Euler Scheme for One-Dimensional SDEs with Time Dependent Reflecting Barriers

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Abstract. We give the rate of mean-square convergence for the Euler scheme for one-dimensional stochastic differential equations with time dependent reflecting barriers. Applications to stock prices models with natural boundaries of Bollinger bands type are considered.

1 Introduction

We consider a market in which fluctuation of stocks prices, and more generally of some economic goods, is given by a stochastic process $S = \{S_t; t \in \mathbb{R}^+\}$ living within the upper- and lower barrier processes $F = \{F_t; t \in \mathbb{R}^+\}$ and $G = \{G_t; t \in \mathbb{R}^+\}$, i.e. $G_t \le S_t \le F_t$, $t \in \mathbb{R}^+$. Such models appear for instance if some institutions may want to prevent prices from leaving interval $[G_t, F_t]$ and prices may have some natural boundaries. Recently, in [8] the simplest case of constant boundaries of the form $[l, d]$ was considered. In this case an option pricing formula was obtained under the assumption that S is a solution of an appropriate stochastic differential equation (SDE). Models of prices fluctuation considered in practice by quantitive analysts are much more general: barriers are stochastic processes depending on the process S. Typical examples of such natural boundaries are the so-called Bollinger bands F, G defined by

$$F_t = A_t + \alpha\Big(\frac{1}{M}\sum_{j=1}^{M}(S_{t-\varepsilon j} - A_t)^2\Big)^{1/2}, \quad G_t = A_t - \alpha\Big(\frac{1}{M}\sum_{j=1}^{M}(S_{t-\varepsilon j} - A_t)^2\Big)^{1/2}$$

and trading bands (envelopes) defined by

$$F_t = (1+\alpha)A_t, \qquad G_t = (1-\alpha)A_t,$$

where A is a moving average process $A_t = \frac{1}{M}\sum_{j=1}^{M} S_{t-\varepsilon j}$, $t \in \mathbb{R}^+$, and $\varepsilon, \alpha > 0$, $M \in \mathbb{N}$ are some parameters.

In [11] existence and uniqueness of solutions of SDE with time dependent reflecting barriers driven by a general semimartingale is proved. In the present

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paper we restrict ourselves to a one-dimensional SDE with reflecting barriers of the form

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + \int_0^t b(X_s)\, ds + K_t, \quad t \in \mathbb{R}^+, \tag{1}$$

where $X_0 \in \mathbb{R}$, $W = \{W_t; t \in \mathbb{R}^+\}$ is a standard Wiener process, $\sigma, b : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions and barrier processes F, G are general Lipschitz operators with delayed argument depending possibly on X (for a precise definition see Section 2). Our aim is to define the Euler scheme $\{\bar{X}^n\}$ for the SDE (1) and to give its rate of mean-square convergence.

The main result of the paper says that under mild assumptions on reflecting barrier processes $G(^{\varepsilon}X), F(^{\varepsilon}X)$ with delayed argument for every $q \in \mathbb{R}^+$ there exists $C > 0$ such that

$$E \sup_{t \le q} |\bar{X}^n_t - X_t|^2 \le C\left(\frac{\ln n}{n} + E\omega^2_{1/n}(G(^{\varepsilon}X), q) + E\omega^2_{1/n}(F(^{\varepsilon}X), q)\right), \tag{2}$$

where $\omega_\delta(x, q) = \sup\{|x_t - x_s|, 0 \le s < t \le q, t - s < \delta\}$, for all $\delta > 0$, $q \in \mathbb{R}^+$ and $x \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ ($\mathbb{D}(\mathbb{R}^+, \mathbb{R})$ is the space of all mappings $x : \mathbb{R}^+ \to \mathbb{R}$ which are right continuous and admit left-hand limits).

From (2) we deduce that in both cases of Bollinger and trading bands for every $q \in \mathbb{R}^+$, $\delta > 0$ there exists $C > 0$ such that

$$E \sup_{t \le q} |\bar{X}^n_t - X_t|^2 \le C\left(\frac{1}{n^{1-\delta}}\right).$$

Moreover, in both cases,

$$E \sup_{t \le q} |\bar{X}^n_t - X_t|^2 \le C\left(\frac{\ln n}{n}\right)$$

if σ, b are bounded.

