Topological Digital Topology

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Abstract. The usefulness of topology in science and mathematics means that topological spaces must be studied, and computers should be used in this study. We discuss how many useful spaces (including all compact Hausdorff spaces) can be approximated by finite spaces, and these finite spaces are completely determined by their specialization orders. As a special case, digital n-space, used to interpret Euclidean n-space and in particular, the computer screen, is also dealt with in terms of the specialization. Indeed, algorithms written using the specialization are comparable in difficulty, storage usage and speed to those which use the traditional (8,4), (4,8) and (6,6) adjacencies, and are of course completely representative of the spaces.

Keywords: Digital topology, general topology, T_0-space, specialization (order), connected ordered topological space (COTS), Alexandroff space, Khalimsky line, digital n-space, metric and polyhedral analogs, chaining maps, calming maps, normalizing maps, inverse limit, Hausdorff reflection, skew (=stable) compactness, (graph) path and arc connectedness and components, (topological) adjacency, Jordan curve, robust scene, cartoon.

1 Introduction: Why Topological Spaces?

During the first calculus or post-calculus course with any intellectual glue, students meet the idea of topology:

Definition 1. *A topological space is a set X, together with a collection τ, of subsets of X, such that:*
(a) if G is a finite subset of τ then its intersection, $\bigcap G \in \tau$, and
(b) if G is any subset of τ then its union, $\bigcup G \in \tau$.
A subset of X is called open if it is in τ, closed if its complement is in τ.

As a result of this definition, since $\emptyset$ is a finite subset of τ, $\emptyset = \bigcup \emptyset$ and $X = \bigcap \emptyset$ are open (are in τ).

Why does topology come up there? First, metrics (distance functions) are noticed in calculus, such as $d(x, y) = |x - y|$, or for vectors, $d(x, y) = \|x - y\|$.

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It is easy to define a topology using from a metric: a set T is open if whenever $x \in T$, then some ball of positive radius, $B_r(x) = \{y \mid d(x,y) < r\}$, is contained in T (for some $r > 0,\ B_r(x) \subseteq T$). Essentially no properties of the distance are used in the proof that this gives a topology, and for metrics satisfying the *triangle inequality:* $d(x,z) \leq d(x,y)+d(y,z)$, $B_r(x)$ is open (if $y \in B_r(x)$ then for $s = r-d(x,y) > 0$, $B_s(y) \subseteq B_r(x)$ (if $d(y,z) < s$ then $d(x,z) \leq d(x,y)+d(y,z) < d(x,y)+(r-d(x,y)) = r$). Good references in general topology include [18] and [19].

Using topology one can easily define:

- Limit (thus derivative), continuous function (at a point or always),
- closure, interior and boundary of sets,
- connected set, compact set.

It then becomes easy to show that each function is continuous at each point where it has a derivative. Also, the connected sets of real numbers are the intervals and the compact sets are the bounded closed sets; thus the closed bounded intervals (sets of the form $[a,b] = \{x \mid a \leq x \leq b\}$) are the connected compact subsets. If $f : X \to Y$ is a function and $A \subseteq X$, the *image of A under f* is $f[A] = \{f(x) \mid x \in A\}$; further, if $B \subseteq Y$, the inverse image of B, $f^{-1}[B] = \{x \mid f(x) \in B\}$. We don't bother with any of these textbook proofs, although we do some later which are related to our particular interest.

Facts: Suppose f is continuous and $A \subseteq X$. If A is connected then $f[A]$ is connected, if A is compact, then $f[A]$ is compact. Thus in particular, if $X = \mathbb{R}$ and $a < b$ then $f[[a,b]]$ is a closed, bounded interval, $[m,M]$, so:

There are $x, y \in [a,b]$ so that $f(x) = m$ and $f(y) = M$ – that is, f achieves a minimum and a maximum on $[a,b]$, so these are worth looking for. This justifies much of differential calculus.

Since $f(a)$ and $f(b)$ are in the interval $f[[a,b]]$, if p is between $f(a)$ and $f(b)$ then $p \in f[[a,b]]$, which is to say that for some $c \in [a,b]$, $p = f(c)$. That is, the equation $p = f(y)$ has a solution in $[a,b]$. This justifies much of the search for roots in algebra.

The above and many similar facts mean that topological questions permeate analysis, thus theoretical science. Therefore, much computing must be done with topological data. We now discuss methods to do this.

2 Finite and Alexandroff Spaces

Definition 2. *A topological space is Alexandroff if:*
(a') if G is ANY subset of τ then $\bigcap G \in \tau$.

(The above is in addition to (b), and implies (a) of Definition 1.) These spaces were studied systematically long ago by the author after whom they are named; see [2].

This is quite atypical of spaces. In $\mathbb{R}$ for example, $\{0\} = \bigcap_1^\infty (-1/n, 1/n) = \bigcap_1^\infty B_{1/n}(0)$ is an intersection of open sets which isn't open. But it is typical of the finite topological spaces that one can completely store in a computer, since then any subset of τ is finite, so its intersection is in τ. The theory of Alexandroff spaces, applied especially to digital topology, is discussed in [11] and [7]. Most of the results in Lemma 2 through Theorem 1 can be found there conveniently (though none originate there).

Alexandroff spaces have a particular property that is extremely useful in computing. Recall that a *preorder* is a relation $\leq$ such that each $a \leq a$ and $a \leq b \& b \leq c \Rightarrow a \leq c$; a *partial order* is a preorder for which $a \leq b \& b \leq a \Rightarrow a = b$. We now work toward a proof that for finite spaces, topology and continuity are completely determined by a preorder (which should be seen as an asymmetric adjacency relation). That is (see Theorem 1 (b), or [11]):

There is a preorder such that the open sets are the *upper* sets; those for which $x \in T \& x \leq y \Rightarrow y \in T$ (lower sets are similarly defined). Furthermore, a function between Alexandroff spaces will be continuous if and only if it preserves the order. Here are some relevant textbook proofs:

Lemma 1. *Given any topological space:*

(a) Finite unions and arbitrary intersections of closed sets are closed.

