Time-Scale Transformations: Effects on VaR Models

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Abstract. This paper investigates the effects of using temporal aggregation rules in the evaluation of the maximum portfolio loss[1]. In particular, we propose and compare different time aggregation rules for VaR models. We implement time-scale transformations for: (i) a EWMA model with Student's t conditional distributions, (ii) a stable sub-Gaussian model, (iii) a stable asymmetric model. All models are subjected to backtest on out-of-sample data in order to assess their forecasting power and to show how these aggregation rules perform in practice.

1 Introduction

Several empirical and theoretical studies on the asymptotic behavior of financial returns (see, among others, [4], [6]) justify the assumption of stable distributed returns. The joint stable sub-Gaussian family is an elliptical family recently used in portfolio theory and risk management (see [8], [10], [11]).

Following these studies, our paper presents and compares some alternative models for the calculation of VaR taking into consideration their time scale transformations. Firstly, we consider EWMA models with conditional elliptical distributed returns and finite variance. Secondly, we describe VaR models in the domain of attraction of stable laws. In particular, we focus our attention on returns either with conditional multivariate Student's t-distributions or with stable Paretian distributions. We describe a time rule for each model and we analyze and compare each performance considering conditional and unconditional coverage tests. We also test a particular temporal rule of VaR for the stable EWMA model in the same way as we did for the elliptical EWMA model with finite variance. In order to consider the asymmetry of financial series, we assume conditional jointly α-stable distributed returns. The asymmetric stable model results from a new conditional version of the stable three fund separation model

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recently proposed in portfolio theory. In this case too, under some regularity conditions, we obtain a time rule of VaR. Finally, we compare the performance of all symmetric and asymmetric VaR time rules proposed. In particular, we evaluate VaR estimates of all models considering different temporal horizons, conditional and unconditional coverage backtesting methods (see, among others, [1]).

The paper is organized as follows: in Section 2 we propose and formalize time rules for elliptical EWMA models with finite variance. Section 3 introduces time rules for returns in the domain of attraction of stable laws. In Section 4 we backtest the proposed VaR models assessing their ability to capture extreme returns. Finally, we briefly summarize the paper.

2 Elliptical EWMA Models with Finite Variance

In some recent approaches (see, among others, [5],[7]) different exponential weighting moving average (EWMA) models were proposed to compute the value at risk of a given portfolio. The EWMA models assume that the conditional distribution of the continuously compounded return is an elliptical law. In particular, the RiskMetrics model is a EWMA model with conditional Gaussian distributed returns. The assumption of conditional elliptical distributed returns simplifies the VaR calculation for those portfolios with many assets. If we denote with $w = [w_1, w_2, \ldots, w_n]'$ the vector of the positions taken in n assets forming the portfolio, its return at time t is given by

$$z_{P,t} = \sum_{i=1}^{n} w_i z_{i,t}, \tag{1}$$

where $z_{i,t} = \log\left(\frac{P_{t,i}}{P_{t-1,i}}\right)$ is the (continuously compounded) return of $i-th$ asset during the period $[t-1, t]$, and $P_{t,i}$ is the price of $i-th$ asset at time t. Generally we assume that within a short period of time the expected return is null and that the return vector

$$z_t = [z_{1,t}, z_{2,t}, \ldots, z_{n,t}]'$$

follows a conditional joint elliptical distribution. We can distinguish two different types of elliptical EWMA models:

1. models with finite variance,
2. models with infinite variance.

