type  $S\vec{u}$  for some  $\vec{u}$ , then  $mc(\Psi, \Pi_S[\vec{u}/\vec{x}]) \in \mathbf{RED}^p_{Bn\vec{u}}[\mathcal{R}, \mathcal{S}]$

<sup>&</sup>lt;sup>4</sup> We also note that McDowell and Miller's proof of cut elimination for  $FO\lambda^{\Delta IN}$  given in [25] appears to contain a small gap in the proof of a main technical lemma. More specifically, they use a similar technical lemma as Lemma 18, but without the extra assumptions about the substitutions  $\delta_1, \ldots, \delta_n, \theta$ . The problem with their formulation of the lemma appears in the case involving the eq $L/\circ L$  reduction rule. This problem is fixed in our cut elimination proof with the more general statement of Lemma 18. See http://www.lix.polytechnique.fr/ dale/papers/tcs00.errata.html for details of the errata in their paper.

Property (1) follows from the fact that reducibility implies normalizability (Lemma 13). Property (2) follows from the fact that reducibility is closed under substitution (Lemma 14). To prove (3), first notice that by Lemma 2,  $\operatorname{indm}(\Pi_S[\vec{u}/\vec{x}]) \leq \operatorname{indm}(\Pi_S) = \operatorname{indm}(\Pi)$  and  $\operatorname{ht}(\Pi_S[\vec{u}/\vec{x}]) \leq \operatorname{ht}(\Pi_S) < \operatorname{ht}(\Pi)$ . Therefore, by the outer induction hypothesis, we have that  $\operatorname{mc}(\Psi, \Pi_S[\vec{u}/\vec{x}])$  is reducible. By Lemma 15, we have that  $\operatorname{mc}(\Psi, \Pi_S[\vec{u}/\vec{x}]) \in \operatorname{\mathbf{RED}}^p_{B\,p\,\vec{u}}[\mathcal{R}, \mathcal{S}]$ . Therefore,  $\mathcal{S}$  is a  $(\mathcal{R}, \Pi_S)$ -saturated set containing  $\Pi'$ , hence  $\Pi$  is reducible.

For n > 0, we analyze all possible cut reductions and show for each case the reduct is reducible. Some cases follow immediately from inductive hypothesis. We show here the non-trivial cases.

 $\supset \mathcal{R}/\supset \mathcal{L}$ : Suppose  $\Pi_1$  and  $\Pi$  are

$$\frac{\Pi_1'}{\Delta_1, B_1' \longrightarrow B_1''} \supset \mathcal{R} \qquad \qquad \frac{\Pi' \qquad \Pi''}{B_2, \dots, \Gamma \longrightarrow B_1' \quad B_1'', B_2, \dots, \Gamma \longrightarrow C} \supset \mathcal{L}$$

The derivation  $\Xi_1$ 

$$\frac{\Delta_2 \delta_2 \longrightarrow B_2 \delta_2}{\Delta_2 \delta_2 \longrightarrow B_2 \delta_2} \quad \dots \quad \Delta_n \delta_n \longrightarrow B_n \delta_n \quad B_2 \gamma, \dots, B_n \gamma, \Gamma \gamma \longrightarrow B'_1 \gamma \\
\qquad \qquad \Delta_2 \delta_2, \dots, \Delta_n \delta_n, \Gamma \gamma \longrightarrow B'_1 \gamma \longrightarrow B'_1 \gamma$$
mc

is reducible by induction hypothesis since  $\operatorname{indm}(\Pi') \leq \operatorname{indm}(\Pi)$  and  $\operatorname{ht}(\Pi') < \operatorname{ht}(\Pi)$ . Since  $\Pi_1$  is reducible, by Definition 15 the derivation  $\Xi_2$ 

$$\frac{\Xi_1}{\Delta_2\delta_2,\dots,\Gamma\gamma\longrightarrow B_1'\gamma\quad B_1'\delta_1,\Delta_1\delta_1\longrightarrow B_1''\delta_1} \frac{\Delta_2\delta_2,\dots,\Gamma\gamma\longrightarrow B_1''\gamma\quad B_1'\delta_1\longrightarrow B_1''\delta_1}{\Delta_1\delta_1,\dots,\Delta_n\delta_n,\Gamma\gamma\longrightarrow B_1''\delta_1} \ mc$$

is a predecessor of  $\Pi_1$  and therefore is reducible. The reduct of  $\Xi$  in this case is the following derivation