(b) For each $A \subseteq X$ there is a smallest closed set containing A called its closure, and defined by $\mathsf{cl}A = \bigcap\{C \text{ closed} |\ A \subseteq C\}$, and a largest open subset of A, its interior, $\mathsf{int}A = \bigcup\{T \text{ open} |\ T \subseteq A\}$.

A function $f : X \to Y$ is defined to be continuous at a point a if whenever $f(a) \in T$ and T is open, then for some open $U \ni a$, $f[U] \subseteq T$. It is continuous if continuous at every point in X.

(c) The following are equivalent: f is continuous $\Leftrightarrow$ for each open T, $f^{-1}[T]$ is open $\Leftrightarrow$ for each closed C, $f^{-1}[C]$ is closed $\Leftrightarrow$ for each A, $f[\,\mathsf{cl}(A)] \subseteq \mathsf{cl}(f[A])$.

Proof. (a) Let G be a collection of closed sets. By de Morgan's laws, $X \setminus \bigcup\{C \mid C \in G\} = \bigcap\{X \setminus C \mid C \in G\}$ so the complement of $\bigcup\{C \mid C \in G\}$ is open if G is finite, thus $\bigcup\{C \mid C \in G\}$ is closed if G is finite; the other proof is similar.

(b) By definition of a topological space, $\bigcup\{T \text{ open} |\ T \subseteq A\}$ is an open set, and is certainly contained in A, and the largest such set (since if $U \subseteq A$ is open, then U is one of the sets whose union is being taken). Thus $\mathsf{int}(A)$ is the largest open set contained in A. By (a), $\bigcap\{C \text{ closed} |\ A \subseteq C\}$ is closed, and the proof that it is the smallest closed set containing A is like the above.

(c) For this proof it's necessary to notice some properties of f^{-1}:

$$
\begin{gathered}
x \in f^{-1}[\bigcup G] \Leftrightarrow f(x) \in \bigcup G \Leftrightarrow \text{for some} \\
B \in G,\ f(x) \in B \Leftrightarrow x \in \bigcup\{f^{-1}[B] \mid B \in G\}, \\
x \in f^{-1}[\bigcap G] \Leftrightarrow f(x) \in \bigcap G \Leftrightarrow \text{for each} \\
B \in G,\ f(x) \in B \Leftrightarrow x \in \bigcap\{f^{-1}[B] \mid B \in G\}, \\
x \in f^{-1}[Y \setminus B] \Leftrightarrow f(x) \in Y \setminus B \Leftrightarrow f(x) \notin B \Leftrightarrow x \notin f^{-1}[B].
\end{gathered}
$$

That is, $f^{-1}[\bigcup G] = \bigcup\{f^{-1}[B \mid B \in G\}]$, $f^{-1}[\bigcap G] = \bigcap\{f^{-1}[B \mid B \in G\}]$, and $f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$. Another useful property is that $A \subseteq f^{-1}[B] \Leftrightarrow f[A] \subseteq B$.

Suppose f is continuous, T is open and $a \in f^{-1}[T]$. Then $f(a) \in T$ so for some open $U_a \ni a$, $f[U_a] \subseteq T$, thus $a \in U_a \subseteq f^{-1}[T]$, therefore $f^{-1}[T] \subseteq \bigcup\{U_a \mid a \in f^{-1}[T]\} \subseteq f^{-1}[T]$, showing $f^{-1}[T]$ to be open.

If the inverse image of each open set is open and C is closed, then $Y \setminus C$ is open, so $f^{-1}[C] = X \setminus f^{-1}[Y \setminus C]$ is closed.

If the inverse image of each closed set is closed, then so is $f^{-1}[\,\mathsf{cl}(f[A])] \supseteq A$. But then as the smallest closed set containing A, $\mathsf{cl}(A) \subseteq f^{-1}[\,\mathsf{cl}(f[A])]$, showing $f[\,\mathsf{cl}(A)] \subseteq \mathsf{cl}(f[A])$.

Finally if each $f[\,\mathsf{cl}(A)] \subseteq \mathsf{cl}(f[A])$ and $f(x)$ is in an open set T, then for each x, $f(x) \notin \mathsf{cl}(Y \setminus T)$, thus $x \notin \mathsf{cl}(f^{-1}[Y \setminus T])$. But this says that for some open set, $x \in U \subseteq X \setminus \mathsf{cl}(f^{-1}[Y \setminus T]) \subseteq X \setminus f^{-1}[Y \setminus T] = f^{-1}[T]$. Therefore f is continuous at x.

The same principles are used to see the key facts for Alexandroff spaces. But we need other definitions first.

Definition 3. *Let X be any set and $\mathcal{B}$ any collection of subsets of X. Then there is a smallest topology $\tau^{\mathcal{B}}$ on X which contains $\mathcal{B}$.*

Let (X, τ) be a topological space. The specialization *is defined by $x \leq_X y \Leftrightarrow x \in \mathsf{cl}\{y\}$.*

The space X is T_0 if whenever $x \in \mathsf{cl}\{y\}$ and $y \in \mathsf{cl}\{x\}$ then $x = y$, and T_1 if each $\{x\}$ is closed.

If $Y \subseteq X$ then the subspace topology $\tau|Y$ is defined by saying that $T \in \tau|Y$ if (and only if) for some $U \in \tau$, $T = U \cap Y$.