In both cases the conditional characteristic function of the return vector $z_t = [z_{1,t}, ..., z_{n,t}]'$ is given by

$$\Phi_{z_t}(u) = E_t(e^{iu'z_t}) = f\left(u'Q_{t/t-1}u\right),$$

and $Q_{t/t-1} = \left[\sigma^2_{ij,t/t-1}\right]$ is either the variance covariance matrix (if it exists as finite), or another dispersion matrix when the return variance is not finite (see [9]).That is, every return conditioned by the forecasted volatility level is distributed like a standardized elliptical distribution: $z_{i,t}/\sigma_{ii,t/t-1} \sim E(0,1)$ and any linear combination of the returns is elliptically distributed, $z_{P,t} \overset{d}{=} E(0, w'Q_{t/t-1}w)$, where $\sigma^2_{P,t/t-1} = w'Q_{t/t-1}w$ is the dispersion of portfolio $z_{P,t}$ and $Q_{t/t-1} = [\sigma^2_{ij,t/t-1}]$ is the forecasted dispersion matrix. When the elliptical distribution admits a finite variance, then we can estimate the variance and covariance matrix $Q_{t/t-1}$ considering the RiskMetrics' EWMA recursive formulas (see [5]).

The explicit modeling of the volatility series captures the time–varying persistent volatility observed in real financial markets. Under the elliptical assumption for the conditional returns, the Value at Risk of $z_{P,t+1} = w'z_{t+1}$ at $(1-\theta)\%$ (denoted by $VaR_{\theta,t+1}$) is given by simply multiplying the volatility $\sigma_{P,t+1/t}$ forecast in the period $[t, t+1]$, times the tabulated value of the corresponding standard elliptical percentile $k_{1-\theta}$ of $E(0,1)$. Therefore,

$$VaR_{\theta,t+1/t}(z_{P,t+1}) = k_{1-\theta}\sigma_{P,t+1/t}. \tag{2}$$

When both vectors of returns $z_t = [z_{1,t}, \dots, z_{n,t}]'$ and $Z_{t+T} = [Z_{1,t+T}, ..., Z_{n,t+T}]'$ (where $Z_{i,t+T} = \log\left(\frac{P_{t+T,i}}{P_{t,i}}\right) = \sum_{s=1}^{T} z_{i,t+s}$) follow the Gaussian EWMA model, then, under further regularity assumptions, the $(1-\theta)\%$ VaR in the period $[t, t+T]$ is given by

$$VaR_{\theta,t+T/t} = \sqrt{T}VaR_{\theta,t+1/t}. \tag{3}$$

This time rule simplifies the computation of the maximum loss that could occur for a given level of confidence in a temporal horizon greater than the unity. In addition, among the elliptical EWMA models with finite variance, the RiskMetrics model is the only one for which the temporal rule (3) can be used. As a matter of fact, the Gaussian law is the unique elliptical distribution with finite variance such that the sum of elliptical i.i.d. random variables belongs to the same family of elliptical random variables, that is, vectors $z_t = [z_{1,t}, \dots, z_{n,t}]'$ and $Z_{t+T} = [Z_{1,t+T}, ..., Z_{n,t+T}]'$ could follow the same elliptical EWMA model only if $Z_{i,t+T} = \sum_{s=1}^{T} z_{i,t+s}$ and $z_{i,m} = \sigma_{ii,m/m-1}\varepsilon_{i,m}$ ($i = 1, ..., n$ and $m = t+1, ..., t+T$) are conditional Gaussian distributed. Thus, the temporal rule (3) cannot be extended to the EWMA models with conditional elliptical non Gaussian distributed returns and finite variance as well as it cannot be extended to the GARCH-type model (see [2]). However, in [7] it is proved a further time aggregation rule when $z_t = [z_{1,t}, z_{2,t}, \dots, z_{n,t}]'$ and $Z_{t+T} = [Z_{1,t+T}, ..., Z_{n,t+T}]'$ follow different EWMA models with conditional elliptical returns $z_{i,m} \overset{d}{=} E_1(0, \sigma_{ii,m/m-1})$, and $Z_{i,t+T} \overset{d}{=} E_2(0, \sigma_{ii,t+T/t})$. Under these assumptions,

$$VaR_{\theta,t+T/t} = \sqrt{T}MVaR_{\theta,t+1/t}, \tag{4}$$

where $M = \frac{\widetilde{k}_{2,1-\theta}}{\widetilde{k}_{1,1-\theta}}$ and $\widetilde{k}_{1,1-\theta}$, $\widetilde{k}_{2,1-\theta}$ are respectively the corresponding $1-\theta$ elliptical $E_1(0,1)$, $E_2(0,1)$ percentiles. Recall that the sum of elliptical i.i.d. random variables is elliptical distributed but it does not necessarily belong to the same elliptical family (see [3]). Then, the sum of q i.i.d. elliptical distributions $E_1(0,1)$ gives another elliptical distribution with variance equal to q, i.e. $\sum_{s=1}^{q} E_1(0,1) \stackrel{d}{=} \sqrt{q}E_2(0,1)$.