$$\frac{\Xi_{2}}{\cdots \longrightarrow B_{1}''\delta_{1}} \begin{cases} \Pi_{i}\delta_{i} \\ \Delta_{i}\delta_{i} \longrightarrow B_{i}\delta_{i} \end{cases} \prod_{i \in \{2...n\}} \frac{\Pi''\gamma}{B_{1}'\gamma, \dots, B_{n}\gamma, \Gamma\gamma \longrightarrow C\gamma} mc} \frac{\Delta_{1}\delta_{1}, \dots, \Delta_{n}\delta_{n}, \Gamma\gamma, \Delta_{2}\delta_{2}, \dots, \Delta_{n}\gamma, \Gamma\gamma \longrightarrow C\gamma}{\Delta_{1}\delta_{1}, \dots, \Delta_{n}\delta_{n}, \Gamma\gamma \longrightarrow C\gamma} c\mathcal{L}$$

which is reducible by induction hypothesis and Definition 15.

 $\forall \mathcal{L}/\forall \mathcal{R}$ : Suppose  $\Pi_1$  and  $\Pi$  are

$$\frac{\Pi'_1}{\Delta_1 \longrightarrow B'_1[y/x]} \quad \forall \mathcal{R} \qquad \frac{B'_1[t/x], B_2, \dots, B_n, \Gamma \longrightarrow C}{\forall x. B'_1, B_2, \dots, B_n, \Gamma \longrightarrow C} \quad \forall \mathcal{L}$$

Since we identify derivations that differ only in the choice of intermediate eigenvariables that are not free in the end sequents, we can choose a variable y such that it is not free in the domains and ranges of  $\delta_1$  and  $\gamma$ . We assume without loss of generality that x is chosen to be fresh with respect to the free variables in the substitutions so we can push the substitutions under the binder. The derivation  $\Xi$  is thus

$$\frac{ \frac{\Pi_1'\delta_1}{\Delta_1\delta_1 \longrightarrow B_1'\delta_1[y/x]}}{\frac{\Delta_1\delta_1 \longrightarrow \forall x.B_1'\delta_1}{\Delta_1\delta_1, \dots, \Delta_n\delta_n, \Gamma\gamma \longrightarrow C\gamma}} \stackrel{\Pi'\gamma}{\forall x.B_1'\gamma, \dots, \Gamma\gamma \longrightarrow C\gamma} \frac{\forall \mathcal{L}}{mc}$$

Let  $\delta_1' = \delta_1 \circ [t\gamma/y]$ . The reduct of  $\Xi$  in this case is

$$\frac{\Delta_1 \delta_1 \longrightarrow B_1' \delta_1' \qquad \qquad \Pi' \gamma}{\Delta_1 \delta_1 \longrightarrow B_1' \delta_1[t \gamma/x] \qquad \dots \qquad B_1' \gamma[t \gamma/x], \dots, \Gamma \gamma \longrightarrow C \gamma}{\Delta_1 \delta_1, \dots, \Delta_n \delta_n, \Gamma \gamma \longrightarrow C \gamma} mc$$

which is reducible by induction hypothesis.

eq $\mathcal{R}/\text{eq}\mathcal{L}$ : Suppose  $\Pi_1$  and  $\Pi$  are

$$\frac{\left\{ \begin{array}{c} \Pi^{\rho} \\ B_{2}\rho, \dots, B_{n}\rho, \Gamma\rho \longrightarrow C\rho \end{array} \right\}_{\rho}}{s = t, \dots, B_{n}, \Gamma \longrightarrow C} \operatorname{eq} \mathcal{L}$$

Then  $\Xi$  is the derivation

$$\underline{\frac{\Delta_1 \delta_1 \longrightarrow (s=t) \delta_1}{\Delta_1 \delta_1 \longrightarrow (s=t) \delta_1}} \stackrel{\text{eqR}}{=} \frac{\left\{ \begin{matrix} \Pi^{\gamma \circ \rho'} \\ B_2 \gamma \rho', \dots, B_n \gamma \rho', \Gamma \rho' \longrightarrow C \gamma \rho' \end{matrix} \right\}_{\rho'}}{(s=t) \gamma, \dots, B_n \gamma, \Gamma \gamma \longrightarrow C \gamma} \stackrel{\text{eqL}}{=}$$

The eq $\mathcal{R}$  tells us that s and t are unifiable via empty substitution (i.e., they are the same normal terms). The reduct of  $\Xi$ 

$$\frac{\Delta_{2}\delta_{2} \longrightarrow B_{2}\delta_{2} \qquad \prod^{\gamma} \qquad C\gamma}{\Delta_{2}\delta_{2}, \dots, \Delta_{n}\delta_{n}, \Gamma\gamma \longrightarrow C\gamma} mc$$

$$\frac{\Delta_{2}\delta_{2}, \dots, \Delta_{n}\delta_{n}, \Gamma\gamma \longrightarrow C\gamma}{\Delta_{1}\delta_{1}, \Delta_{2}\delta_{2}, \dots, \Delta_{n}\delta_{n}, \Gamma\gamma \longrightarrow C\gamma} wL$$

is therefore reducible by induction hypothesis.