Given a collection of spaces, (X_i, τ_i), $i \in I$, the product topology *on the set $\prod_{i \in I} X_i$* [1]*, is the smallest one containing each set of the form $\{x \mid x_i \in U\}$, where $i \in I$ and $U \in \tau_i$.*

Lemma 2. *(a) For each X, $\leq_X$ is a preorder. It is a partial order iff the space is T_0, and equality if and only if the space is T_1.*

(b) Each closed set is a $\leq_X$ lower set and each open set is a $\leq_X$-upper set. For each continuous $f : X \to Y$, $x \leq_X y \Rightarrow f(x) \leq_Y f(y)$.

(c) Given a subspace Y of a space X, for $x, y \in Y$, $x \leq_Y y \Leftrightarrow x \leq_X y$. In a product, for $x, y \in \prod_{i \in I} X_i$, $x \leq_{\prod_{i \in I} X_i} y$ if and only if for every coordinate, $x_i \leq_{X_i} y_i$.

Proof. (a) Of course, $x \in \mathsf{cl}\{x\}$. Next notice that $x \in \mathsf{cl}\{y\}$ if and only if $\mathsf{cl}\{x\} \subseteq \mathsf{cl}\{y\}$; it is immediate that $\leq_X$ is transitive. The assertion about partial order is immediate from our slightly non-standard definition of T_0, and that about equality is immediate from our standard definition of T_1.

(b) If $x \in C$, C is closed, and $y \leq_X x$, then $y \in \mathsf{cl}\{x\} \subseteq C$, so $y \in C$, thus C is lower. Therefore each open set is upper since its complement is lower. If f is continuous and $y \leq_X x$, then $y \in \mathsf{cl}\{x\}$ so $f(y) \in f[\,\mathsf{cl}\{x\}] \subseteq \mathsf{cl}(f[\{x\}])$, which is to say, $f(y) \leq_Y f(x)$.

(c) Notice that in the subspace topology, $C \subseteq Y$ is closed if and only if $C = Y \cap D$ for some closed $D \subseteq X$. Thus $Y \cap \mathsf{cl}\{y\}$ is closed in $\tau|Y$ and if $y \in C$ closed in $\tau|Y$ then for some closed $D \subseteq X$, $y \in D$ (thus $\mathsf{cl}\{y\} \subseteq D$) and

[1] Recall that the product is the set of all maps x on I such that each $x(i) \in X_i$. Usually $x(i)$ is called the i'th coordinate, and denoted x_i.

$C = Y \cap D$. Thus $C \supseteq Y \cap \mathsf{cl}\{y\}$. This shows that $Y \cap \mathsf{cl}\{y\}$ is the smallest closed set in $\tau|Y$, containing y, and of course, for $x \in Y$, $x \in \mathsf{cl}\{y\} \Leftrightarrow x \leq_X y$.

Notice that if each $C_i \subseteq X_i$ is closed, then $\prod_{i\in I} C_i = \{x \in \prod_{i\in I} X_i \mid \text{each } x_i \in C_i\} = \bigcap_{i\in I}\{x \in \prod_{i\in I} X_i \mid x_i \in C_i\}$ and is thus closed in $\prod_{i\in I} X_i$ since for each i, the complement $\{x \in \prod_{i\in I} X_i \mid x_i \in X_i \setminus C_i\} \in \tau_i$. Thus for each y, $\prod_{i\in I} C_i = \{x \in \prod_{i\in I} X_i \mid \text{each } x_i \in \mathsf{cl}\{y_i\}\}$ is the smallest closed set containing y, and of course x is in this set iff each $x_i \leq_{X_i} y_i$.

The converses of (a) and (b) above are not true: Notice that each function must preserve $=$, the specialization order of T_1 spaces, while most are not continuous. For similar reasons, each set in a T_1 space is both upper and lower, but the only sets in $\mathbb{R}$ which are both open and closed are $\emptyset$ and $\mathbb{R}$. But the converses hold for Alexandroff spaces:

Theorem 1. *(a) A space is Alexandroff if and only if all unions of closed sets are closed; equivalently, if and only if each A is contained in a smallest open set, which we call $\mathsf{n}(A)$.*

(b) For an Alexandroff space (X, τ), the closed sets are precisely the $\leq_\tau$-lower sets, and the $\leq_\tau$-upper sets are exactly the open sets. Further, the continuous functions are simply the specialization order preserving functions.

Proof. (a) The first assertion is shown using de Morgan's laws, exactly as Lemma 1 (a) was shown. For the second, the existence of $\mathsf{n}(A)$ in Alexandroff spaces is shown just like that of $\mathsf{cl}(A)$ in all topological spaces, in Lemma 1 (b). Conversely, if $\mathsf{n}(A)$ always exists and G is a collection of open sets, then for each $T \in G$, $\mathsf{n}(\bigcap G) \subseteq T$; therefore $\mathsf{n}(\bigcap G) \subseteq \bigcap G$; but since in general $A \subseteq \mathsf{n}(A)$, we have that $\bigcap G = \mathsf{n}(\bigcap G)$, an open set.

(b) One direction of each assertion in the first sentence holds by Lemma 2. For the converses, if C is a lower set in an Alexandroff space, then $C = \bigcup\{\,\mathsf{cl}(\{x\}) \mid x \in C\}$, a closed set. Thus if T is an upper set then its complement is lower, so closed, thus T is open.