A typical multivariate elliptical distribution with null mean and finite variance is the multivariate Student's t-distribution with $v > 2$ degrees of freedom $\mathbf{MV\text{-}t}(\mathbf{0}, \sum_v)$. These distributions were often used in literature in order to justify the leptokurtic behavior of conditional returns (see, among others, [3]). Therefore, we can assume that the return vector $z_s = [z_{1,s}, ..., z_{n,s}]'$ follows a EWMA model with conditional t-distributed returns and $v > 2$ degrees of freedom. Under this assumption every return $z_{i,s}$ admits the following conditional density function

$$t(x/\sigma_{ii,s/s-1}, v) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sigma_{ii,s/s-1}\left((v-2)\pi\right)^{1/2}\Gamma\left(\frac{v}{2}\right)}\left(1 + \frac{x^2}{\sigma_{ii,s/s-1}^2(v-2)}\right)^{-\frac{v+1}{2}}.$$

We refer to [7] for further details about the properties of the EWMA model with conditional t-distributed returns.

3 Alternative Models with Stable Distributions

In this section we present some alternative models to compute VaR. In particular, we focus our attention on two different stable models for the profit/loss distribution:

1. the stable sub-Gaussian EWMA (SEWMA) model,
2. the stable asymmetric model.

3.1 The SEWMA Model

The SEWMA model assumes that the conditional distribution of the continuously compounded returns vector $z_t = [z_{1,t}, ..., z_{n,t}]'$ is α-stable sub-Gaussian ($\alpha > 1$) with characteristic function

$$\Phi_{z_t}(u) = E_t(e^{iu'z_t}) = \exp\left(-\left(u'Q_{t/t-1}u\right)^{\alpha/2} + iu'\mu_t\right),$$

where $Q_{t/t-1} = \left[\sigma_{ij,t/t-1}^2\right]$ is the conditional dispersion matrix, and $\mu_t = E(z_t)$, even if we assume that within a short period of time the expected return is null. This model is an elliptical EWMA model with infinite variance. In particular, we observe that for any $i, j = 1, ..., n$ the elements of the dispersion matrix can be defined

$$\sigma^2_{ij,t/t-1} = (A(\alpha, p))^{\frac{2}{p}} f(p, \widetilde{z}_{i,t}, \widetilde{z}_{j,t}) \text{ for every } p \in [1, \alpha)$$

where $\widetilde{z}_{i,t} = z_{i,t} - \mu_{i,t}$, $A(\alpha, p) = \frac{\Gamma\left(1-\frac{p}{2}\right)\sqrt{\pi}}{2^p \Gamma\left(1-\frac{p}{\alpha}\right)\Gamma\left(\frac{p+1}{2}\right)}$, $\widetilde{z_j}^{\langle p-1\rangle} = sgn\left(\widetilde{z_j}\right)|\widetilde{z_j}|^{p-1}$ and $f(p, z_{i,t}, z_{j,t}) = E_{t-1}\left(\widetilde{z}_{i,t}\left(\widetilde{z}_{j,t}\right)^{\langle p-1\rangle}\right)\left(E_{t-1}(|\widetilde{z}_{j,t}|^p)\right)^{\frac{2-p}{p}}$. We refer to [7] for further details on the estimation of the elements of the dispersion matrix $Q_{t/t-1}$. Under the assumptions of the SEWMA model, the $(1-\theta)\%$ VaR in the period $[t-1, t]$ is obtained by multiplying the corresponding percentile, $k_{1-\theta,\alpha}$, of the standardized α-stable $S_\alpha(1, 0, 0)$, times the forecast volatility $\sigma_{P,t/t-1} = \sqrt{w' Q_{t/t-1} w}$, that is