Note that if $G = -\infty$, $F = +\infty$ then $\{\bar{X}^n\}$ is the classical Euler scheme introduced in [6]. In the case $G = 0$, $F = +\infty$ and $G = l$, $F = d$ the rate of mean-square convergence was examined earlier by many authors (see, e.g., [2,4, 5,7,9,10]).

In the paper no attemps has been made to obtain option pricing formulas for markets with dynamics of prices given by (1). This question deserves an independent study.

2 SDEs with Time Dependent Reflecting Barriers

We begin with a definition of the Skorokhod problem with time dependent reflecting barriers.

Definition 1. Let $y, f, g \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ with $g \le f$ and $g_0 \le y_0 \le f_0$. We say that a pair $(x, k) \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^2)$ is a solution of the Skorokhod problem associated with y and barriers f, g (and write $(x, k) = SP(y, f, g)$) if

(i) $x_t = y_t + k_t$, $t \in \mathbb{R}^+$,
(ii) $g_t \le x_t \le f_t$, $t \in \mathbb{R}^+$.
(iii) $k_t = k_t^{(-)} - k_t^{(+)}$, $t \in \mathbb{R}^+$, where $k^{(-)}$, $k^{(+)}$ are nondecreasing, right continuous functions with $k_0 = k_0^{(-)} = k_0^{(+)} = 0$ such that $k^{(-)}$ increases only on $\{t; x_t = g_t\}$ and $k^{(+)}$ increases only on $\{t; x_t = f_t\}$.

Theorem 1. *([11]) Assume that $f, g \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ satisfy the condition*

$$\inf_{t \le q}(f_t - g_t) > 0, \quad q \in \mathbb{R}^+.$$

Then for every $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$ with $g_0 \le y_0 \le f_0$, there exists a unique solution (x, k) of the Skorokhod problem associated with y and barriers f, g.

□

Note that in the case of continuous function y similar definitions of the Skorokhod problem were earlier given in [3] and [1]. These papers contain also results on existence and uniqueness of solutions of the Skorokhod problem in the case of continuous y and continuous barriers f, g (see e.g. [3, Lemma 4.1]).

The following theorem, where Lipschitz continuity of solutions of the Skorokhod problem is stated will prove to be very useful in Section 3.

Theorem 2. *([11]) Assume that $y^i, f^i, g^i \in \mathbb{D}(\mathbb{R}^+, \mathbb{R})$, $g_0^i \le y_0^i \le f_0^i$ and*

$$\inf_{t \le q}(f_t^i - g_t^i) > 0, \quad q \in \mathbb{R}^+$$

for $i = 1, 2$. Let $(x^i, k^i) = SP(y^i, f^i, g^i)$, $i = 1, 2$. Then for every $q \in \mathbb{R}^+$

$$\sup_{t \le q}|x_t^1 - x_t^2| \le 3 \sup_{t \le q}|y_t^1 - y_t^2| + \sup_{t \le q}|f_t^1 - f_t^2| + \sup_{t \le q}|g_t^1 - g_t^2|$$

and

$$\sup_{t \le q}|k_t^1 - k_t^2| \le 2 \sup_{t \le q}|y_t^1 - y_t^2| + \sup_{t \le q}|f_t^1 - f_t^2| + \sup_{t \le q}|g_t^1 - g_t^2|.$$

□

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space.

Definition 2. Let $\mathcal{D}$ denote the space of all $(\mathcal{F}_t)$ adapted processes with trajectories in $\mathbb{D}(\mathbb{R}^+, \mathbb{R})$. We say that an operator $H : \mathcal{D} \to \mathcal{D}$ is Lipschitz if

(i) $H(X) \in \mathcal{D}$ for any $X \in \mathcal{D}$,
(ii) for any $X, Y \in \mathcal{D}$ and any stopping time τ,
$X^{\tau-} = Y^{\tau-}$ implies $H(X)^{\tau-} = H(Y)^{\tau-}$,
(iii) there exists $L > 0$ such that for any $X, Y \in \mathcal{D}$

$$|H(X)_t - H(Y)_t| \le L \sup_{s \le t}|X_t - Y_t|, \quad t \in \mathbb{R}^+.$$

Given $X \in \mathcal{D}$ and $\varepsilon > 0$ set ${}^{\varepsilon}X = \{X_{t-\varepsilon}; t \in \mathbb{R}^+\}$ (with the convention that $X_t = X_0$ for $t \in [-\varepsilon, 0)$). In what follows barriers of the form $F({}^{\varepsilon}X), G({}^{\varepsilon}X)$, where F, G are Lipschitz operators we will call barrier operators with delayed arguments.