\*/I $\mathcal{L}$ : Suppose  $\Pi$  is the derivation

$$\frac{DS\vec{x} \longrightarrow S\vec{x} \quad S\vec{t}, \Gamma \longrightarrow C}{n\vec{t}, \Gamma \longrightarrow C} \text{ IL}$$

where  $p\vec{x} \stackrel{\mu}{=} D p\vec{x}$ . Let  $p\vec{u}$  be the result of applying  $\delta_1$  to  $p\vec{t}$ . Then  $\Xi$  is the derivation

$$\frac{\prod_{1}\delta_{1}}{\Delta_{1}\delta_{1} \longrightarrow p\vec{u}} \cdots \frac{\prod_{n}\delta_{n}}{\Delta_{n}\delta_{n} \longrightarrow B_{n}\delta_{n}} \frac{DS\vec{x} \longrightarrow S\vec{x}}{p\vec{u}, \dots, \Gamma\gamma \longrightarrow C\gamma} \xrightarrow{mc} IL$$

The derivation  $\Xi$  reduces to the derivation  $\Xi'$ 

$$\frac{\mu(\Pi_1,\Pi_S)\delta_1}{\Delta_1\delta_1 \longrightarrow S\vec{u} \cdots \Delta_n\delta_n \longrightarrow B_n\delta_n} \frac{\Pi'\gamma}{S\vec{u},\Gamma\gamma \longrightarrow C\gamma} mc$$

$$\frac{\Delta_1\delta_1 \longrightarrow S\vec{u} \cdots \Delta_n\delta_n,\Gamma\gamma \longrightarrow C\gamma}{\Delta_1\delta_1,\dots,\Delta_n\delta_n,\Gamma\gamma \longrightarrow C\gamma} mc$$

Notice that we have used the fact that

$$\mu(\Pi_1\delta_1,\Pi_S) = \mu(\Pi_1,\Pi_S)\delta_1$$

in the derivation above, which follows from Lemma 5. Therefore, in order to prove that  $\Xi'$  is reducible, it remains to show that the unfolding of  $\Pi_1$  produces a reducible derivation. This will be proved using Lemma 16, but we shall first prove the following properties, which are the conditions for applying Lemma 16:

- 1. For every derivation  $\Psi$  of  $\Delta \longrightarrow D p \vec{s}$ , if  $\mu(\Psi, \Pi_S)$  is reducible, then the derivation  $mc(\mu(\Psi, \Pi_S), \Pi_S[\vec{s}/\vec{x}])$  is reducible.
- 2. For every reducible derivation  $\Psi$  of  $\Delta \longrightarrow S\vec{u}$  the derivation  $mc(\Psi, \mathrm{Id}_{S\vec{u}})$  is reducible.
- 3. The derivation  $ind(\Pi_S, Id_{S\vec{u}})$  is reducible, for every  $\vec{u}$  of the appropriate types.

To prove (1), we observe that  $\operatorname{indm}(\Pi_S[\vec{u}/\vec{x}]) \leq \operatorname{indm}(\Pi_S) < \operatorname{indm}(\Pi)$ , so by the outer induction hypothesis, the derivation  $mc(\mu(\Xi,\Pi_S),\Pi_S[\vec{u}/\vec{x}])$  is reducible. Property (2) is proved similarly, by observing that  $\operatorname{indm}(\operatorname{Id}_{S\vec{u}}) < \operatorname{indm}(\Pi)$  (since identity derivations do not use the  $\operatorname{IL}$  rule; c.f. Lemma 4). Property (3) follows from the fact that  $\operatorname{Id}_{S\vec{u}}$  is reducible and that  $\Pi_S$  is reducible (hence, also normalizable). Having shown these three properties, using Lemma 16 we conclude that  $\mu(\Pi_1,\Pi_S)$  is reducible, hence, by the outer induction ( $\Pi'$  is smaller than  $\Pi$ ), the reduct  $\Xi'$  is reducible.

