# A common algebraic description for probabilistic and quantum computations* 

Martin Beaudry ${ }^{\dagger}$<br>Université de Sherbrooke

José M. Fernandez<br>Université de Montréal

Markus Holzer ${ }^{\ddagger}$<br>Technische Universität München


#### Abstract

We study the computational complexity of the problem SFT (Sum-free Formula partial Trace) : given a tensor formula $F$ over a subsemiring of the complex field $(\mathbb{C},+, \cdot)$ plus a positive integer $k$, under the restrictions that all inputs are column vectors of $\mathrm{L}_{2}$-norm 1 and norm-preserving square matrices, and that the output matrix is a column vector, decide whether the $k^{\text {th }}$ partial trace of $\mathrm{FF}^{\dagger}$ is superior to $1 / 2$. The $k^{\text {th }}$ partial trace of a matrix is the sum of its lowermost $k$ diagonal elements. We also consider the promise version of this problem, where the $1 / 2$ threshold is an isolated cutpoint. We show how to encode a quantum or reversible gate array into a tensor formula which satisfies the above conditions, and vice-versa; we use this to show that the promise version of SFT is complete for the class BPP for formulas over the semiring $\left(\mathbb{Q}^{+},+, \cdot\right)$ of the positive rational numbers, for BQP in the case of formulas defined over the field $(\mathbb{Q},+, \cdot)$, and for P in the case of formulas defined over the Boolean semiring, all under logspace-uniform reducibility. This suggests that the difference between probabilistic and quantum polynomial-time computers may ultimately lie in the possibility, in the latter case, of having destructive interference between computations occuring in parallel.


## 1 Introduction

The "algebraic approach" in the theory of computational complexity consists in characterizing complexity classes within unified frameworks built around a computational model or problem involving an algebraic structure (usually finite or finitely generated) as the main parameter. In this way, various complexity classes are seen to share the same definition, up to the choice of the underlying algebra. Successful examples of this approach include the description of $N C^{1}$ and its subclasses $A C^{0}$ and $A C C^{0}$ in terms of polynomial-size programs over finite monoids [4] , and analogous results for PSPACE, the polynomial hierarchy and the polytime mod-counting classes, through the use of polytime leaf languages [14]. A more recent example is the complexity of problems whose input is a tensor formula, i.e. a fully parenthetized expression where the inputs are matrices (given in full) over some finitely generated algebra

[^0]and the allowed operations are matrix addition, multiplication, and tensor product (also known as outer, or direct, or Kronecker product). Depending on the semiring over which the formula is defined, the problem of deciding whether the output matrix contains at least one nonzero entry is complete for NP (Boolean semiring) and $\mathrm{MOD}_{\mathrm{q}}-\mathrm{P}$ (modulo semiring $\mathbb{Z}_{q}$ ) [7]. Other common-sense computational problems on tensor formulas were analyzed in [7] , 5]. Tensor formulas are a compact way of specifying very large matrices. As such, they immediately find a potential application in the description and the behavior of circuits, be they classical Boolean, arithmetic (tensor formulas over the appropriate semiring) or quantum (formulas over the complex field, or an adequately chosen subsemiring thereof). In this paper, we formalize and confirm this intuition, in that we define a meaningful computational problem over tensor formulas which enables us to capture the significant complexity classes P, BPP, and BQP. Looking at variants of the problem enables us to capture further complexity classes; a table in the last section summarizes our results.
Apart from offering a first application of the algebraic approach to quantum computing, our paper reasserts the point made by Fortnow [12], that for the classes BPP and BQP, the jump from classical to quantum polynomial-time computation consists in allowing negative matrix entries for the evolution operators, which means the possibility of having destructive interference between different computations done in parallel.

## 2 Background on circuits and complexity

We use standard notions and notations from computational complexity, see for example [2, 20]. In particular we assume that the reader is familiar with the deterministic and probabilistic Turing machine models, with the usual notion of a Boolean circuit, and with logspace many-one reducibility: a set K is logspace time many-one reducible to a set $L$ if there is a logspace computable mapping $f$ such that for all $x, x \in K$ iff $f(x) \in L$.
To handle the three types of computation discussed in this paper (deterministic, probabilistic and quantum), we use gate arrays as a common setting. From now on, we reserve the word circuit to the traditional idea of an acyclic network with a unique output bit, and we use gate array to describe those computational networks which satisfy the following definition.

Definition 2.1. Let $\mathrm{n}, \mathrm{d} \geq 1$. A width n , d -leveled gate array is $a \mathrm{n} \times \mathrm{d}$ array where each line is called $a$ wire and each column a level. The size of a gate array is the number nd. A gate is a set of array entries from the same level (corresponding to the wires involved in the gate's operation) together with a square matrix which describes its action. Gates on a given level act on disjoint sets of entries from this level. Let the levels be numbered 1 to d from left to right. Each wire carries a bit from a level to the next in the left-to-right direction; the value entering column 1 from the left is called an input the value exiting level d to the right is an output.

A gate of $k$ binary inputs operates on the set of $k$-bit vectors by mapping each of the $2^{k}$ possible combinations of input values to a combination of output values. The extra constraint, that all gates act on neighboring wires, can be enforced on an arbitrary array at the cost of inserting a quadratic number of extra levels with "swap" gates, which interchange the values carried by two adjacent wires.
Gate arrays are used in particular to describe reversible classical computations. A computation is reversible iff knowledge of its output is sufficient to be able to deterministically reconstruct the input. It has been shown that for any polynomial-time deterministic algorithm there exists an equivalent polynomial-time reversible algorithm; in other words, from every polynomial-size Boolean circuit an equivalent reversible gate array [13] can be constructed, by

- modifying the circuit so that the numbers of input and output bits are equal;
- replacing the usual one-output gates with reversible gates;
- making sure that an especially identified "decision" bit takes value 1 at the output level iff the original circuit's output is 1 .

From the description of the original circuit, its equivalent reversible gate array can be constructed in deterministic logspace; circuit size and depth are increased only by a polynomial factor; usually, a polynomial number of extra input bits initialized at 0 , called ancillary bits, also has to be added in the process. It has been shown that this array can be built solely with the one- and two-bit reversible operations, plus either one of the "Toffoli" ( $\Theta$ ) or "Fredkin" $(\Phi)$ gates, where

$$
\Theta=\left[\begin{array}{llllllll}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right] \quad \text { and } \quad \Phi=\left[\begin{array}{llllllll}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] ;
$$

here the top left position corresponds to bit values 000 and the bottom right to 111 .
Standard techniques can therefore be used in sequence to transform the description of a polytime deterministic Turing machine and its input $x$ into an instance of the Circuit Value Problem with constant inputs (where $x$ is hardwired) [16], then to turn this circuit into a reversible gate array, in order to give the following definition for the class P . (Alternatively, one can start from the definition of P as the class of those languages decided by logspace-uniform families of polynomial-size Boolean circuits.)

Definition 2.2. P is the class of those languages $\mathrm{L} \subset \Sigma^{*}$ for which there exist a logspace-computable function which, given an input $\chi \in \Sigma^{*}$, computes the encoding of a reversible gate array $\mathrm{C}(\mathrm{x})$ with constant inputs, whose decision bit takes value 1 at the output level iff $x \in L$.