For functions, we show more than stated in (b): a function $f : X \to Y$, X, Y Alexandroff, is continuous at $x \in X \Leftrightarrow$ whenever $x \leq_X y$ then $f(x) \leq_Y f(y)$. To see this, note that "$x \leq_X y \Rightarrow f(x) \leq_Y f(y)$" is equivalent to $f[\,\mathsf{n}\{x\}] \subseteq \mathsf{n}(f[\{x\}])$, and if the latter holds and $f(x) \in T$, an open set, then $\mathsf{n}\{f(x)\} \subseteq T$, so for $U = \mathsf{n}\{x\}$, $x \in U$ and $f[U] \subseteq T$.

From the last paragraph, it results that a function between Alexandroff spaces is continuous if and only if it is specialization preserving.

The results in Theorem 1 essentially say that for all Alexandroff spaces, (including each space, X, that can be completely stored in a computer), all the information about X can be learned from the "asymmetric adjacency" $\leq_X$. We use this below.

3 The Computer Screen

Since the execution of programs and the computer screen are "discrete", programs for the computer screen operate in terms of adjacencies, that is, binary

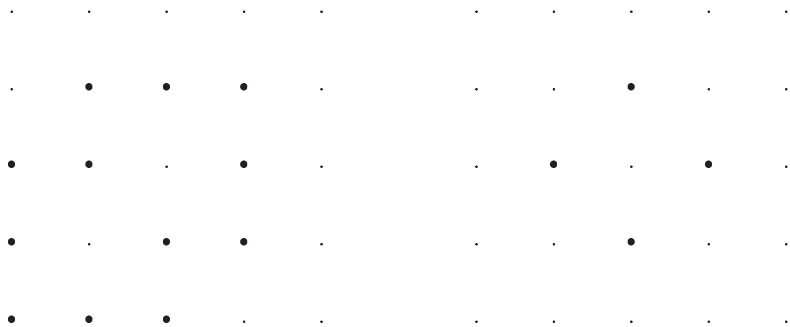


Fig. 1. (4,4) and (8,8) violations of the Jordan curve theorem.

relations that are symmetric and irreflexive; the most popular are 4-adjacency, where each $(x, y) \in \mathbb{Z}^2$ is adjacent to $(x, y\pm 1)$ and $(x\pm 1, y)$ and 8-adjacency, in which each $(x, y) \in \mathbb{Z}^2$ is adjacent to $(x\pm 1, y\pm 1)$ and the above 4 points. This very well known theory is discussed in [5] and [12], and many other places.

Given an adjacency A on X and a subset S of X, *an A-path in S (from y to z),* is a finite sequence $x_1, \ldots, x_n \in S$ such that for each $1 \leq k < n$, $(x_k, x_{k+1}) \in A$ (and $y = x_1,\ z = x_n$). The subset S is *A-connected* if for each $y, z \in S$, there is an A-path in S from y to z. An *A-component* is a maximal A-connected subset. Further, an *an A-arc* is an A-path $x_1, \ldots, x_n$ such that whenever $1 \leq k, m \leq n$ and $(x_k, x_m) \in A$, then $m = k\pm 1$, and an *A-Jordan curve* is an A-arc, except that $(x_n, x_1) \in A$.

But adjacencies that seem to respect nearness need not mirror topological reality. For example, Figure 1 shows well-known, easy examples of a 4-Jordan curve whose complement has 3 4-components, and an 8-Jordan curve whose complement is 8-connected.

But: if $\{k, m\} = \{4, 8\}$ then whenever J is a k-Jordan curve, then $\mathbb{Z}^2 \setminus J$ has exactly two m-components. This suggests the care needed in selecting an adjacency to represent Euclidean space.

With the help of the earlier discussion, we discuss the solution of putting a topology on the finite computer screen which behaves like that on a the rectangle in the plane that it is supposed to represent. This raises several issues:

Finite T_1-spaces are discrete (each singleton is the finite intersection of the complements of the other singletons; thus singletons are open, and therefore all sets are open). Thus they can't be connected if they have more than one point.

When a space (X, τ) isn't T_1, its specialization order becomes important. For us, the specialization is centrally important; it will be the tool for writing algorithms which, by Theorem 1, fully represent the topology of the space. It isn't difficult to see that if $\leq$ is any preorder, then the collection of $\leq$-upper sets, $\alpha(\leq)$, is an Alexandroff topology, and by Theorem 1 (a), for each Alexandroff space, $\tau = \alpha(\leq_\tau)$.

For the moment, we take dimension in its most trivial sense: an object will surely be k-dimensional if it is the product of k 1-dimensional objects. The computer screen certainly looks like the product of two such spaces – in fact, it looks like the product of two intervals. Recall that a topological space is *connected* if whenever $A \subseteq X$ is both open and closed, then $A = X$ or $A = \emptyset$. We take the following to be the essence of 1-dimensionality in $\mathbb{R}$ and intervals: a *connected ordered topological space (COTS)* is a connected space such that among any three points is one whose deletion leaves the other two in separate components of the remainder. Certainly the reals and intervals have this property; $\mathbb{R}^2$ doesn't since the deletion of any singleton leaves the remainder connected. But figure 2 shows a finite COTS.

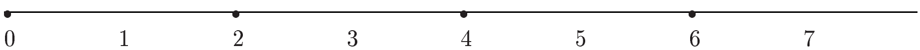


Fig. 2. A COTS with 8 points: 4 open, 4 closed.

The diagram uses two conventions which enable us to draw "Euclidean" pictures and interpret them as finite T_0-spaces:

- apparently featureless sets represent points,
- sets which 'look' open are open.

Figure 3 below uses these conventions, to show products of 2 and 3 COTS, looking appropriately 2 and 3-dimensional.

The computer screen seems reasonably, to be the product of two long finite COTS; in it, the open points can be seen (are the 'pixels') and the others are invisible addresses that might be used in programs. (In fact, would it be reasonable to think of space as the product of 3 long finite COTS?)