 $CI\mathcal{R}/CI\mathcal{L}$ : Suppose  $\Pi_1$  and  $\Pi$  are

$$\frac{\Pi'_1}{\Delta_1 \longrightarrow S\vec{t}} \frac{\Pi_S}{S\vec{x} \longrightarrow DS\vec{x}} \text{ CIR} \qquad \frac{D\vec{pt}, B_2, \dots, \Gamma \longrightarrow C}{\vec{pt}, B_2, \dots, \Gamma \longrightarrow C} \text{ CIL}$$

where  $p\vec{x} \stackrel{\vee}{=} Dp\vec{x}$ . Suppose  $(p\vec{t})\delta_1 = (p\vec{t})\gamma = p\vec{u}$ . Then  $\Xi$  is the derivation

Let  $\mathcal{R} = \bigcup \{ \mathbf{RED}_F \mid lvl(F) < lvl(p) \}$ . Since  $\Pi_1$  is reducible, there exists a  $(\mathcal{R}, \Pi_S)$ -saturated set S such that  $\Pi'_1 \in S$ . Let  $\Xi_1$  be the derivation

$$\frac{\Pi'_1 \delta_1}{\Delta_1 \delta_1 \longrightarrow S \vec{u}} \frac{\Pi_S[\vec{u}/\vec{x}]}{S \vec{u} \longrightarrow D S \vec{u}} mc$$

$$\frac{\Delta_1 \delta_1 \longrightarrow D S \vec{u}}{\Delta_1 \delta_1 \longrightarrow D S \vec{u}} mc$$

Since S is a  $(\mathcal{R},\Pi_S)$ -saturated set, by Definition 14,  $\Xi_1 \in \mathbf{RED}^p_{Dp\vec{u}}[\mathcal{R},S]$ . It then follows from Lemma 17 that  $\nu(\Xi_1,\Pi_S)$  is reducible.

The reduct of  $\Xi$  is the derivation

$$\frac{\nu(\Xi_1,\Pi_S)}{\Delta_1\delta_1 \longrightarrow D p \vec{u}} \frac{\Pi_n\delta_n}{\cdots \Delta_n\delta_n \longrightarrow B_n\delta_n} \frac{\Pi'\gamma}{D p \vec{u}, \dots, B_n\gamma, \Gamma\gamma \longrightarrow C\gamma} mc.$$

Its reducibility follows from the reducibility of  $v(\Xi_1, \Pi_S)$  and the outer induction hypothesis.

 $\supset \mathcal{L}/\circ \mathcal{L}$ : Suppose  $\Pi_1$  is

$$\frac{\Delta_1' \longrightarrow D_1' \quad D_1'', \Delta_1' \longrightarrow B_1}{D_1' \supset D_1'', \Delta_1' \longrightarrow B_1} \supset \mathcal{L}$$

Since  $\Pi_1$  is reducible, it follows from Definition 15 that  $\Pi'_1$  is normalizable and  $\Pi''_1$  is reducible. Let  $\Xi_1$  be the derivation

$$\frac{D_1''\delta_1}{D_1''\delta_1,\Delta_1'\delta_1 \longrightarrow B_1\delta_1} \frac{\Pi_2\delta_2}{\Delta_2\delta_2 \longrightarrow B_2\delta_2} \frac{\Pi\gamma}{\cdots B_1\delta_1,\dots,\Gamma\gamma \longrightarrow C\gamma} mc$$

 $\Xi_1$  is reducible by induction hypothesis on the reduction of  $\Pi_1$  ( $\Pi_1''$  is a predecessor of  $\Pi_1$ ). The reduct of  $\Xi$  in this case is the derivation

$$\frac{D_1'\delta_1}{\frac{\Delta_1'\delta_1 \longrightarrow D_1'\delta_1}{\Delta_1'\delta_1, \Delta_2\delta_2, \dots, \Gamma\gamma \longrightarrow D_1'\delta_1}} \overset{w\mathcal{L}}{=} \frac{\Xi_1}{D_1''\delta_1, \Delta_1'\delta_1, \Delta_2\delta_2, \dots, \Gamma\gamma \longrightarrow C\gamma} \supset \mathcal{L}$$

Since  $\Pi'_1$  is normalizable and substitutions preserve normalizability, by Definition 11 the left premise of the reduct is normalizable, and hence the reduct is reducible.