An encoding for $C(x)$ is suitable for this definition if it consists of a reasonable description of the array's inputs, wiring and gates; the latter can wlog be restricted to have constant fan-in/fan-out, so that the action of each gate can be specified with a constant-size Boolean matrix.
Complexity classes for polynomial-time probabilistic computation are usually defined in terms of a polytime Turing machine which picks a random bit at every step of its computation, and otherwise acts deterministically (see e.g. [侮). An equivalent circuit is built from this Turing machine and its input, in which an appropriate number of random bits are fed in alongside the (constant) input bits; whether the input belongs to $L$ is verified by counting those combinations of random bits for which the output bit takes value 1 . All random bit combinations have equal length and are equally likely.

Definition 2.3. PP is the class of those languages $\mathrm{L} \subset \Sigma^{*}$ for which there exist a logspace-computable function which, given an input $\chi \in \Sigma^{*}$, yields the encoding of a reversible gate array $\mathrm{C}(\mathrm{x})$ with a combination of constant and random inputs, such that $x \in L$ iff $\mathrm{f}_{\mathrm{C}}(\mathrm{x})>\frac{1}{2}$ and $\mathrm{x} \notin \mathrm{L}$ iff $\mathrm{f}_{\mathrm{C}}(\mathrm{x})<\frac{1}{2}$, where $\mathrm{f}_{\mathrm{C}}(\mathrm{x})$ denotes the probability that
$\mathrm{C}(\mathrm{x})$ 's decision bit takes value 1 at the output level.
BPP is defined with the extra condition that there exists a parameter $\varepsilon, 0<\varepsilon<\frac{1}{2}$, such that $x \in \operatorname{Liff} \mathrm{f}_{\mathrm{C}}(\mathrm{x})>\frac{1}{2}+\varepsilon$ value 1 at the output level.
The class NP can be similarly defined, with the condition that $x \in \operatorname{L}$ iff $\mathrm{f}_{\mathrm{C}}(\mathrm{x})>0$.
The definition of BPP includes the implicit constraint, that the proportion of accepting computations can never fall inside the interval $\left[\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon\right]$; in other words, $\frac{1}{2}$ is an isolated cutpoint. Note that both PP and BPP can be redefined with a cutpoint other than $\frac{1}{2}$.

Polynomial-time quantum computation was defined originally in terms of quantum Turing machines [8]: the data handled by this machine (qubits) are formally represented as a vector whose complex components give the distribution of amplitudes for the probability that the qubits be in a certain combination of values; each transition of the machine acts as a unitary transformation on this vector.
It was later shown [21] that a quantum Turing machine and its input can be encoded in deterministic polynomial time into an array of quantum gates, if one is allowed a small probability of error. Each wire in a quantum gate array represents a path of a single qubit (in time or space, forward from left to right), and is described by a state in a two dimensional Hilbert space with basis $|0\rangle$ and $|1\rangle$. Just as classical bit strings can represent the discrete states of arbitrary finite dimensionality, so a string of $n$ qubits can be used to represent quantum states in any Hilbert space of dimensionality up to $2^{n}$. The action of a gate of $k$ inputs is a unitary operation of the group $U\left(2^{k}\right)$, i.e., a generalized rotation in a Hilbert space of dimension $2^{k}$. It has been shown that a small set of one- and two-qubit gates suffices to build quantum arrays, in that any $n$-qubit gate can be simulated by a subarray consisting of two-qubit gates, and the number thereof is at most an exponential in $n$ (see for example [3, 9, 18, 17]). As two-qubit gates it suffices to take the controlled-not N . Because of its usefulness we also mention the two-qubit "swap" gate T .

$$
\mathrm{N}=\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array}\right], \quad \mathrm{T}=\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]
$$

The vector of qubits received as input by a quantum gate array can be regarded as a linear combination of pure states. There is a measurement done on the array's output, which consists in projecting the output vector onto a subspace, usually defined by setting a chosen subset of the qubits to $|1\rangle$ ("accepting subspace"). If the qubits are numbered 1 to $n$, then a $k$-qubit accepting subset can be chosen to be qubits 1 to $k$, at the cost of inserting a quadratic number of extra swap gates. For the sake of simplicity, we can assume that the final output state will be such that all qubits other than the decision qubit have value $|0\rangle$. This is without loss of generality, as it will be possible to "uncompute" the circuit while keeping the value of the decision bit. Thus, the accepting subspace has dimension 1 , and contains only one base vector, and similarly for the rejecting subspace.

Definition 2.4. BQP is the class of those languages $\mathrm{L} \subset \Sigma^{*}$ for which there exist a logspace-computable function which, given an input $x \in \Sigma^{*}$, yields the encoding of a quantum gate array $\mathrm{C}(\mathrm{x})$ with constant inputs, and a parameter $\varepsilon, 0<\varepsilon<\frac{1}{2}$, such that $x \in L$ iff $\mathrm{f}_{\mathrm{C}}(\mathrm{x})>\frac{1}{2}+\varepsilon$ and $x \notin L$ iff $\mathrm{f}_{\mathrm{C}}(\mathrm{x})<\frac{1}{2}-\varepsilon$, where $\mathrm{f}_{\mathrm{C}}(\mathrm{x})$ denotes the probability that the qubits of $\mathrm{C}(\mathrm{x})$ be projected onto the accepting subspace at the output level.

The remark on parameter $\varepsilon$ made after the definition of BPP also holds here. The definition of BQP still holds if we restrict the gates to implement unitary operators with entries taken in a small set of rationals [1]], and to determine acceptance or rejection by the value of a single qubit [6].

The same definition, with unitary operators and input vectors having rational entries and without the condition that $\frac{1}{2}$ be an isolated cutpoint, yields a "quantum" version of the (classical) class PP. However, this "new" class is in fact no different than PP itself, as can be shown by a simple counting complexity theory.

For any language $L$ in this class, there exists a quantum circuit that accepts it, for which we can define the nonnegative functions $f(x)$ and $g(x)$, as the sum of all the positive and negative contributions, respectively, to the total amplitude for the accepting configuration on a given input $x$. The amplitude of this unique accepting configuration is $f(x)-g(x)$. Similarly, define $f^{\prime}(x)$ and $g^{\prime}(x)$ for the rejecting configuration, with the corresponding rejecting amplitude being $f^{\prime}(x)-g^{\prime}(x)$. It is easy to see that $f, g, f^{\prime}$, and $g^{\prime}$ are all \#P functions. The difference between the probability of accepting and rejecting of this circuit is thus

$$
(f-g)^{2}-\left(f^{\prime}-g^{\prime}\right)^{2}=f^{2}+g^{2}+2 f^{\prime} g^{\prime}-\left(f^{\prime 2}+g^{\prime 2}+2 f g\right)
$$

which is a GapP function, since \#P is closed under (finite) sum and product. This function will be positive if and only $x$ is in $L$, which is another way of characterizing languages in the class PP [11].

On the other hand, the languages defined with quantum gate arrays where unitary operators have rational entries and such $x \in L$ iff $f_{C}(x)>0$ form the complexity class NQP, the quantum analogue to NP, which coincides with the (classical) class coC=P [10].

## 3 Tensor Algebra

A semiring is a tuple $(\mathbb{K},+, \cdot)$ with $\{0,1\} \subseteq \mathbb{K}$ and binary operations $+, \cdot: \mathbb{K} \times \mathbb{K} \rightarrow \mathbb{K}$ (sum and product), such that $(\mathbb{K},+, 0)$ is a commutative monoid, $(\mathbb{K}, \cdot, 1)$ is a monoid, multiplication distributes over sum, and $0 \cdot a=a \cdot 0=0$ for every $a$ in $\mathbb{K}$ (see, e.g., [15]). A semiring is a ring if and only if $(S,+, 0)$ is a group. In this paper we consider the following semirings: the Booleans $(\mathbb{B}, \vee, \wedge)$, the field of rational numbers $(\mathbb{Q},+, \cdot)$, the semiring $\left(\mathbb{Q}^{+},+, \cdot\right)$ of positive rational numbers, and the field of complex numbers $(\mathbb{C},+, \cdot)$.