These diagrams suggest that COTS are natural 1-dimensional spaces. Here is a theorem which reinforces that idea:

Theorem 2. *A topological space X is a COTS if and only if there is a linear order $<$ on X such that for each $x \in X$, (x, ∞) [2] and $(-\infty, x)$ are the two components of $X \setminus \{x\}$. In this case there are exactly two such total orders, the other being $<^{-1}$.*

In $\mathbb{Z}$ or $\mathbb{R}$, the orders which satisfy Theorem 2 are the usual order and its reverse; note that the specialization order, $\leq_{\mathbb{Z}}$, discussed after Proposition 2, is quite differenct, relating only adjacent numbers (and not all of them). Although we haven't assumed any separation, the following result tells us that our spaces are T_0, and shows the generality of Figure 2:

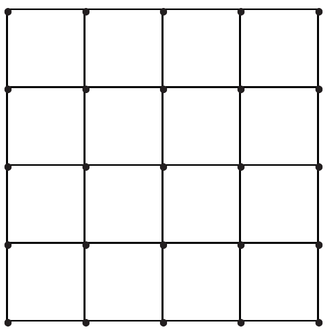
Proposition 1. *For a COTS at least 3 points:*

(a) Each point is open or closed, but never both. The space is T_0.

(b) Distinct points $x, y \in X$ are adjacent (with respect to $<$) if and only if $\{x, y\}$ is connected.

(c) X is T_1 if and only if it has no adjacent points; in this case, X infinite.

[2] For $x, y \in X$, $(x, \infty) = \{z \mid x < z\}, (-\infty, y) = \{z \mid z < y\}$, and $(x, y) = (x, \infty) \cap (-\infty, y)$.



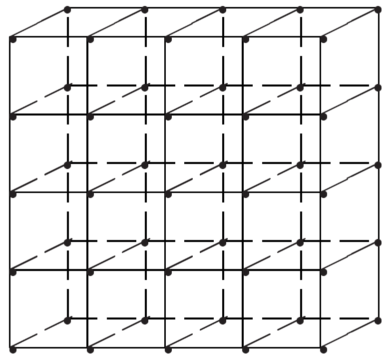


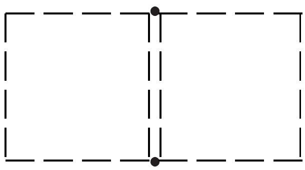
Fig. 3. A product of 2 9-point COTS A $9 \times 9 \times 3$ "3-space".

Proposition 2. *The set $\mathbb{Z}$ of integers, with the smllest topology in which each $\{2n-1, 2n, 2n+1\}$, $n \in \mathbb{Z}$ is open, is a T_0 COTS such that each finite T_0 COTS is homeomorphic to some (x, y), $x, y \in \mathbb{Z}$.*

In fact, the numbers in Figure 2 indicated one of many ways that finite COTS could be imbedded in $\mathbb{Z}$. The space of Proposition 2 is often called the *Khalimsky line.* In it, a set T is open if and only if whenever it contains an even number, it contains the odd numbers adjacent, $2n \in T \leftrightarrow 2n-1, 2n+1 \in T$. that is, $2n \in T \leftrightarrow 2n-1, 2n+1 \in T$. Thus a set C is closed if and only if, whenever it contains an odd number, it contains the two even numbers adjacent, that is, $2n+1 \in T \leftrightarrow 2n, 2n+2 \in T$. As a result, $x \leq_{\mathbb{Z}} y$ if and only if $x = y$, or for some n, $x = 2n \& y = 2n \underline{+} 1$. By Lemma 2 (c), the specialization in *diital n-space,* $\mathbb{Z}^n$, is found coordinatewise by the rule: for $x, y \in \mathbb{Z}^n$, $x \leq_{\mathbb{Z}^k} y$ if and only if for each $i = 1, \ldots, k$, $x_i = y_i$, or for some n, $x_i = 2n \& y_i = 2n \underline{+} 1$.

With Theorem 1 (b) and the usefulness of adjacencies in mind, we define *the adjacency $A(\tau)$ induced by τ* by $(x, y) \in A(\tau)$ if $\{x, y\}$ is a set connected in τ (that is, if and only if, $x \leq_\tau y$ or $y \leq_\tau x$), and x, y are distinct. We also let $\mathcal{A}(p)$ denote the set of points which are $A(\tau)$-adjacent to p. Note that this adjacency depends only on the topological space, and not on the "background" and "foreground". In $\mathbb{Z}^k$, for example, $\mathcal{A}(p)$ depends on how many of the coordinates are odd and how many are even. For example, if both coordinates are even: $\mathcal{A}(2n, 2m) = \mathsf{cl}\{(2n, 2m)\} \cup \mathsf{n}\{(2n, 2m)\} \setminus \{(2n, 2m)\} = \{(2n, 2m)\} \cup \{2n-1, 2n, 2n+1\} \times \{2m-1, 2m, 2m+1\} \setminus \{(2n, 2m)\}$, the points 8-adjacent to $(2n, 2m)$, and similarly (but exchanging the roles of cl, n) each $\mathcal{A}(2n+1, 2m+1)$ (both coordinates odd), is again the set of points 8-adjacent to $(2n+1, 2m+1)$. For a point where 1 coordinate is even (the other odd), we have $\mathcal{A}(2n+1, 2m) = \mathsf{cl}\{(2n+1, 2m)\} \cup \mathsf{n}\{(2n+1, 2m)\} \setminus \{(2n, 2m)\} = \{2n+1\} \times \{2m-1, 2m, 2m+1\} \cup \{2n, 2n+1, 2n+2\} \times \{2m\} \setminus \{(2n+1, 2m)\}$, the points 4-adjacent to $(2n+1, 2m)$. Figure 4 below illustrates some typical cases in $\mathbb{Z}^2$, $\mathbb{Z}^3$.