 $eq \mathcal{L} / \circ \mathcal{L}$ : Suppose  $\Pi_1$  is

$$\frac{\left\{\Delta_1'\rho \xrightarrow{\Pi^{\rho}} B_1\rho\right\}_{\rho}}{s = t, \Delta_1' \longrightarrow B_1} \text{ eq} \mathcal{L}$$

Then  $\Xi$  is the derivation

$$\frac{\left\{ \begin{array}{c} \Pi^{\delta_{1} \circ \rho'} \\ \Delta'_{1} \delta_{1} \rho' \longrightarrow B_{1} \delta_{1} \rho' \right\}_{\rho'}}{(s=t)\delta_{1}, \Delta'_{1} \delta_{1} \longrightarrow B_{1} \delta_{1}} \operatorname{eq} \mathcal{L} \quad \Pi_{2} \delta_{2} \quad \Pi_{\gamma} \\ \underline{(s=t)\delta_{1}, \Delta'_{1} \delta_{1} \longrightarrow B_{1} \delta_{1}} \quad \Delta_{2} \delta_{2} \longrightarrow B_{2} \delta_{2} \quad \cdots \quad B_{1} \gamma, \dots, \Gamma \gamma \longrightarrow C \gamma \\ \underline{(s=t)\delta_{1}, \Delta'_{1} \delta_{1}, \Delta_{2} \delta_{2}, \dots, \Gamma \gamma \longrightarrow C \gamma} \quad mc$$

Notice that each premise derivation  $\Pi^{\delta_1 \circ \rho'}$  of  $\Pi_1 \delta_1$  is a also a premise derivation of  $\Pi_1$ , since for every unifier  $\rho'$  of  $(s=t)\delta_1$ , there is a unifier of s=t, i.e., the substitution  $\delta_1 \circ \rho'$ . Therefore every  $\Pi^{\delta_1 \circ \rho'}$  is a predecessor of  $\Pi_1$ . Let  $\Xi^{\rho'}$  be the derivation

$$\frac{\Pi_1^{\delta_1 \circ \rho'} \quad \Pi_2 \delta_2 \rho' \quad \Pi_{\gamma \rho'}}{\Delta_1' \delta_1 \rho' \longrightarrow B_1 \delta_1 \rho' \quad \Delta_2 \delta_2 \rho' \longrightarrow B_2 \delta_2 \rho' \quad B_1 \gamma \rho', \dots, \Gamma \gamma \rho' \longrightarrow C \gamma \rho'}{\Delta_1' \delta_1 \rho', \Delta_2 \delta_2 \rho', \dots, \Gamma \gamma \rho' \longrightarrow C \gamma \rho'} mc.$$

The reduct of  $\Xi$ 

$$\frac{\left\{ \Delta_{1}'\delta_{1}\rho', \dots, \Gamma\gamma\rho' \longrightarrow C\gamma\rho' \right\}_{\rho'}}{(s=t)\delta_{1}, \Delta_{1}'\delta_{1}, \dots, \Gamma\gamma \longrightarrow C\gamma} \text{ eq} \mathcal{L}$$

is then reducible by Definition 15.

 $I\mathcal{L}/\circ\mathcal{L}$ : Suppose  $\Pi_1$  is

$$\frac{DS\vec{x} \longrightarrow S\vec{x} \quad S\vec{t}, \Delta'_1 \longrightarrow B_1}{p\vec{t}, \Delta'_1 \longrightarrow B_1} \text{ I.L.}$$

Since  $\Pi_1$  is reducible, it follows from the definition of reducibility that  $\Pi'_1$  is reducible predecessor of  $\Pi_1$  and  $\Pi_S$  is normalizable. Suppose  $p\vec{u} = (p\vec{t})\delta_1 = (p\vec{t})\gamma$ . Let  $\Xi_1$  be the derivation

$$\frac{S\vec{u}, \Delta_1'\delta_1 \qquad \Pi_n\delta_n \qquad \Pi\gamma}{S\vec{u}, \Delta_1'\delta_1 \longrightarrow B_1\delta_1 \qquad \Delta_n\delta_n \longrightarrow B_n\delta_n \quad B_1\gamma, \dots, B_n\gamma, \Gamma\gamma \longrightarrow C\gamma} mc$$

 $\Xi_1$  is reducible by induction on the reduction of  $\Pi_1$ , therefore the reduct of  $\Xi$ 

$$\frac{DS\vec{x} \longrightarrow S\vec{x} \quad S\vec{u}, \Delta'_{1}\delta_{1}, \dots, \Delta_{n}\delta_{n}, \Gamma\gamma \longrightarrow C\gamma}{p\vec{u}, \Delta'_{1}\delta_{1}, \dots, \Delta_{n}\delta_{n}, \Gamma\gamma \longrightarrow C\gamma} \text{ I.C.}$$

is reducible.