Let $\mathbb{M}_{\mathbb{K}}$ denote the set of all matrices over $\mathbb{K}$, and define $\mathbb{M}_{\mathbb{K}}^{k, \ell} \subseteq \mathbb{M}_{\mathbb{K}}$ to be the set of all matrices of order $k \times \ell$. Let $[k]$ denote the set $\{1,2, \ldots, k\}$; for a matrix $A$ in $\mathbb{M}_{\mathbb{K}}^{k, \ell}$ and $(i, j) \in[k] \times[\ell]$, the $(i, j)^{\text {th }}$ entry of $A$ is denoted by $a_{i, j}$ or $(A)_{i, j}$. Addition and multiplication of matrices in $\mathbb{M}_{\mathbb{K}}$ are defined in the usual way. Additionally we consider the tensor product $\otimes: \mathbb{M}_{\mathbb{K}} \times \mathbb{M}_{\mathbb{K}} \rightarrow \mathbb{M}_{\mathbb{K}}$ of matrices, also known as Kronecker product, outer product, or direct product, which is defined as follows: for $A \in \mathbb{M}_{\mathbb{K}}^{k, \ell}$ and $B \in \mathbb{M}_{\mathbb{K}}^{m, n}$ let $A \otimes B \in \mathbb{M}_{\mathbb{K}}^{k m, \ell n}$ be

$$
A \otimes B:=\left[\begin{array}{ccc}
a_{1,1} \cdot B & \ldots & a_{1, \ell} \cdot B \\
\vdots & \ddots & \vdots \\
a_{k, 1} \cdot B & \ldots & a_{k, \ell} \cdot B
\end{array}\right]
$$

Hence $(A \otimes B)_{i, j}=(A)_{q, r} \cdot(B)_{s, t}$ where $i=k \cdot(q-1)+s$ and $j=\ell \cdot(r-1)+t$.

The following notation is used: let $I_{n}$ be the order $n$ identity matrix, $e_{i}^{n}$ the $n \times 1$ column vector whose $i^{\text {th }}$ entry has value 1 and the others 0 . and let $A^{\otimes n}$ stand for the $n$-fold iteration $A \otimes A \otimes \cdots \otimes A$.

Stride permutations, which play a crucial role in the implementation of efficient parallel programs for block recursive algorithms such as the fast Fourier transform (FFT) and Batcher's bitonic sort (see [19]) will be useful in our proofs. The mn-point stride $\mathfrak{n}$ permutation $P_{n}^{m n} \in \mathbb{M}_{\mathbb{K}}^{m n}, m n$ is defined as

$$
P_{n}^{m n} e_{i}^{m} \otimes e_{j}^{n}=e_{j}^{n} \otimes e_{i}^{m}
$$

where $e_{i}^{m} \in \mathbb{M}_{\mathbb{K}}^{m}, 1$ and $e_{j}^{n} \in \mathbb{M}_{\mathbb{K}}^{n, 1}$. In other words, the matrix $P_{n}^{m n}$ permutes the elements of a vector of length mn with stride distance $n$. We will make use of the following identities on stride permutations.

Proposition 3.1. The following holds for all $\ell, \mathfrak{m}, \mathfrak{n}$ :

1. $\left(\mathrm{P}_{\mathrm{n}}^{\mathrm{mn}}\right)^{-1}=\mathrm{P}_{\mathrm{m}}^{m n}$;
2. $P_{\mathfrak{m n}}^{\ell m n}=P_{m}^{\ell m n} \cdot P_{n}^{\ell m n}$;
3. $\quad P_{n}^{\ell m n}=\left(P_{n}^{\ell n} \otimes I_{m}\right) \cdot\left(I_{\ell} \otimes P_{n}^{m n}\right)$.

### 3.1 Tensor formulas

Definition 3.2. The tensor formulas over a semiring $\mathbb{K}$ and their order are recursively defined as follows.

1. Every matrix F from $\mathbb{M}_{\mathbb{K}}^{k, \ell}$ with entries from $\mathbb{K}$ is a (atomic) tensor formula of order $k \times \ell$.
2. If F and G are tensor formulas of order $\mathrm{k} \times \ell$ and $\mathrm{m} \times \mathrm{n}$, respectively, then
$(\mathrm{F}+\mathrm{G})$ is a tensor formula of order is $\mathrm{k} \times \ell$ if $\mathrm{k}=\mathrm{m}$ and $\ell=\mathrm{n}$;
$(F \cdot G)$ is a tensor formula of order $k \times n$ if $\ell=m$;
$(\mathrm{F} \otimes \mathrm{G})$ is a tensor formula of order $\mathrm{km} \times \ell \mathrm{n}$.

## 3. Nothing else is a tensor formula.

We say that a tensor formula F is sum-free whenever none of F and its subformulas has the form $\mathrm{G}+\mathrm{H}$. Let $\mathbb{T}_{\mathbb{K}}$ denote the set of all tensor formulas over $\mathbb{K}$, and define $\mathbb{T}_{\mathbb{K}}^{k, \ell} \subseteq \mathbb{T}_{\mathbb{K}}$ to be the set of all tensor formulas of order $k \times \ell$.

In this paper we only consider semiring elements whose value can be given with a standard encoding over some finite set $\mathcal{G}$. Input matrices can therefore be string-encoded using list notation such as " $[[001][101]]$." Nonatomic tensor formula can be encoded over the alphabet $\Sigma=\mathcal{G} \cup\{[],,(),, \cdot,+, \otimes\}$. Strings over $\Sigma$ which do not encode valid formula are deemed to represent the trivial tensor formula 0 of order $1 \times 1$.
The size of a tensor formula $F$ is 1 if $F$ is atomic, otherwise $F=G \circ H$ for $\circ \in\{+, \cdot, \otimes\}$ and the size of $F$ is 1 plus the sizes of $G$ and $H$. The diameter of tensor formula $F$, denoted by $|F|$, is $\max \{k, \ell\}$ if $F$ is atomic of order $k \times \ell$; otherwise we have that $F=G \circ H$ is of order $k \times \ell$, and $|F|=\max \{k, \ell,|G|,|H|\}$.
It will sometimes be convenient to speak of a tensor formula in graph-theoretical terms: in this context, a tensor formula is a binary tree whose edges are directed toward the root ("output node"), whose leaves ("input nodes") are labelled with atomic formulas and each of whose interior nodes is labelled with an operation from the set $\{+, \cdot, \otimes\}$. The depth of a tensor formula is the maximum root-leaf distance.

Definition 3.3. For each semiring $\mathbb{K}$ and each k and each $\ell$ we define $\mathrm{val}_{\mathbb{K}}^{\mathrm{k}, \ell}: \mathbb{T}_{\mathbb{K}}^{\mathrm{k}, \ell} \rightarrow \mathbb{M}_{\mathbb{K}}^{\mathrm{k}, \ell}$, that is, we associate with node f of order $\mathrm{k} \times \ell$ of a tensor formula F its $\mathrm{k} \times \ell$ matrix "value," which is defined as follows:

1. $\quad \operatorname{val}_{\mathbb{K}}^{k, \ell}(f)=F$ if $f$ is an input node labeled with $F$,

2. $\quad \operatorname{val}_{\mathbb{K}}^{k, \ell}(\mathrm{f})=\operatorname{val}_{\mathbb{K}}^{k, m}(\mathrm{~g}) \cdot \operatorname{val}_{\mathbb{K}}^{\mathfrak{m}, \ell}(\mathrm{h})$ if $\mathrm{f}=(\mathrm{g} \cdot \mathrm{h})$, and
3. $\quad \operatorname{val}_{\mathbb{K}}^{k, \ell}(f)=\operatorname{val}_{\mathbb{K}}^{k / m, \ell / n}(g) \otimes \operatorname{val}_{\mathbb{K}}^{m, n}(h)$ if $f=(g \otimes h)$.
4. For completeness, recall that $\mathrm{val}_{\mathbb{K}}^{\mathbb{k}, \ell}(f)=0$ whenever the formula is not valid.