$$VaR_{\theta,t/t-1} = k_{1-\theta,\alpha}\sigma_{P,t/t-1}. \tag{5}$$

Moreover, just like in the case of the elliptical EWMA model , we obtain a time rule for the dispersion measure $Q_{t+T/t} = \left[\sigma^2_{ij,t+T/t}\right] = T^{\frac{2}{\alpha}} Q_{t+1/t}$ and under some regularity conditions, it follows the temporal aggregation rule :

$$VaR_{\theta,t+T/t} = T^{\frac{1}{\alpha}} VaR_{\theta,t+1/t}. \tag{6}$$

Observe that among the elliptical distributions, the $\alpha-$stable sub-Gaussian distributions with $\alpha \in (0, 2]$ (where with $\alpha = 2$ we obtain the Gaussian case) are the unique elliptical distributions such that the sum of i.i.d. elliptical random variables belongs to the same family of elliptical random variables. That is, vectors $z_t = [z_{1,t}, ..., z_{n,t}]'$ and $Z_{t+T} = [Z_{1,t+T}, ..., Z_{n,t+T}]'$ could follow the same elliptical EWMA model only if $Z_{i,t+T} = \sum_{s=1}^{T} z_{i,t+s}$ and $z_{i,m} = \sigma_{ii,m/m-1}\varepsilon_{i,m}$ ($i = 1, ..., n$ and $m = t+1, ..., t+T$) are conditional $\alpha-$stable sub-Gaussian distributed with $\alpha \in (0, 2]$.

3.2 An α-Stable Model with Asymmetric Distributed Returns

As an alternative to the previous model, we can consider the asymmetry of stable distributions generalizing the model proposed in [8]. In particular, we can consider the following three-fund separation model of conditional centered returns:

$$\widetilde{z}_{i,t} = z_{i,t} - \mu_{i,t} = b_{i,t} Y_t + \sigma_{ii,t/t-1}\varepsilon_{i,t}; \;\; i = 1, ..., n, \tag{7}$$

where $\mu_{i,t} = E(z_{i,t})$, the values $b_{i,t}$ will be determined with an OLS method, while the random vector $\varepsilon_t = (\varepsilon_{1,t}, \varepsilon_{2,t}, ..., \varepsilon_{n,t})'$ is α-stable sub-Gaussian distributed with zero mean and it is independent of $Y_t \sim S_\alpha\left(\sigma_{Y_t}, \beta_{Y_t}, 0\right)$. In particular, we assume that the centered return vector $\widetilde{z}_{t+1} = [\widetilde{z}_{1,t+1}, ..., \widetilde{z}_{n,t+1}]'$ is conditional jointly α-stable distributed with conditional characteristic function

$$\Phi_{\tilde{z}_{t+1}}(u) = E_t(e^{iu'z_{t+1}}) = \exp\left(-\left(\left(u'Q_{t+1/t}u\right)^{\alpha/2} + \left|u'b_{t+1}\sigma_{Y_{t+1}}\right|^{\alpha}\right) \times \right. \quad (8)$$
$$\left. \times \left(1 - i\frac{\left|u'b_{t+1}\sigma_{Y_{t+1}}\right|^{\alpha} sgn(u'b_{t+1})\beta_{Y_{t+1}}}{\left(u'Q_{t+1/t}u\right)^{\alpha/2} + \left|u'b_{t+1}\sigma_{Y_{t+1}}\right|^{\alpha}} \tan\left(\frac{\pi\alpha}{2}\right)\right)\right),$$

where $\sigma_{Y_{t+1}}$ and $\beta_{Y_{t+1}}$ are respectively the dispersion and the skewness of the factor $Y_{t+1} \stackrel{d}{=} S_\alpha\left(\sigma_{Y_{t+1}}, \beta_{Y_{t+1}}, 0\right)$, that is an α-stable asymmetric (i.e. $\beta_{Y_{t+1}} \neq 0$) centered index return. Moreover, just like for the SEWMA model, we obtain the following time rule when the parameters α, β_{Y_t}, σ_{Y_t}, b_t are constant over the time

$$VaR_{\theta,t+T/t} = T^{\frac{1}{\alpha}} VaR_{\theta,t+1/t}. \quad (9)$$

We again refer to [7] for further details on properties of this stable VaR model.