 $-/\supset \mathcal{L}$ : Suppose  $\Pi$  is

$$\frac{B_1,\ldots,B_n,\Gamma'\longrightarrow D'\quad B_1,\ldots,B_n,D'',\Gamma'\longrightarrow C}{B_1,\ldots,B_n,D'\supset D'',\Gamma'\longrightarrow C}\supset \mathcal{L}$$

Let  $\Xi_1$  be

$$\frac{\Delta_1 \delta_1 \longrightarrow B_1 \delta_1 \qquad \prod_n \delta_n \qquad \prod' \gamma}{\Delta_1 \delta_1 \longrightarrow B_1 \delta_1 \qquad \cdots \qquad \Delta_n \delta_n \longrightarrow B_n \delta_n \qquad B_1 \gamma, \dots, B_n \gamma, \Gamma' \gamma \longrightarrow D' \gamma}{\Delta_1 \delta_1, \dots, \Delta_n \delta_n, \Gamma' \gamma \longrightarrow D' \gamma} mc$$

and  $\Xi_2$  be

$$\frac{\Delta_1 \delta_1 \longrightarrow B_1 \delta_1 \qquad \Pi_n \delta_n \qquad \Pi'' \gamma}{\Delta_1 \delta_1 \longrightarrow B_1 \delta_1 \qquad \cdots \qquad \Delta_n \delta_n \longrightarrow B_n \delta_n \qquad B_1 \gamma, \dots, B_n \gamma, D'' \gamma, \Gamma' \gamma \longrightarrow C \gamma}{\Delta_1 \delta_1, \dots, \Delta_n \delta_n, D'' \gamma, \Gamma' \gamma \longrightarrow C \gamma} mc$$

Both  $\Xi_1$  and  $\Xi_2$  are reducible by induction hypothesis. Therefore the reduct of  $\Xi$ 

$$\frac{\Delta_1\delta_1,\ldots,\Delta_n\delta_n,\Gamma'\gamma\longrightarrow D'\gamma}{\Delta_1\delta_1,\ldots,\Delta_n\delta_n,(D'\supset D'')\gamma,\Gamma'\gamma\longrightarrow C\gamma}\supset \mathcal{L}$$

is reducible (reducibility of  $\Xi_1$  implies its normalizability by Lemma 12).

 $-/CI\mathcal{R}$ : Suppose  $\Pi$  is

$$\frac{B_1, \dots, B_n, \Gamma \longrightarrow S\vec{t} \quad S\vec{x} \longrightarrow DS\vec{x}}{B_1, \dots, B_n, \Gamma \longrightarrow p\vec{t}} \text{ CIR}$$

where  $p\vec{x} \stackrel{\vee}{=} Dp\vec{x}$ . Suppose  $p\vec{u} = (p\vec{t})\delta_1 = (p\vec{t})\gamma$ . Let  $\Xi_1$  be the derivation

$$\frac{\Delta_1 \delta_1 \longrightarrow B_1 \delta_1 \qquad \prod_n \delta_n \qquad \prod' \gamma}{\Delta_1 \delta_1 \longrightarrow B_1 \delta_1 \qquad \cdots \qquad \Delta_n \delta_n \longrightarrow B_n \delta_n \qquad B_1 \gamma, \dots, B_n \gamma, \Gamma \gamma \longrightarrow S \vec{u}}{\Delta_1 \delta_1, \dots, \Delta_n \delta_n, \Gamma \gamma \longrightarrow S \vec{u}} mc$$

The derivations  $\Pi'\gamma$ ,  $\Pi_S$ ,  $\Xi_1$  and the derivation

$$\frac{\Delta' \longrightarrow S\vec{w} \quad S\vec{w} \longrightarrow DS\vec{w}}{\Delta' \longrightarrow DS\vec{w}} \ mc$$

where  $\Psi$  is any reducible derivation, are all reducible by induction hypothesis on the length of  $\Pi$ . Again, we use the same arguments as in the case where n=0 to construct a  $(\mathcal{R},\Pi_S)$ -saturated set S such that  $\Xi_1 \in S$ . Therefore by Definition 15, the reduct of  $\Xi$ :

$$\frac{\Delta_1 \delta_1, \dots, \Delta_n \delta_n, \Gamma \gamma \longrightarrow S \vec{u} \quad S \vec{x} \longrightarrow D S \vec{x}}{\Delta_1 \delta_1, \dots, \Delta_n \delta_n, \Gamma \gamma \longrightarrow p \vec{u}} \text{ CIR}$$

is reducible.