The value $\operatorname{val}_{\mathbb{K}}^{k}, \ell(F)$ of a tensor formula $F$ of order $k \times \ell$ is defined to the value of the unique output node.

### 3.2 The sum-free partial trace problem

A column vector $v$ with complex coefficients is a unit vector iff its $L_{2}$-norm is 1 , that is, iff $v^{\dagger} v=1$. In this paper, we work on probabilistic and quantum computations where the probability amplitudes are encoded in unit column vectors, and the foremost requirement on the computing model is that the inner product (hence also the $\mathrm{L}_{2}$ norm) be preserved at each step of a computation. The action of each such step on the various combinations of values transported by the wires is described with a square matrix; our requirement is equivalent to asking that each matrix preserves the inner product (unitary matrices).
A square matrix $M$ over the complex numbers is unitary iff $M^{\dagger}=M^{-1}$. For a matrix $M$ over the real numbers, this translates into $M^{\top}=M^{-1}$; which means that $M$ is orthogonal. It is an easily verified fact that an orthogonal matrix contains only nonnegative entries if, and only if, it is a permutation matrix (i.e., exactly one entry per line and column is 1 and all others are 0 ).
In the sequel, whenever we deal simultaneously with the cases where matrices with real or complex coefficients, we use the notations and vocabulary from the real case alone, in order to make the text easier to read.
The trace of a square matrix is the sum of its diagonal elements ; for $k>0$, its $k^{\text {th }}$ partial trace is the sum of its last $k$ diagonal elements, counting upwards from the lower right corner. For completeness, if $k$ exceeds the diameter of the matrix, then the $\mathrm{k}^{\mathrm{th}}$ partial trace coincides with the usual trace.

Definition 3.4. A sum-free tensor formula is OSL if and only if it satisfies the conditions:

- all inputs are orthogonal square matrices and/or unit column vectors;
- the output matrix is a column vector.
(We choose the term "orthogonal-system-like" because as we will show, such a formula can be reorganized as a product $M \cdot V$ of an orthogonal matrix with a column vector, i.e. as the specification of an orthogonal system of linear equations.)

Definition 3.5. Let K be a finitely generated semiring. An instance of problem $\operatorname{SFT}(\mathbb{K})$ ("sum-free formula partial trace") consists of an order $\mathrm{N} \times 1$ OSL tensor formula F over semiring $\mathbb{T}$ and a positive integer $k$; the problem consists in deciding whether the $\mathrm{k}^{\text {th }}$ partial trace of $\left(\operatorname{val}_{\mathbb{K}}^{\mathbb{N}, 1}(\mathrm{~F})\right) \cdot\left(\operatorname{val}_{\mathbb{K}}^{N}, 1(\mathrm{~F})\right)^{\top}$ is greater than some predetermined constant $\alpha, 1 / 2 \leq \alpha<1$. In the "promise version" of $\operatorname{SFT}(\mathbb{K})$, no instance can yield $a \mathrm{k}^{\text {th }}$ partial trace which evaluates in the interval $[1-\alpha, \alpha]$.
We also define a "nonzero version" to $\operatorname{SFT}(\mathbb{K})$, as the problem which consists in deciding whether the $\mathrm{k}^{\text {th }}$ partial trace of $\left(\operatorname{val}_{\mathbb{K}}^{N}, 1(\mathrm{~F})\right) \cdot\left(\operatorname{val}_{\mathbb{K}}^{\mathbb{N}, 1}(\mathrm{~F})\right)^{\top}$ is nonzero.

The following propositions show that basic questions on inputs for problem $\mathrm{SFT}(\mathbb{K})$ can be answered in polynomial time.

Proposition 3.6. [5] If F is a tensor formula of depth d which has input matrices of diameter at most p , then $|\mathrm{F}| \leq \mathrm{p}^{2^{\mathrm{d}}}$, and there exists a formula which outputs a matrix of exactly this diameter. (Proof omitted.)

Proposition 3.7. [5] Testing whether a string encodes a valid tensor formula and if so, computing its order, is feasible in deterministic polynomial time. (Proof omitted.)

## 4 From gate arrays to tensor formulas to gate arrays

In this section we show how to encode the description of a reversible or quantum gate array into a OSL tensor formula over the appropriate semiring, and conversely, how to compute from an OSL formula $F$ a gate array which will later used as a mean to solve an SFT instance built from F.

### 4.1 From arrays to formulas

Lemma 4.1. Let C be a gate array operating on n wires, whose gates can be described with orthogonal matrices over semiring $\mathbb{K}$. There is a logspace computable function which, given a suitable coding of C , computes a tensor formula $\mathrm{F}(\mathrm{C})$ of logarithmic depth such that for each $x=\left(x_{1}, \ldots, x_{n}\right) \in\{0,1\}^{n}$,

$$
C(x)=\operatorname{val}_{\mathbb{K}}^{n}, 1\left(F(C) \cdot d_{x}\right)
$$

where $d_{x}=\bigotimes_{i=1}^{n} \chi_{i}, \chi_{i}=e_{2}^{1}$ if $x_{i}=0$, and $\chi_{i}=e_{2}^{2}$ otherwise.

Proof. Let $C$ have $m$ levels and let $C_{i}$ denote the $i^{\text {th }}$ level, with $C_{1}$ the left-most and $C_{m}$ the right-most. We describe how to construct an equivalent tensor formula $M(C)$ from $C$ assuming that 0 and 1 are encoded by $e_{2}^{1}$ and $e_{2}^{2}$, respectively (for quantum arrays, that $|0\rangle$ and $|1\rangle$ are encoded by $e_{2}^{1}$ and $e_{2}^{2}$, respectively). We distinguish two cases.
(i) If each gate of $C_{i}$ acts on consecutive wires, that is, if $C_{i}$ contains $\ell \geq 1$ gates $H_{1}, \ldots, H_{\ell}$, acting on wires $j_{1}$ to $k_{1}, \ldots, j_{\ell}$ to $k_{\ell}$, with $j_{1} \leq k_{1}<j_{2} \cdots k_{\ell-1}<j_{\ell} \leq k_{\ell}$, then

$$
M\left(C_{i}\right)=\left(I_{2}^{\otimes \mathfrak{j}_{1}-1} \otimes \mathrm{H}_{1} \otimes \mathrm{I}_{2}^{\otimes \mathfrak{j}_{2}-\mathrm{k}_{1}-1} \otimes \cdots \otimes \mathrm{H}_{\ell} \otimes \mathrm{I}_{2}^{\otimes \mathfrak{n}-\mathrm{k}_{\ell}}\right)
$$

is the orthogonal matrix of order $2^{n} \times 2^{n}$ describing the action of the $i^{\text {th }}$ level of $C$.
(ii) If $C_{i}$ contains gates acting on nonadjacent wires, then choose a permutation $\sigma$ of the wires which brings next to each other those wires which are involved in the same gate. Denote by $D_{i}$ the $i^{\text {th }}$ level reorganized in this way; its action on the (permuted) wires is described with a formula $M\left(D_{i}\right)$ built as in case $(i)$ above. The permutation is implemented by inserting between levels $i-1$ and $i$ extra depth levels consisting of swap gates, which are collectively described by a formula $P_{\sigma}$; it is undone with other extra levels, inserted between $i$ and $i+1$ and described by $P_{\sigma^{-1}}$. Any permutation can be expressed as a product of a polynomial number of cycles of the form $(j, j+1, \ldots, k-1, k)$,