We then have the notions of τ-path, etc., and:



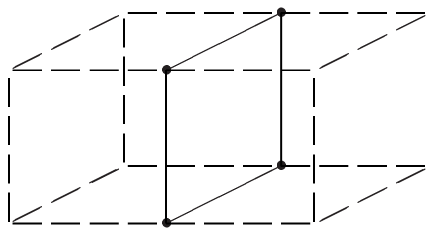


Fig. 4. $\mathcal{A}(2,1)$ $\mathcal{A}(1,2,1)$.

Proposition 3. *Let (X,τ) be an Alexandroff space.*

(a) A subset $S \subseteq X$ is an $A(\tau)$-path if and only if it is the continuous image of a COTS (equivalently, of an interval in $\mathbb{Z}$). It is an $A(\tau)$-arc if and only if it is a COTS.

(b) A subset $S \subseteq X$ is connected if and only if it is $A(\tau)$-connected (also, if and only if for each $x, y \in S$ there is an $A(\tau)$-arc in S from x to y).

(c) If $J \subseteq \mathbb{Z}^2$ is a Jordan curve then $\mathbb{Z}^2 \setminus J$ has two connected components.

Boundary-tracking is another concern of digital topology. The plane is often about a million pixels, and a region in it has comparable magnitude, but a relatively straight boundary might be a few thousand bytes in size. So considerable savings in storage is often achieved by replacing regions by their boundaries. Not all Jordan curves are closed sets, so not all can be boundaries, and not every set has as its boundary a Jordan curve. (Examples: the boundary of the set of closed points is the set itself, and that of the set of open points is its complement.) But these issues are overcome in a natural way:

A set S is *regular* if $\mathsf{int}(\mathsf{cl}(S)) \subseteq S \subseteq \mathsf{cl}(\mathsf{int}(S))$. A *robust scene* is a partition of $\mathbb{Z}^2$ into regular sets whose interiors are connected. A *cartoon* is a finite union of Jordan curves. Then (see [8]):

Theorem 3. *(a) For any finite $S \subseteq \mathbb{Z}^2$, ∂S is a (closed) Jordan curve if and only if S is regular and $\mathsf{int}(S)$, $\mathsf{int}(\mathbb{Z}^2 \setminus S)$ are both connected.*

(b) The union of the boundaries of the sets in a robust scene is a cartoon, and every cartoon is such a union.

Although we have only discussed the two-dimensional case, most of these results extend to arbitrary (finite) dimensions. An important fact however, is that while the proofs in the two-dimensional case are all appropriately digital (carried out, for example, by induction on the lengths of the shortest paths with certain properties), those now known in higher dimensions require uses of other techniques.

Problem 1: Find digital proofs in higher dimensions.

There are algorithms written in terms of the topological adjacency, but in overwhelming number, they are in terms of the traditional adjacencies and some newer ones that have the advantage of providing a great deal of guidance by being "small" – since boundaries are traced by going from point to adjacent point, adjacencies in which few points are adjacent require fewer steps to carry out. Thus the best that can be hoped, is:

Problem 2: Are the sound algorithms in digital topology those that can be shown sound by comparison to some finite T_0-space?

For example, soundness of the $(4, 8)$, $(8, 4)$ and $(6, 6)$ algorithms can be shown this way (see [13]).

4 Comparing to Polyhedra

The following basic tool is developed in [10], from which most results in this section come.

Definition 4. *A metric analog of a topological space X with base point x_0, is a metric space M with base point m_0, together with an open quotient map $q : M \to X$, such whenever A is a metric space with base point a_0:*

for any map $f : A \to X$ there is a map $\hat{f} : A \to M$ such that $f = q\hat{f}$.

for any maps $f, g : A \to M$ so that $qf = qg$ there is a homotopy $F : A \times [0, 1] \to M$ such that whenever $x \in A$ and $t, u \in [0, 1]$: $F(x, 0) = f(x)$; $F(x, 1) = g(x)$; $F(a_0, t) \equiv m_0$, and for each $qF(x, t) = qF(x, u)$ ($t \to qF(x, t)$ is constant).

Composition by the open quotient q induces a bijection between the path components (see [6]) of M and those of X, and this composition induces isomorphisms between the homotopy groups of M and those of X; that is to say, q is a weak homotopy equivalence between M and X.

A homotopy which, like the above, has the property that $t \to qF(x, t)$ is constant, is said to *ignore* the quotient q. Further, suppose (M, q) is a pair such that M is a metric space and $q : M \to X$ is an open quotient, and suppose A is a metric space; then for any maps $F, G : A \to M$, F and G are *quotient homotopic* if there is a homotopy $H : A \times [0, 1] \to M$ between F and G such that $qH(x, t) = qH(x, 0) = qH(x, 1)$ for all x in A and t in $[0, 1]$; this relation is denoted $F \simeq G$.

Theorem 4. *Each T_0 countable join*[3] *of Alexandroff topologies has a metric analog.*

By the following, any two metric analogs of the same space are homotopy equivalent. (Below, let $\mathbf{1}_A$ denote the identity map on A.)

Theorem 5. *Suppose (M, q) is a metric analog of a space X. If (N, r) is another metric analog of X, then there are maps $F : M \to N$ and $G : N \to M$ such that $GF \simeq \mathbf{1}_M$ and $FG \simeq \mathbf{1}_N$.*

Conversely, if N is a metric space, $r : N \to X$ is an open quotient, and there are maps $g : M \to N$, $h : N \to M$ so that $gh \simeq \mathbf{1}_N$ and $hg \simeq \mathbf{1}_M$, then (N, r) is metric analog of X.