 $mc/\circ L$ : Suppose  $\Pi_1$  ends with a mc. Then any reduct of  $\Pi_1\delta_1$  corresponds to a predecessor of  $\Pi_1$  by Lemma 7. Therefore the reduct of  $\Xi$  is reducible by induction on the reduction of  $\Pi_1$ .

-/init:  $\Xi$  reduces to  $\Pi_1\delta_1$ . Since  $\Pi_1$  is reducible, by Lemma 14,  $\Pi_1\delta_1$  is reducible and hence  $\Xi$  is reducible.

**Corollary 1.** Every derivation is reducible.

*Proof.* The proof follows from Lemma 18, by setting n = 0.

Since reducibility implies cut-elimination, it follows that every proof can be transformed into a cut-free proof.

**Corollary 2.** Given a fixed stratified definition, a sequent has a proof in Linc<sup>-</sup> if and only if it has a cut-free proof.

The consistency of Linc<sup>-</sup> is an immediate consequence of cut-elimination. By consistency we mean the following: given a fixed stratified definition and an arbitrary formula C, it is not the case that both C and  $C \supset \bot$  are provable.

**Corollary 3.** The logic Linc<sup>-</sup> is consistent.

*Proof.* Suppose otherwise, that is, there is a formula C such that there is a proof  $\Pi_1$  of C and another proof  $\Pi_2$  for  $C \supset \bot$ . Since cut elimination holds, we can assume, without loss of generality, that  $\Pi_1$  and  $\Pi_2$  are cut free. By inspection of the inference rules of Linc<sup>-</sup>, we see that  $\Pi_2$  must end with  $\supset \mathcal{R}$ , that is,  $\Pi_2$  is

$$\frac{C \longrightarrow \bot}{\longrightarrow C \supset \bot} \supset \mathcal{R}$$

Cutting  $\Pi_1$  with  $\Pi_2'$  we get a derivation of  $\cdot \longrightarrow \bot$ , and applying the cut-elimination procedure we get a cut-free derivation of  $\cdot \longrightarrow \bot$ . But there cannot be such a derivation since there is no right-introduction rule for  $\bot$ , contradiction.

## 6 Related Work

Of course, there is a long association between mathematical logic and inductive definitions [2] and in particular with proof-theory, starting with the Takeuti's conjecture, the earliest relevant entry for our purposes being Martin-Löf's original formulation of the theory of *iterated inductive definitions* [24]. From the impredicative encoding of inductive types [7] and the introduction of (co)recursion [16, 29] in system F, (co)inductive types became common and made it into type-theoretic proof assistants such as Coq [37], first via a primitive recursive operator, but eventually in the let-rec style of functional programming languages, as in Gimenez's *Calculus of Infinite Constructions* [18]; here termination (resp. productivity) is ensured by a syntactic check known as *guarded by destructors* [17]. Note that Coq forbids altogether the introduction of blocks of mutually dependent types containing both inductive and co-inductive ones, even though they could be stratified. Moreover, while a syntactic check has obvious advantages, it tends to be too restrictive, as observed and improved upon in [6] by using type based termination. The same can be said about *Agda* [36], where size types termination will eventually supersede guardedness [28].

Baelde and Miller have recently introduced an extension of linear logic with least and greatest fixed points [5]. However, cut elimination is proved indirectly via a second-order encoding of the least and the greatest fixed point operators into higher-order linear logic and via an appeal to completeness of focused proofs for higher-order linear logic.

Circular proofs are also connected with the emerging proof-theory of of fixed point logics and process calculi [48, 55], as well as in traditional sequent calculi such as in [8]. The issue is the equivalence between systems with local vs global induction, that is, between fixed point rules vs. well-founded and guarded induction (*i.e.* circular proofs). In the sequent calculus it is unknown whether every inductive proof can be obtained via global induction.