Figure 1: Simulating an arbitrary controlled-not by a controlled-not acting on neighboring wires.
with $\mathfrak{j}<k$; therefore it suffices to describe the formulas $P_{j, k}(C)$ and $\bar{P}_{j, k}(C)$ which implement this cycle and its inverse, respectively. $\overline{\mathrm{P}}_{\mathrm{j}, \mathrm{k}}(\mathrm{C})$ which implements its inverse. The reader can verify that $\square$

$$
P_{j, k}(C)=\left(I_{2}^{\otimes j-1} \otimes T_{j, k} \otimes I_{2}^{\otimes n-k}\right), \quad \text { where } \quad T_{j, k}=\prod_{i=1}^{k-j-1}\left(I_{2}^{\otimes k-j-i} \otimes T \otimes I_{2}^{\otimes i-1}\right)
$$

and

$$
\overline{\mathrm{P}}_{\mathrm{j}, \mathrm{k}}(\mathrm{C})=\left(\mathrm{I}_{2}^{\otimes \mathrm{j}-1} \otimes \overline{\mathrm{~T}}_{\mathrm{j}, \mathrm{k}} \otimes \mathrm{I}_{2}^{\otimes n-\mathrm{k}}\right), \quad \text { where } \quad \overline{\mathrm{T}}_{\mathrm{j}, \mathrm{k}}=\prod_{\mathrm{i}=1}^{\mathrm{k}-\mathfrak{j}-1}\left(\mathrm{I}_{2}^{\otimes \mathrm{i}-1} \otimes \mathrm{~T} \otimes \mathrm{I}_{2}^{\otimes \mathrm{k}-\mathrm{j}-\mathrm{i}}\right) ;
$$

with $\sigma=\left(\left(j_{1} \cdots \mathrm{k}_{1}\right) \cdots\left(\mathrm{j}_{\ell} \cdots \mathrm{k}_{\ell}\right)\right)^{-1}$, this yields

$$
P_{\sigma}(C)=\bar{P}_{j_{1}, k_{1}}(C) \cdots \bar{P}_{j_{\ell}, k_{\ell}}(C) \quad \text { and } \quad P_{\sigma^{-1}}(C)=P_{j_{\ell}, k_{\ell}}(C) \cdots P_{j_{1}, k_{1}}(C)
$$

so that

$$
M\left(C_{i}\right)=P_{\sigma^{-1}}(C) \cdot M\left(D_{i}\right) \cdot P_{\sigma}(C)
$$

A sample construction for $\mathfrak{j}=1$ and $k=4$ is depicted in Figure 1.
The complete tensor formula $F(C)$ is given by

$$
F(C)=\prod_{i=1}^{m} M\left(C_{i}\right)
$$

which can be parenthesized in order to have logarithmic depth. It is readily verified that for each $x \in\{0,1\}^{n}$

$$
C(x)=\operatorname{val}_{\mathbb{K}}^{n, 1}\left(F(C) \cdot d_{x}\right) .
$$

Formula $\mathrm{F}(\mathrm{C})$ is logspace constructible from C : in particular, a permutation $\sigma$ suitable for case (ii) can be built by choosing a reorganization $D_{i}$ of level $C_{i}$ in which the gates $H_{1}, \ldots, H_{\ell}$, act on wires $j_{1}$ to $k_{1}, \ldots, j_{\ell}$ to $k_{\ell}$, such that $1=j_{1}, k_{1}+1=j_{2}, k_{\ell-1}+1=j_{\ell}$; then the cyclic decomposition of $\sigma$ has the form $\left(1,2,3, \ldots, h_{1}\right)\left(2,3, \ldots, h_{2}\right) \ldots$ where for each $i \geq 2$, the wires $1,2, \ldots, i-1$ are left untouched by the $i^{\text {th }}$ cycle.

[^1]
### 4.2 From formulas to arrays

In the formula-to-array part, one must deal with the fact that an OSL formula may contain matrices of various sizes, and column vectors at atypical locations. The latter may be regarded a nonstandard or disorderly manner of specifying the array's inputs. Matrices of nonstandard orders, however, cannot be readily interpreted in terms of Boolean or quantum computation: one may accept to work with many-valued bits and qubits, or the matrices may be padded in order to turn their orders into powers of 2 , which is the option we choose in this paper.

Lemma 4.2. There exists a polynomial-time algorithm which turns an OSL tensor formula F over semiring $\mathbb{K}$ into a formula $\Pi(\mathrm{F})$ where all subformula sizes are powers of 2 , and whose output is

$$
\left[\begin{array}{c}
\operatorname{van}_{\mathbb{K}}^{n}, 1(F) \\
0
\end{array}\right],
$$

where 0 denotes a (possibly empty) null block.
Proof. For an integer $n \geq 0$, let $\pi(n)$ denote the smallest power of 2 greater than or equal to $n$. We also define a unary operator $\pi$ which acts as follows on a matrix $A$ :

- if $A$ is a $n \times n$ square matrix, then $\pi(A)$ is a $\pi(n) \times \pi(n)$ block-diagonal square matrix consisting in a copy of $A$ at the top left position and a copy of the identity matrix $I_{\pi(n)-n}$ at the bottom right;
- if $A$ is a $n \times 1$ column vector, then $\pi(A)$ is $\pi(n) \times 1$ with the entries of $A$ at the first $n$ positions, and value 0 in the $\pi(n)-n$ others;
- if $A$ is neither of the above, then $\pi(A)$ is undefined.

Whenever $A \cdot B, \pi(A)$ and $\pi(B)$ are defined, we have $\pi(A \cdot B)=\pi(A) \cdot \pi(B)$, so that in the simple case where $F$ does not contain any occurrence of the Kronecker product, $\Pi(F)$ is built by replacing each atomic subformula of $F$ with its image by $\pi$.
This does not work in general. Consider for example the formula $(A \otimes B) \cdot(V \otimes W)$ where $A$ and $B$ are $33 \times 33$ and $35 \times 35$, respectively, and $V$ and $W$ are $21 \times 1$ and $55 \times 1$, respectively: the orders of $(\pi(A) \otimes \pi(B))$ and $(\pi(V) \otimes \pi(W))$ do not match. There also exist cases where the orders match but the entries of $(A \otimes B) \cdot(V \otimes W)$ are not consecutive in the column vector $(\pi(A) \otimes \pi(B)) \cdot(\pi(V) \otimes \pi(W))$. Some subformulas may even yield matrices which are neither square nor column vectors.
Nevertheless, we claim that if matrices $\Pi(A)$ and $\Pi(B)$ are available, then there exists permutations $Q$ and $Q^{\prime}$ and a block H such that

$$
Q \cdot(\Pi(A) \otimes \Pi(B)) \cdot Q^{\prime}=\left[\begin{array}{cc}
A \otimes B & 0 \\
0 & H
\end{array}\right]
$$

where Q and $\mathrm{Q}^{\prime}$ can be specified with polynomial-size sum-free tensor formulas. (Note that H is orthogonal whenever both $A$ and $B$ are.) In the special case where both $A$ and $B$ are column vectors, $Q^{\prime}=I_{1}$ and the claim reads

$$
\mathrm{Q} \cdot(\Pi(\mathrm{~A}) \otimes \Pi(\mathrm{B}))=\left[\begin{array}{c}
\mathrm{A} \otimes \mathrm{~B} \\
0
\end{array}\right] .
$$