Fix $\varepsilon > 0$. Let W be an $(\mathcal{F}_t)$ adapted Wiener process and let F, G be two Lipschitz operators such that for any $X \in \mathcal{D}$

$$\inf_{t \le q}(F({}^{\varepsilon}X)_t - G({}^{\varepsilon}X)_t) > 0, \quad q \in \mathbb{R}^+. \tag{3}$$

We will say that a pair (X, K) of $(\mathcal{F}_t)$ adapted processes is a strong solution of the SDE (1) with barrier operators $F({}^{\varepsilon}X), G({}^{\varepsilon}X)$ with delayed argument if $(X, K) = SP(Y, F({}^{\varepsilon}X), G({}^{\varepsilon}X))$, where

$$Y_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + \int_0^t b(X_s)\, ds, \quad t \in \mathbb{R}^+. \tag{4}$$

Theorem 3. *([11]) Let $\varepsilon > 0$. Assume that σ, b are Lipschitz continuous functions and F, G are Lipschitz operators satisfying (3) with $G(X_0)_0 \le X_0 \le F(X_0)_0$. Then there exists a unique strong solution (X, K) of the SDE (1).*

□

Theorem 4. *Let $\varepsilon > 0$ and let σ, b, F, G, X_0 satisfy the assumptions of Theorem 3. If*

$$E \sup_{t \le q}(|F(X_0)_t| + |G(X_0)_t|)^{2p} < +\infty, \quad q \in \mathbb{R}^+, p \in \mathbb{N}$$

then

(i) $E \sup_{t \le q} |X_t - X_0|^{2p} < +\infty$, $p \in \mathbb{N}$, $q \in \mathbb{R}^+$,

(ii) *if moreover, $F({}^{\varepsilon}X)_s = F({}^{\varepsilon}X^s)_t$ and $G({}^{\varepsilon}X)_s = G({}^{\varepsilon}X^s)_t$ for any $0 \le s \le t$, then for every $p \in \mathbb{N}$, $q \in \mathbb{R}^+$ there exists $C > 0$ such that*

$$E|X_t - X_s|^{2p} \le C(t - s)^p$$

for $s \le t \le q$.

□

Corollary 1. *Under assumptions of Theorems 3 and 4 for any $q \in \mathbb{R}^+$, $\delta > 0$ there exists $C > 0$ such that*

$$E(\omega_{1/n}(F({}^{\varepsilon}X), q))^2 + E(\omega_{1/n}(G({}^{\varepsilon}X), q))^2 \le C(\frac{1}{n^{1-\delta}}). \tag{5}$$

If moreover, σ, b are bounded then

$$E(\omega_{1/n}(F({}^{\varepsilon}X), q))^2 + E(\omega_{1/n}(G({}^{\varepsilon}X), q))^2 \le C(\frac{\ln n}{n}). \tag{6}$$

□

3 Euler Scheme for SDEs with Time Dependent Reflecting Barrier Operators

Let $(\mathcal{F}_t^n)$ denote the discretization of $(\mathcal{F}_t)$, i.e. $\mathcal{F}_t^n = \mathcal{F}_{k/n}$ for $t \in [k/n, (k+1)/n)$ and let F^n, G^n denote the discretizations of operators F, G, i.e. any process X, $F^n(X)_t = F(X)_{k/n}$, $G^n(X)_t = G(X)_{k/n}$ for any $t \in [k/n, (k+1)/n)$.