[3] The join of a collection of topologies is the smallest topology containing them all.

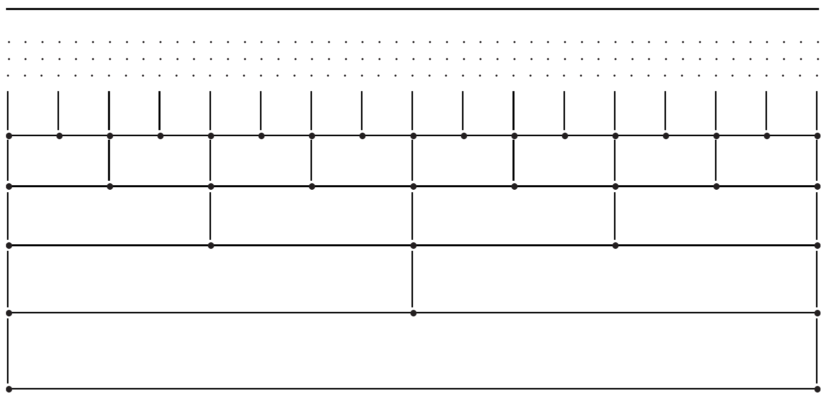


Fig. 5. Approximation of the unit interval by finite COTS.

The converse is useful in creating other metric analogs from a given one. In particular, it is used in showing the existence, for each finite T_0 space K, of a *polyhedral analog:* a subset $|K|$ of a finite dimensional Euclidean space, with a vertex for each point in K, and whose simplices are the convex hulls of the specialization order chains in K, together with the quotient map which takes each point of this metric space into the specialization-largest vertex of the smallest simplex in which the point lies.

Two results shown using polyhedral analogs are the Jordan surface theorem for three-dimensional digital spaces and that the product topology on $\mathbb{Z}^n$ is the only simply-connected one whose connected sets include all $2n$-connected sets but no $3^n - 1$-disconnected sets.

This last result (of [9]) is a two-edged sword: it gives a complete representation of topological adjacencies that emulate finite dimensional Euclidean space topologies. In doing so, it points out their scarcity among all adjacencies. There are other adjacencies which emulate many of the properties of Euclidean space, and give rise to faster algorithms.

5 Finite Approximation of Compacta

Now we will use finite spaces to approximate others. Figure 5 illustrates such an approximation and motivates the mathematics that is needed. Its top horizontal line represents the unit interval, but those at the bottom are meant to be finite COTS: $D_n = \{\frac{i}{2^n} \mid 0 \leq i \leq 2^n\} \cup \{(\frac{i}{2^n}, \frac{i+1}{2^n}) \mid 0 \leq i < 2^n\}$, with $2^{n+1} + 1$ points and the quotient topology induced from $[0, 1]$. The vertical lines indicate maps going down, for which a closed point is the image of the one directly above it, while an open point is that of the three above it.

Recall that a topological space X is *compact* if whenever $X = \bigcup G$ for some collection of open sets, then there is a finite subcollection $H \subseteq G$ such that $X = \bigcup H$. It is *Hausdorff (T_2)* if whenever $x \neq y$ there are $T, U \in \tau$ such that $x \in T$, $y \in U$ and $T \cap U = \emptyset$.

The following result has long been known ([Al]):

Theorem 6. *A T_2 space X is compact, if and only if there is an inverse system of finite spaces and continuous maps such that X is the largest T_2 continuous image of the limit of the system.*

The largest T_2 continuous image of a space is called its *Hausdorff reflection.* Also, recall that an inverse system of topological spaces and continuous maps is a directed set $(\Gamma, \leq)$ together with a space X_γ for each $\gamma \in \Gamma$ and whenever $\delta \geq \gamma$, a continuous $f_{\delta\gamma} : X_\delta \to X_\gamma$, such that each $f_{\gamma\gamma} = \mathbf{1}_{X_\gamma}$ and if $\delta \geq \gamma \geq \beta$ then $f_{\delta\beta} = f_{\gamma\beta} f_{\delta\gamma}$. Its inverse limit (unique to homeomorphism) is an X_Γ, together with, for each α, $p_\alpha : X_\Gamma \to X_\alpha$, such that for whenever $\alpha \geq \beta$, $p_\beta = f_{\alpha\beta} p_\alpha$, and minimal among such spaces, in that whenever we have a Y and for each α, a $g_\alpha : Y \to X_\alpha$ such that $\alpha \geq \beta$, $g_\beta = f_{\alpha\beta} g_\alpha$, then there is a unique $g : Y \to X_\Gamma$ such that for each α, $g_\alpha = p_\alpha g$. This inverse limit can be represented as the subspace of the product $\prod_{\gamma\in\Gamma} X_\gamma$ whose elements are those x in the product such that whenever $\alpha \geq \beta$, $x_\beta = f_{\alpha\beta}(x_\alpha)$.

In the case of the diagram above, the inverse limit is essentially $[0, 1] \cup \{d^+ \mid d = m/2^n,\ 0 \leq m < 2^n\} \cup \{d^- \mid d = m/2^n,\ 0 < m \leq 2^n\}$, where $d^{\pm}(k) = (m{\pm}1)/2^n$. This space is rarely Hausdorff, thus rarely the X we set out to approximate. It is for this reason that we need to use the Hausdorff reflection.