We first show how to reorder the lines of $\Pi(A) \otimes \Pi(B)$ where both $A$ and $B$ are column vectors. With $A=$ $\left[x_{1} \cdots x_{m}\right]^{\top}$ and $B=\left[y_{1} \cdots y_{n}\right]^{\top}$, let $\mu=2^{j} \geq \pi(m), \sigma=\mu-m, v=2^{k} \geq \pi(n)$, and $\tau=v-n$. We start
with

$$
\begin{aligned}
& \Pi(A)=\left[x_{1} \cdots x_{m} \bar{x}_{m+1} \cdots \bar{x}_{\mu}\right]^{\top}, \quad \Pi(B)=\left[y_{1} \cdots y_{n} \bar{y}_{n+1} \cdots \bar{y}_{v}\right]^{\top} \\
& \text { and } \Pi(A) \otimes \Pi(B)=\left[\begin{array}{llllll}
x_{1} y_{1} & x_{1} y_{2} & \cdots & x_{1} \bar{y}_{v} & x_{2} y_{1} & x_{2} y_{2}
\end{array} \cdots \bar{x}_{\mu} \bar{y}_{v}\right]^{\top} ;
\end{aligned}
$$

the $\bar{x}_{i}$ 's and $\bar{y}_{i}$ 's are the elements added by padding. Multiplying to the left with the stride permutation $P_{\gamma}^{\mu \nu}$ gives

$$
P_{v}^{\mu \nu} \cdot(\Pi(A) \otimes \Pi(B))=\left[\begin{array}{llllll}
x_{1} y_{1} & x_{2} y_{1} & \cdots & x_{\mu} \bar{y}_{1} & x_{1} y_{2} & x_{2} y_{2}
\end{array} \cdots \bar{x}_{\mu} \bar{y}_{v}\right]^{\top} .
$$

Next we multiply with the matrix

$$
R_{n}^{\mu \nu}=\left[\begin{array}{cc}
P_{\mu}^{n \mu} & 0 \\
0 & \left(N^{\mu \tau}\right)^{k}
\end{array}\right]
$$

where $\mathrm{N}^{\mu \tau}=\mathrm{I}_{\tau} \otimes \mathrm{P}_{2}^{\mu}$. The reader can verify that

$$
R_{n}^{\mu \nu} \cdot P_{v}^{\mu \nu} \cdot(\Pi(A) \otimes \Pi(B))=\left[x_{1} y_{1} x_{1} y_{2} \cdots x_{1} y_{n} \cdots x_{m} y_{n} H\right]^{\top}=[(A \otimes B) H]^{\top}
$$

where $H$ is a size $\mu \nu-m n$ block whose first $n \sigma$ entries are $\bar{x}_{m+1} y_{1}, \ldots, \bar{x}_{\mu} y_{n}$ and the other positions contain a permutation of $x_{1} \bar{y}_{n+1}, \ldots, x_{1} \bar{y}_{v}, \ldots, \bar{x}_{\mu} \bar{y}_{v}$.
There remains to show how to build matrices $P_{\nu}^{\mu \nu}$ and $R_{n}^{\mu \nu}$ with polynomial-size sum-free tensor formulas. By Proposition 3.1, it is readily verified that $\mathrm{P}_{\nu}^{\mu \nu}=\left(\mathrm{P}_{2}^{\mu \nu}\right)^{k}$, and that for any $\ell \geq 1$, the induction formula $\mathrm{P}_{2}^{2^{\ell+2}}=$ $\left(P_{2}^{2^{\ell+1}} \otimes I_{2}\right) \cdot\left(I_{2^{\ell}} \otimes P_{2}^{4}\right)$ yields for the matrix $P_{2}^{\mu \nu}$ a quadratic-size tensor formula with input nodes for $I_{2}$ and $P_{2} 4$. Meanwhile, $R_{n}^{\mu \nu}=\left(S_{n}^{\mu \nu}\right)^{k}$, where

$$
S_{n}^{\mu \nu}=\left[\begin{array}{cc}
P_{2}^{n \mu} & 0 \\
0 & N^{\mu \tau}
\end{array}\right] .
$$

In order to build this matrix, let

$$
\mathrm{U}=\left[\begin{array}{cc}
\mathrm{P}_{2}^{2 n} & 0 \\
0 & \mathrm{I}_{2 \tau}
\end{array}\right]
$$

and $P_{2}^{n \mu}=\left(P_{2}^{2 n} \otimes I_{2^{j-1}}\right) \cdot\left(I_{n} \otimes P_{2}^{\mu}\right)$ by Proposition 3.1; observe that

$$
\left(U \otimes I_{2 j-1}\right) \cdot\left(I_{v} \otimes P_{2}^{\mu}\right)=\left[\begin{array}{cc}
P_{2}^{2 n} \otimes I_{2^{j-1}} & 0 \\
0 & I_{\tau \mu}
\end{array}\right] \cdot\left[\begin{array}{cc}
I_{n} \otimes P_{2}^{\mu} & 0 \\
0 & I_{\tau} \otimes P_{2}^{\mu}
\end{array}\right]=\left[\begin{array}{cc}
P_{2}^{n \mu} & 0 \\
0 & I_{\tau} \otimes P_{2}^{\mu}
\end{array}\right]=S_{n}^{\mu \nu}
$$

Expressed in this way, matrix $R_{n}^{\mu v}$ can be built with a polynomial-size sum-free tensor formula, where matrix $U$ is either given explicitly by a made-to-purpose gate if $n$ is the diameter of an input matrix, or built inductively in the case where $n=\pi(p)$ for some $p$, because in this case $U=P_{2}^{2 n}$.
The same technique applies to reorder the lines for arbitrary matrices $A$ and $B$; in this case the $x_{i}$ 's and $y_{i}$ 's are lines and each $x_{i} y_{j}$ in the above equations must be read as $x_{i} \otimes y_{j}$. The claim for the existence of a matrix $Q^{\prime}$ which reorders the columns is proved in a dual manner.
Let $F$ be an OSL formula; the following algorithm builds a formula $\Pi(F)$ which satisfies the conditions of the Lemma, by recursively defining $\Pi(G)$ for each subformula $G$ of $F$.

- For each atomic subformula G , let $\Pi(\mathrm{G})=\pi(\mathrm{G})$.
- Repeat recursively from the leaves toward the root of $F$ : for each subformula $G=H \circ K$ for which $\Pi(H)$ and $\Pi(\mathrm{K})$ have already been computed and $\circ \in\{\cdot, \otimes\}$ :
- if $\circ$ is " $\otimes$ " then let $\Pi(\mathrm{G})=\mathrm{Q} \cdot(\Pi(\mathrm{H}) \otimes \Pi(\mathrm{K})) \cdot \mathrm{Q}^{\prime}$ and insert the appropriate subformulas for Q and $\mathrm{Q}^{\prime}$ (note that $\Pi(\mathrm{Q})=\mathrm{Q}$ and $\left.\Pi\left(\mathrm{Q}^{\prime}\right)=\mathrm{Q}^{\prime}\right)$;
- otherwise $\circ$ is ".": if the orders of $\Pi(\mathrm{H})$ and $\Pi(\mathrm{K})$ match, then let $\Pi(\mathrm{G})=\Pi(\mathrm{H}) \cdot \Pi(\mathrm{K})$; else they differ by a power of 2 and the smaller matrix must undergo some padding, that is, either $\Pi(G)=\left(I_{2}^{\otimes i} \otimes \Pi(H)\right)$. $\Pi(\mathrm{K})$, or $\Pi(\mathrm{G})=\Pi(\mathrm{H}) \cdot\left(\left(e_{2}^{1}\right)^{\otimes \mathfrak{i}} \otimes \Pi(\mathrm{K})\right)$, for an appropriate $i$.