The Euler scheme for the SDE (1) is given by the following recurrent formula

$$\begin{cases} \bar{X}_0^n &= X_0, \\ \bar{X}_{(k+1)/n}^n &= \max\big[\min\big(\bar{X}_{k/n}^n + \sigma(\bar{X}_{k/n}^n)(W_{(k+1)/n} - W_{k/n}) \\ & \qquad + b(\bar{X}_{k/n}^n)\frac{1}{n}, F({}^{\varepsilon}\bar{X}^{n,k/n})_{(k+1)/n}\big), G({}^{\varepsilon}\bar{X}^{n,k/n})_{(k+1)/n}\big], \\ \bar{X}_t^n &= \bar{X}_{k/n}^n, \quad t \in [k/n, (k+1)/n), \end{cases}$$

where ${}^{\varepsilon}\bar{X}_t^{n,k/n} = {}^{\varepsilon}\bar{X}_t^n$ if $t \le k/n$ and ${}^{\varepsilon}\bar{X}_t^{n,k/n} = {}^{\varepsilon}\bar{X}_{k/n}^n$ if $t > k/n$. Set

$$\bar{Y}_t^n = X_0 + \int_0^t \sigma(\bar{X}_{s-}^n)\, dW_s^n + \int_0^t b(\bar{X}_{s-}^n)\, d\rho_s^n, \quad t \in \mathbb{R}^+,$$

where $\rho_t^n = k/n$ for $t \in [k/n, (k+1)/n)$ and W^n is a disretization of Wiener process W, that is $W_t^n = W_{\rho_t^n}$, $t \in \mathbb{R}^+$. Note that $(\bar{X}^n, \bar{K}^n = \bar{X}^n - \bar{Y}^n)$ is a pair of $(\mathcal{F}_t^n)$ adapted processes such that $(\bar{X}^n, \bar{K}^n) = SP(\bar{Y}^n, F^n({}^{\varepsilon}\bar{X}^n), G^n({}^{\varepsilon}\bar{X}^n))$.

Theorem 5. *Assume that σ, b are Lipschitz continuous functions and F, G are Lipschitz operators satysfying (3) such that $G(X_0)_0 \le X_0 \le F(X_0)_0$*

$$E\sup_{t\le q}(|F(X_0)_t| + |G(X_0)_t|)^{2+\delta} < +\infty, \quad q \in \mathbb{R}^+,$$

for some $\delta > 0$. If (X, K) is a solution of the SDE (1) then for every $q \in \mathbb{R}^+$ there exists $C > 0$ such that (2) holds true.

Proof. We begin by proving that

$$\sup_n E\sup_{t\le q}|\bar{X}_t^n - X_0|^{2+\delta} < \infty. \tag{7}$$

Without loss of generality we may and will assume that

$$\sup_n E\sup_{t\le q-\varepsilon}|\bar{X}_t^n - X_0|^{2+\delta} < \infty.$$

Since $(X_0, 0) = SP(X_0, F^n(X_0)_0, G^n(X_0)_0)$, it follows from Theorem 2 that

$$\sup_{t\le q}|\bar{X}_t^n - X_0| \le 3\sup_{t\le q}\left|\int_0^t \sigma(\bar{X}_{s-}^n)\, dW_s^n + \int_0^t b(\bar{X}_{s-}^n) d\rho_s^n\right| \\ + \sup_{t\le q}(|F^n({}^{\varepsilon}\bar{X}^n)_t - F^n(X_0)_t| + |G^n({}^{\varepsilon}\bar{X}^n)_t - G^n(X_0)_t|) + H_q,$$

where $H_q = \sup_{t\le q}(|G^n(X_0)_t - G^n(X_0)_0| + |F^n(X_0)_t - F^n(X_0)_0|)$. Clearly

$$E\sup_{t\le q}(|F^n({}^{\varepsilon}\bar X^n)_t - F^n(X_0)_t| + |G^n({}^{\varepsilon}\bar X^n)_t - G^n(X_0)_t|)^{2+\delta}$$
$$\le CE\sup_{t\le q-\varepsilon}|\bar X^n_t - X_t|^{2+\delta} < +\infty,$$

and $EH_q^{2+\delta} < \infty$. Therefore, by the Burkholder–Davis–Gundy and Hölder inequalities we have

$$E\sup_{t\le q}|\bar X^n_t - X_0|^{2+\delta} \le C\{\int_0^q E\sup_{u\le s}|\bar X^n_{u-} - X_0|^{2+\delta}\,ds + 1\},$$

and hence (7). Since σ, b are Lipschitz continuous (7) yields

$$\sup_n E\sup_{t\le q}|\sigma(\bar X^n)_t|^{2+\delta} < \infty, \quad \sup_n E\sup_{t\le T}|b(\bar X^n)_t|^{2+\delta} < \infty. \tag{8}$$