In higher order logic (co)inductive definitions are obtained via the usual Tarski fixed point constructions, as realized for example in Isabelle/HOL [38]. As we mentioned before, those approaches are at odd with HOAS even at the level of the syntax. This issue has originated a research field in its own that we can only try to mention the main contenders: in the Twelf system [41] the LF type theory is used to encode deductive systems as judgments and to specify metatheorems as relations (type families) among them; a logic programming-like interpretation provides an operational semantics to those relations, so that an external check for totality (incorporating termination, well-modedness and coverage [42,53]) verifies that the given relation is indeed a realizer for that theorem. Coinduction is still unaccounted for and may require a switch to a different operational semantics for LF. There exists a second approach to reasoning in LF that is built on the idea of devising an explicit (meta-)meta-logic ( $\mathcal{M}_{\omega}$ ) for reasoning (inductively) about the framework, in a fully automated way [52]. It can be seen as a constructive first-order inductive type theory, whose quantifiers range over possibly open LF objects over a signature. In this calculus it is possible to express and inductively prove meta-logical properties of an object level system.  $\mathcal{M}_0$  can be also seen as a dependently-typed functional programming language, and as such it has been refined first into the Elphin programming language [54] and more recently in Delphin [47]. In a similar vein the context modal logic of Pientka, Pfenning and Naneski [34] provides a basis for a different foundation for programming with HOAS and dependent types based on hereditary substitutions, see the programming language Beluga ([43,44]). Because all of these systems are programming languages, we refrain from a deeper discussion. We only note that systems like Delphin or Beluga separate data from computations. This means they are always based on eager evaluation, whereas co-recursive functions should be interpreted lazily. Using standard techniques such as thunks to simulate lazy evaluation in such a context seems problematic (Pientka, personal communication).

Weak higher-order abstract syntax [11] is an approach that strives to co-exist with an inductive setting, where the positivity condition for datatypes and hypothetical judgments must be obeyed. The problem of negative occurrences in datatypes is handled by replacing them with a new type. The approach is extended to hypothetical judgments by introducing distinct predicates for the negative occurrences. Some axioms are needed to reason about hypothetical judgments, to mimic what is inferred by the cut rule in our architecture. Miculan *et al.*'s framework [22] embraces this *axiomatic* approach extending Coq with the "theory of contexts" (ToC). The theory includes axioms for the the reification of key properties of names akin to *freshness*. Furthermore, higher-order induction and recursion schemata on expressions are also assumed. *Hybrid* [3] is a  $\lambda$ -calculus on top of Isabelle/HOL which provides the user with a *Full* HOAS syntax, compatible with a classical (co)-inductive setting. Linc<sup>-</sup> improves on the latter on several counts. First it disposes of Hybrid notion of *abstraction*, which is used to carve out the "parametric" function space from the full HOL space. Moreover it is not restricted to second-order abstract syntax, as the current Hybrid version is (and as ToC cannot escape from being). Finally, at higher types, reasoning via  $def \mathcal{L}$  is more powerful than inversion, which does not exploit higher-order unification.

ToC can be seen as a stepping stone towards Gabbay and Pitts *nominal logic*, which aims to be a foundation of programming and reasoning with *names*. It can be presented as a first-order theory [45], which includes primitives for variable renaming and variable freshness, and a (derived) new "freshness" quantifier. Using this theory, it is possible to prove properties by structural induction and also to define functions by recursion over syntax [46]. Urban *et al.*'s have engineered a *nominal datatype package* inside Isabelle/HOL [35] analogous to the standard datatype package but defining equivalence classes of term constructors. In more recent versions, principles of primitive recursion and strong induction have been added [60]. Coinduction on nominal datatypes is not available, but to be fair it is also absent from Isabelle/HOL due to some technical limitations in the automation of the inductive package

#### 7 Conclusion and Future Work

We have presented a proof theoretical treatment of both induction and co-induction in a sequent calculus compatible with HOAS encodings. The proof principle underlying the explicit proof rules is basically fixed point (co)induction. We have shown some examples where informal (co)inductive proofs using invariants and simulations are reproduced formally in Linc<sup>-</sup>.

Consistency of the logic is an easy consequence of cut-elimination. Our proof system is, as far as we know, the first which incorporates a co-induction proof rule with a direct cut elimination proof. This schema can be used as a springboard towards cut elimination procedures for more expressive (conservative) extensions of Linc<sup>-</sup>, for example in the direction of  $FO\lambda^{\nabla}$  [31], or more recently, the logic  $LG^{\omega}$  [57] by Tiu and the logic G by Gacek *et al.* [14].

As far as future work, we may investigate loosening the stratification condition for example in the sense of *local* stratification, possibly allowing to encode proofs such as type preservation in operational semantics directly in Linc<sup>-</sup> rather than with the 2-level approach [26, 32]. More general notions of stratifications are already allowed in practice, see the proof by logical relations in [15], but not formally justified.

Another interesting problem is the connection with *circular proofs*, which is particularly attractive from the view-point of proof search, both inductively and co-inductively. This could be realized by directly proving a cut-elimination result for a logic where circular proofs, under termination and guardedness conditions completely replace (co)inductive rules. Indeed, the question whether "global" proofs are equivalent to "local" proofs [8] is still unsettled.

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