Lemma 4.3. There is a polytime computable function which, from a OSL tensor formula F over semiring $\mathbb{K}$, computes a polynomial-size gate array $\mathrm{C}(\mathrm{F})$ whose input is represented with a unit vector V , whose action over the inputs is given by an orthogonal matrix $M$, and such that matrices $M V$ and $\operatorname{val}_{\mathbb{K}}^{n, 1}(F)$ satisfy

$$
M V=\left[\begin{array}{c}
\operatorname{val}_{\mathbb{K}}^{n, 1}(\mathrm{~F}) \\
0
\end{array}\right],
$$

where 0 denotes a (possibly empty) null block.

Proof. The formula $\Pi(F)$ is used as a specification for a gate array $C(F)$. For each atomic subformula $G$ of $F$, either $G$ is $m \times m$ for some $m \leq|F|$, where $|F|$ is the diameter of $F$, and $\Pi(G)$ is interpreted as the specification of a gate with $\log _{2} \pi(\mathrm{~m})=\left\lceil\log _{2} \mathrm{~m}\right\rceil$ inputs, or G is $\mathrm{m} \times 1$ and $\Pi(\mathrm{G})$ specifies the probability amplitudes for all possible combinations of values of $\log _{2} \pi(m)=\left\lceil\log _{2} m\right\rceil$ input bits or qubits. In the former case, a polynomial-size array of elementary gates implements the operation specified by $\Pi(G)$; in the latter case, a size $\mathrm{m}^{\mathrm{O}(1)}$ array is built to take as input some constant unit vector (say $e_{\Pi(\mathfrak{m})}^{1}$ ) and yield as output the vector $\Pi(G)$. Next, working recursively from the leaves toward the root of $\Pi(F)$, the interior nodes are interpreted as specifications for combining the subarrays either in a sequential (nodes labelled ".") or parallel (nodes labelled " $\otimes$ ") manner. The resulting gate array has polynomial size and satisfies the conditions of the lemma.

## 5 Complexity results

Over the Boolean semiring, a column vector is a unit vector as soon as it is nonzero, so that the standard, promise and nonzero versions of problem SFT coincide.

Theorem 5.1. Over the Boolean semiring, problem SFT is P -complete under logspace reducibility.
Proof. Given a size $n$ instance $(F, k)$ of $\operatorname{SFT}(\mathbb{B})$, we use Lemma 4.3 to build an equivalent reversible gate array $C(F)$ over $\mathrm{N}=\mathrm{n}^{\mathrm{O}(1)}$ bits, and we compute the output value of each of these bits (i.e. we solve N instances of the usual Boolean circuit value problem). This yields a combination of N values which corresponds to a given position along the diagonal of

$$
\left(\operatorname{val}_{\mathbb{B}}^{2^{\mathrm{N}}, 1}(\mathrm{~F})\right) \cdot\left(\operatorname{val}_{\mathbb{B}}^{2^{\mathrm{N}}, 1}(\mathrm{~F})\right)^{\top},
$$

under the convention that combinations $00 \cdots 0, \ldots, 11 \cdots 1$ correspond to lines (and columns) $1, \ldots, 2^{\mathrm{N}}$, respectively. The hardness part consists in using Lemma 4.1 to reduce the P-complete circuit value problem [16] to an instance of $\operatorname{SFT}(\mathbb{B})$.

For the quantum and probabilistic cases we are mainly interested in the promise version of SFT, which gives us a striking description for the difference between complexity classes BPP and BQP.

## Theorem 5.2. The promise version of problem $\operatorname{SFT}(\mathbb{Q})$ is complete for the class BQP , under logspace reducibility.

Proof. The hardness part is a generic reduction. Using Definition 2.3, we start with a m-leveled gate array $C$ on $n$ qubits numbered 1 to $n$ whose accepting subspace is defined by setting qubit 1 to $|1\rangle$, and whose gates are defined with unitary matrices over $\mathbb{Q}$. Denote by $f_{C}$ the probability that qubit 1 be projected to $|1\rangle$ when the measurement takes place. We use Lemma 4.1 to build from $C$ an equivalent tensor formula $F(C)=\prod_{i=1}^{m} M\left(C_{i}\right)$. Meanwhile we define for the array's input qubits a tensor product $V$ of $n$ unit vectors of size $2 \times 1$. An easy induction on $j$ shows that

$$
\operatorname{val}_{\mathbb{Q}}^{2^{n}, 1}\left(\prod_{i=1}^{j} M\left(C_{i}\right) \cdot V\right)
$$

is exactly the vector of amplitudes after level $j$ in $C$. Thus the last $2^{n-1}$ entries along the diagonal of

$$
\left(\operatorname{val}_{\mathbb{Q}}^{2^{n}, 1}(F(C) \cdot V)\right) \cdot\left(\operatorname{val}_{\mathbb{Q}}^{2^{n}}, 1(F(C) \cdot V)\right)^{\top}
$$

add up to the value of $f_{C}$, and the original array's input is accepted iff this partial trace exceeds the threshold by which acceptance by $C$ was defined. Scrutiny of the reduction shows that the constraint on $f_{C}$ is transported intact from the description of $C$ to the $\operatorname{SFT}(\mathbb{Q})$ instance $F(C) \cdot V$.
In the other direction, we use Lemma 4.3 to translate an instance $(F, k)$ for $\operatorname{SFT}(\mathbb{Q})$ into the description of a quantum gate array over $m$ qubits, $m \geq \log _{2} n$, and of its inputs; the $k^{\text {th }}$ partial trace of

$$
\left(\operatorname{val}_{\mathbb{Q}}^{2^{m}, 1}(\mathrm{~F})\right) \cdot\left(\operatorname{val}_{\mathbb{Q}}^{2^{m}, 1}(\mathrm{~F})\right)^{\top}
$$

represents the probability that the output qubits of this array be projected onto the direct sum of the dimension- 1 subspaces generated by $\left|2^{m}-1\right\rangle=|1 \cdots 11\rangle,\left|2^{m}-2\right\rangle=|1 \cdots 10\rangle,\left|2^{m}-3\right\rangle=|1 \cdots 01\rangle, \ldots$, and $\left|2^{m}-k\right\rangle$. The promise on the partial trace is transported unmodified from the input tensor formula to the quantum gate array.

The argument described above can be used to prove that the "standard" (non-promise) version of problem SFT( $\mathbb{Q}$ ) is complete for PP, defined by removing the constraint from definition 2.4. Finally, when the proof is applied to the "nonzero" version of problem $\operatorname{SFT}(\mathbb{Q})$, a completeness statement is obtained for the class NQP.

Finally, we consider problem SFT over the semiring of the nonnegative rational numbers. Note that, just as in the quantum case, the entries in the column vectors are regarded as probability amplitudes. All the gates do in a classical reversible array is permute the different vector components without ever mixing or combining them; no interference ever takes place and it does not matter in terms of the final result, whether the probabilities are represented as such or as amplitudes.