4 Backtest Models

This section presents an analysis through backtest in order to assess the reliability of the models proposed to compute VaR. We propose three different methods for evaluating the Value at Risk estimates of 25 random portfolios and they are: a basic backtest method to verify if the average coverage of the VaR is equal to the nominal coverage; the conditional and the unconditional coverage tests proposed by [1].

During the period 15/11/93–30/01/98 we have examined daily, 10 days, and 60 days returns of Gaussian distribution, Student's t distributions, Stable sub-Gaussian distribution, stable asymmetric distribution and distributions in the domain of attraction of stable laws. We use some of the most representative index returns of the international market (Brent crude, CAC40, Corn n.2, DAX100, Dow Jones Industrial, FTSE all Share, Goldman Sachs, Nikkei 500, S&P500, Reuters) and their relative exchange rates whose values we converted into USD. Over a period of 769 days, we have computed the interval forecasts using the time aggregation rules and considering $\theta = 95\%$ and $\theta = 99\%$.

4.1 The Basic Backtest Method

In the first backtest analysis proposed we determined how many times during the period taken into account the profits/losses fall outside the confidence interval. In particular, for $\theta = 95\%$ and $\theta = 99\%$, the expected number of observations outside the confidence interval must not exceed respectively 5% and 1%.

The first empirical analysis compares the results obtained from the backtest carried out among the elliptical EWMA models and the stable asymmetric model for $\theta = 95\%$ and $\theta = 99\%$. In view of this comparison, we assume the same parameters of daily models analyzed in [7]. Then, we apply the different time rules (3), (4), (6), and (9) in order to forecast VaR estimates and compare their performance.

Among the alternative models for the VaR calculation, we could observe that the stable and the Student's models and their time rules are more reliable than the RiskMetrics one, in terms of confidence interval $\theta = 99\%$ and 10 days returns particularly. The advantage of using stable models as an alternative to the normal one is reduced when the percentiles are higher than 5% and we consider three months returns.

4.2 Conditional and Unconditional Coverage Tests

Under every distributional hypothesis and for every portfolio $z_{P,t+1}$ we have evaluated daily $VaR_{\theta,t+1/t}(z_{P,t+1})$. Following the interval forecast method proposed in [1], we can propose the following tests:

1. a likelihood ratio test for unconditional coverage LR_{uc} with an asymptotic $\chi^2(1)$ distribution,
2. a likelihood ratio test for conditional coverage LR_{cc} with an asymptotic $\chi^2(2)$ distribution.

These tests partially confirm the previous basic backtest analysis. In particular, we observe that generally the Gaussian time rule does not offer good performance, whilst the time rules (4) and (6) present the best performance. Further tables describing our empirical analysis in details are available by the authors if requested [2].

5 Concluding Remarks

This paper proposes and compares alternative models for the VaR calculation. In the first part we describe several elliptical and stable Paretian exponential weighted moving average models. In the second part, we compare the efficiency of different time aggregation rules to forecast VaR . The empirical comparison confirms that when the percentiles are below 5%, the hypothesis of normality of the conditional return distribution determines intervals of confidence whose forecast ability is low. In particular, the stable Paretian and the Student's t time aggregation rules have shown very good performance to predict future losses when we assume a temporal horizon of 10 days. Whereas, when we consider 60 days returns all the models do not present very good performances.

[2] For tables reporting conditional and unconditional tests on the above time aggregation rules, please refer to the following e-mail address: sol@unibg.it.

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