Set

$$\widehat Y^n_t = X_0 + \int_0^t \sigma(\bar X^n_{s-})\,dW_s + \int_0^t b(\bar X^n_{s-})\,ds, \quad t\in\mathbb{R}^+,$$

and $(\hat X^n, \hat K^n) = SP(\hat Y^n, F^n({}^{\varepsilon}\bar X^n), G^n({}^{\varepsilon}\bar X^n))$. Then

$$\hat Y^n_t - \bar Y^n_t = \sigma(\bar X^n_{k/n})(W_t - W_{k/n}) + b(\bar X^n_{k/n})(t-k/n) \tag{9}$$

for $t\in[k/n,(k+1)/n)$. Therefore, by Theorem 2,(8),(9) and [9, Lemma A4]

$$\begin{aligned}
E\sup_{t\le q}|\bar X^n_t - \hat X^n_t|^2 &\le 9E\sup_{t\le q}|\bar Y^n_t - \hat Y^n_t|^2\\
&\le 9\big\{(E\sup_{t\le q}|\sigma(\bar X^n_{t-})|^{2+\delta})^{2/(2+\delta)}(E(\omega_{1/n}(W,q))^{(2\delta+4)/\delta})^{\delta/(\delta+2)}\\
&\qquad +(\frac{1}{n})^2 E\sup_{t\le q}|b(\bar X^n_{t-})|^2\big\}\\
&\le C\big(\frac{\ln n}{n} + (\frac{1}{n})^2\big) \le C\big(\frac{\ln n}{n}\big).
\end{aligned} \tag{10}$$

Clearly

$$\begin{aligned}
E\sup_{t\le q}|\hat X^n_t - X_t|^2 &\le C\big\{E\sup_{t\le q}|\hat Y^n_t - Y_t|^2\\
&\qquad +E(\sup_{t\le q}|F^n({}^{\varepsilon}\bar X^n)_t - F^n({}^{\varepsilon}X)_t| + |G^n({}^{\varepsilon}\bar X^n)_t - G^n({}^{\varepsilon}X)_t|)^2\\
&\qquad +E(\sup_{t\le q}|F^n({}^{\varepsilon}X)_t - F({}^{\varepsilon}X)_t| + |G^n({}^{\varepsilon}X)_t - G({}^{\varepsilon}X)_t|)^2\big\}\\
&\le C\big\{\int_0^q E\sup_{u\le s}|\bar X^n_u - X_u|^2\,ds + E\sup_{t\le q-\varepsilon}|\bar X^n_t - X_t|^2 + \delta_n\big\},
\end{aligned} \tag{11}$$

where $\delta_n = E\omega^2_{1/n}(F({}^{\varepsilon}X),q) + E\omega^2_{1/n}(G({}^{\varepsilon}X),q)$. Since without loss of generality we may and will assume

$$E\sup_{t\le q-\varepsilon}|\bar X^n_t - X_t|^2 \le C\left(\frac{\ln n}{n} + E\omega^2_{1/n}(G({}^{\varepsilon}X),q-\varepsilon) + E\omega^2_{1/n}(F({}^{\varepsilon}X),q-\varepsilon)\right),$$

from (10) and (11) we obtain

$$E\sup_{t\le q}|\bar{X}^n_t - X_t|^2 \le 2E\sup_{t\le q}|\bar{X}^n_t - \hat{X}^n_t|^2 + 2E\sup_{t\le q}|\hat{X}^n_t - X_t|^2$$
$$\le C\left\{\frac{\ln n}{n} + \int_0^t E\sup_{u\le s}|\bar{X}^n_u - X_u|^2 du + \delta_n\right\}.$$