Theorem 5.3. Problem $\operatorname{SFT}\left(\mathbb{Q}^{+}\right)$is PP-complete under logspace reducibility.
Proof. For a generic reduction, we start with a reversible gate array $C$ whose input is a string of $N=s(n)+t(n)$ bits, where the initial $s(n)$ bits are the ancillary bits, all set to 0 , and the other $t(n)$ bits are random. By Lemma 4.1, $C$ and its input can be encoded into $F(C) \cdot V$, where the $2^{N} \times 1$ unit vector $V$ specifies the inputs, i.e. a bit string $c_{1} \cdots c_{s(n)} d_{1} \cdots d_{t(n)}$ which satisfies the conditions
i. $\quad c_{i}=0$ for all $i \leq i \leq s(n)$, and
ii. all combinations of values for the random bits $d_{1} \cdots d_{t(n)}$ are equally likely.

The corresponding $2^{\mathfrak{t}(n)}$ entries in the vector val $\mathbb{Q}^{2^{+}}, 1(\mathrm{~V})$ carry value $1 / \sqrt{2^{\mathfrak{t}(n)}}$; all others contain 0 . We demand wlog that $t(n)$ be even; dealing with the random bits pairwise enables us to ensure that no irrational values are necessary. Then

$$
V=\left(e_{1}^{1}\right)^{\otimes s(n)} \otimes\left[\begin{array}{l}
1 / \sqrt{2} \\
1 / \sqrt{2}
\end{array}\right]^{\otimes \mathfrak{t}(n)}=\left(e_{1}^{1}\right)^{\otimes s(n)} \otimes\left[\begin{array}{l}
1 / 2 \\
1 / 2 \\
1 / 2 \\
1 / 2
\end{array}\right]^{\otimes \frac{t(n)}{2}} .
$$

Let the acceptance condition be that bit $\mathrm{c}_{1}$ has value 1 at the output level. This corresponds to the first $2^{\mathrm{N}-1}$ positions along the diagonal of $\left(\operatorname{val}_{\mathbb{Q}^{+}}^{2^{\mathrm{N}}, 1}(\mathrm{~F}(\mathrm{C}) \cdot \mathrm{V})\right) \cdot\left(\operatorname{val}_{\mathbb{Q}^{+}}^{\mathrm{N}^{\mathrm{N}}, 1}(\mathrm{~F}(\mathrm{C}) \cdot \mathrm{V})\right)^{\top}$.
In the other direction, consider an instance $(F, k)$ for $\operatorname{SFT}\left(\mathbb{Q}^{+}\right)$. We have discussed in Section 4.2 how the column vectors and square matrices are interpreted as "inputs" and "gates" in the equivalent array, through the construction of a formula $\Pi(F)$ where all matrices have orders which are powers of 2 . We add extra steps to the construction of $\Pi(F)$ in order to enforce the further condition, that all fractions have a power of 2 as denominator.
Consider a $n \times 1$ unit vector $v_{i}=\left[\frac{a_{1}}{d} \cdots \frac{a_{n}}{d}\right]^{\top}$, where $a_{1}^{2}+\cdots+a_{n}^{2}=d^{2}$. Let $d$ not be a power of 2: $d<\pi(d)$. The reader can verify that there exist integers $b_{1}, \ldots, b_{p}$ such that $\pi(d)^{2}=a_{1}^{2}+\cdots+a_{n}^{2}+b_{1}^{2}+\cdots+b_{p}^{2}$ and $p \leq 3\left\lceil\log _{2} d\right\rceil$. Let $q=\min \left\{2^{2 j}: 2^{2 j}>n+3\left\lceil\log _{2} d\right\rceil\right\}$, and embed $v$ into the $q \times 1$ vector

$$
\left[\frac{a_{1}}{\pi(d)} \cdots \frac{a_{n}}{\pi(d)} 0 \cdots 00 \frac{b_{1}}{\pi(d)} \cdots \frac{b_{p}}{\pi(d)}\right]^{T}
$$

which can be interpreted as a distribution of probability amplitudes for $\log _{2} q$ input bits. Denote by $\delta_{i}$ the fraction $d / \pi(d)$. Repeating this process on each input column vector yields an instance ( $G, k$ ) where the resulting partial trace is the same one obtained from ( $\mathrm{F}, \mathrm{k}$ ), times a factor $\Delta^{2}=\prod_{i} \delta_{i}^{2}$. If we accept instance ( $\mathrm{F}, \mathrm{k}$ ) whenever the partial trace is above a threshold $\alpha$, then there exists a probabilistic polytime Turing machine $M$ which accepts ( $G, k$ ) with probability above $\frac{\alpha}{\Delta^{2}}$.
The algorithm of $M$ is divided into three phases; the first consists in building the new instance ( $G, k$ ) from the original ( $\mathrm{F}, \mathrm{k}$ ), the second in choosing nondeterministically a column vector to give as input to the equivalent array $\mathrm{C}(\mathrm{G})$, and the third in deterministically simulating $C(G)$ on its input. In the second step $M$ nondeterministically selects values for the bits in the string $d_{1} \cdots d_{t(n)}$; the preprocessing step has organized their probability distribution in order to ensure that this can be done with a sequence of nondeterministic binary choices, followed by a look-up into a table which is linear in size and is computed from the column vectors in ( $\mathrm{F}, \mathrm{K}$ ).

The reader can verify that this proof can be rewritten in terms of the promise problem $\operatorname{SFTP}\left(\mathbb{Q}^{+}\right)$and the complexity class BPP; in the second part of the proof the cutpoint and the size of the empty interval can be modified, however. Meanwhile, the complexity of the nonzero version is obtained with a straightforward application of the above argument.

| Semiring/Version | Standard | Promise | Nonzero |
| :---: | :---: | :---: | :---: |
| $(\mathbb{Q},+, \cdot)$ | PP | BQP | NQP |
| $\left(\mathbb{Q}^{+},+, \cdot\right)$ | PP | BPP | NP |
| $(\mathbb{B}, \vee, \wedge)$ | P |  |  |

Figure 2: Summary of completeness results

Corollary 5.4. The promise and nonzero versions of problem $\operatorname{SFT}\left(\mathbb{Q}^{+}\right)$are BPP -complete and NP -complete, respectively, under logspace reducibility.

## 6 Conclusion

Through the study of problem SFT, we have developed a common algebraic description for polynomial-time complexity classes, where the choice of the semiring determines the complexity class. For the inclusion chain $\mathrm{P} \subseteq \mathrm{BPP} \subseteq$ $B Q P$, in particular, the classical model of polytime probabilistic computation turns out to be a special case of polytime quantum computation where interference between computations is ruled out.

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    ${ }^{\dagger}$ Corresponding author.
    ${ }^{\ddagger}$ Part of the work was done while the author was at Département d'I.R.O., Université de Montréal.
    Beaudry: Département de mathématiques et d'informatique, Université de Sherbrooke, 2500 boul. Université, Sherbrooke, Québec, J1K 2R1 Canada. email: beaudry@dmi.usherb.ca
    Fernandez: Département d’I.R.O. Université de Montréal, C.P. 6128, succ. Centre-Ville, Montréal, Québec, H3C 3J7 Canada. email: fernandz@iro.umontreal.ca
    Holzer: Institut für Informatik, Technische Universität München, Arcisstraße 21, D-80290 München, Germany. email: holzer@informatik.tu-muenchen.de

[^1]:    ${ }^{1}$ Note that according to the usual convention, the input-to-output direction in a gate array is left-to-right, while in its matrix representation, the array's action on its input is given as a product of orthogonal matrices with a column vector, and is read right-to-left.