To complete the proof it suffices now to use Gronwall's lemma. □

Corollary 2. *Assume that σ, b are Lipschitz continuous functions and F, G are $(\mathcal{F}_t)$ adapted processes such that $G_0 \le X_0 \le F_0$, $\inf_{t\le q}(F_t - G_t) > 0$, $q \in \mathbb{R}^+$ and*

$$E\sup_{t\le q}(|F_t| + |G_t|)^{2+\delta} < +\infty, \quad q \in \mathbb{R}^+,$$

for some $\delta > 0$. If (X, K) is a solution of the SDE (1) then for every $q \in \mathbb{R}^+$ there exists $C > 0$ such that

$$E\sup_{t\le q}|\bar{X}^n_t - X_t|^2 \le C\left(\frac{\ln n}{n} + E\omega^2_{1/n}(G, q) + E\omega^2_{1/n}(F, q)\right),$$

□

Corollary 3. *If σ, b are Lipschitz continuous and F, G are Bollinger or trading bands then for every $q \in \mathbb{R}^+$, $\delta > 0$ there exists $C > 0$ such that*

$$E\sup_{t\le q}|\bar{X}^n_t - X_t|^2 \le C\left(\frac{1}{n^{1-\delta}}\right).$$

Proof. Due to Theorem 5 and (5) it is sufficient to prove that the respective barrier operators F, G are Lipschitz.

First we will consider the case of Bolinger bands. We restrict our attention to the operator F. Observe that it has the following form:

$$F(X)_t = B(X)_t + \alpha(\frac{1}{M}\sum_{j=0}^{M-1}(X_{t-\varepsilon j} - B(X)_t)^2)^{1/2},$$

where $B(X)_t = \frac{1}{M}\sum_{j=0}^{M-1} X_{t-\varepsilon j}$, $t \in \mathbb{R}^+$, $X \in \mathcal{D}$. From the above formula it follows immediately that F posseses the properties (i) and (ii) of Definition 2. Moreover, for any $X, Y \in \mathcal{D}$ and $t \le q \in \mathbb{R}^+$,

$$|F(X)_t - F(Y)_t| \le |B(X)_t - B(Y)_t|$$
$$+\alpha|(\frac{1}{M}\sum_{j=0}^{M-1}(X_{t-\varepsilon j} - B(X)_t)^2)^{1/2} - (\frac{1}{M}\sum_{j=0}^{M-1}(Y_{t-\varepsilon j} - B(Y)_t)^2)^{1/2}|$$
$$\le \sup_{t\le q}|X_t - Y_t| + \alpha(\frac{1}{M}\sum_{j=0}^{M-1}(X_{t-\varepsilon j} - Y_{t-\varepsilon j} + B(X)_t - B(Y)_t)^2)^{1/2}$$
$$\le \sup_{t\le q}|X_t - Y_t| + \alpha\max_{0\le j\le M-1}|X_{t-\varepsilon j} - Y_{t-\varepsilon j} + B(X)_t - B(Y)_t|$$
$$\le (1 + 2\alpha)\sup_{t\le q}|X_t - Y_t|,$$

which shows that F is Lipschitz.

In the case of trading bands F has the form:

$$F(X)_t = (1+\alpha)B(X)_t, \quad t \in \mathbb{R}^+.$$

Hence F posseses properties (i) and (ii) of Definition 2 and for $X, Y \in \mathcal{D}$, $q \in \mathbb{R}^+$ we have

$$\sup_{t \le q} |F(X)_t - F(Y)_t| \le (1+\alpha) \sup_{t \le q} |X_t - Y_t|,$$

so F is Lipschitz. □

Corollary 4. *If σ, b are Lipschitz continuous and bounded functions and F, G are Bollinger or trading bands then for every $q \in \mathbb{R}^+$ there exists $C > 0$ such that*

$$E \sup_{t \le q} |\bar{X}_t^n - X_t|^2 \le C\left(\frac{\ln n}{n}\right).$$

Proof. It follows from (6), Theorem 5 and the fact that in both cases F, G are Lipschitz. □

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