We now look at cases of this construction that are sufficiently general to study all compact Hausdorff spaces, but relatively easy to understand; these are studied in [16], [15] and [14] (related earlier constructions can be found in [1], [2], [4] and [3]). First we look at the method used to get the inverse system, which dates from [Al] and is in our notation in [KW]. Suppose (X, τ) is our compact Hausdorff space. Whenever F is a finite set of open sets, we get a partition of X into a finite number subsets: for each of the finite number of subsets G of F, let $P_G = \{x \in X \mid \text{for } T \in F,\ x \in T \Leftrightarrow T \in G\}$. Let $X_F = \{P_G \neq \emptyset\}$, with the map $\pi_F : X \to X_F$ defined by $f(x)$ be the element of the partition in which x lies. Also, let τ_F be the quotient topology resulting from π_G (that is, $U \in \tau_F \Leftrightarrow \pi_F^{-1}[U]$ is open in X. Each X_F is a T_0 space. Also, we get increasingly fine partitions of X by taking more and more open sets; that is, if $F \subseteq F'$, then $f_{F'F}(P'_G) = P_{F\cap G'}$ defines a map $f_{F'F} : X_{F'} \to X_F$, such that $\pi_F = f_{F'F}\pi_{F'}$. Certainly, $\{F \subseteq \tau \mid F \text{ finite}\}$ is directed by $\subseteq$, and it can be checked that $f_{FF} = \mathbf{1}_{X_F}$ and the $f_{F'F}$ are continuous maps such that if $F \subseteq F' \subseteq F'$ then $f_{F"F} = f_{F'F} f_{F"F'}$. Thus this method of considering partitions by larger and larger finite sets of open sets, yields a natural inverse system of finite spaces and maps.

The above has been refined to cases that are easy to handle, but the refinement is best understood if we work with *bitopological spaces:* sets with two topologies (X, τ, τ^*). A bitopological space is *pseudoHausdorff (pH)* if whenever $x \notin \mathsf{cl}_\tau(y)$ then there is a $T \in \tau$ and $U \in \tau^*$ which are disjoint and such that $x \in T$ and $y \in U$. It is *pairwise* Q if both it and its *dual,* (X, τ^*, τ), are Q. It is *joincompact* if it is pairwise pH and the join, $\tau \vee \tau^*$ is compact and T_0.

A topological space (X, τ) is *skew compact* if there is a second topology τ^* on X such that (X, τ, τ^*) is joincompact.

For example, if $X = [0,1]$ then $\tau = \{(a,1] \mid 0 \leq a \leq 1\} \cup \{X\}$ is skew compact, using $\tau^* = \{[0,a) \mid 0 \leq a \leq 1\} \cup \{X\}$, and each compact Hausdorff space is skew compact, with $\tau^* = \tau$.

In what follows, $\mu(X)$ will denote the set of specialization-minimal elements of X - that is, those $x \in X$ such that $\{x\}$ is closed. Further, m will denote the relation $\{(x,y) \mid y \in \mathsf{cl}(\{x\}),\ \{y\} \text{ closed}\}$.

Proposition 4. *Suppose X is skew compact.*

(a) $\mu(X)$ is a compact subspace of (X,τ).

(b) If each $x \in X$ lies above a unique element $m_x \in \mu(X)$, then m is a continuous map from (X,τ) onto $(\mu(X), \tau|\mu(X))$.

(c) Suppose $T \cap U \neq \emptyset$ whenever $x \in T, y \in U$ and $T, U \in \tau$. Then there is a $z \in X$ such that $x, y \in \mathsf{cl}(z)$.

(d) If each element of X has a unique minimal element in its closure, then $\mu(X)$ is a Hausdorff subspace of (X,τ).

A topological space is *normal* if disjoint closed sets are contained in disjoint open sets.

Theorem 7. *The following are equivalent for a skew compact space X:*

(a) X is normal,

(b) Each point of X has a unique closed point in its closure,

(c) m is a retract from X to its subspace $\mu(X)$.

If any of these hold, then $(\mu(X), m)$, is the Hausdorff reflection of (X,τ).

While there are many finite normal spaces, normality is best built up in the approximation:

Definition 5. *Suppose that X and Y are T_0-spaces; we say that a map $f : X \to Y$ is:*

normalizing if inverse images of disjoint closed sets are contained in disjoint open sets,

chaining if $f[\,\mathsf{cl}\{x\}]$ is a specialization chain for each x.

Then a space X is normal if and only if the identity map $\mathbf{1}_X$ on X is a normalizing map.

An inverse system of topological spaces and continuous maps $(X_\alpha, f_{\beta\alpha})$ whose inverse limit is X, is *eventually normalizing (resp. chaining)* if for each $\alpha \in I$ there is some $\gamma \geq \alpha$ such that $f_{\gamma\alpha}$ is normalizing (resp. chaining).

Theorem 8. *(a) The limit of an inverse system of finite T_0-spaces and continuous maps is normal if and only if the system is eventually normalizing.*

(b) Each compact Hausdorff space is the Hausdorff reflection of the inverse limit of a spectrum of an eventually chaining inverse system of finite T_0-spaces and continuous maps. Also, every chaining map is normalizing, so the same holds for normalizing maps.

The *simplicialization* of a finite T_0-space X is the set X^C of nonempty chains (totally ordered subsets) of $(X, \leq)$, with the Alexandroff topology $\mathcal{A}(\subseteq)$ whose specialization order is containment (that is, if $S, T \in X^C$, then $S \in \mathsf{cl}\{T\}$ if and only if $S \subseteq T$). Define the *simplicial quotient* $p_X : X^C \to X$, by $p_X(S) = \max(S)$.

Proposition 5. *The map $p_X : X^C \to X$ is continuous, open, and chaining. Furthermore, a continuous map $f : X \to Y$ is chaining if and only if there is a continuous map $\tilde{f} : X \to Y^C$ such that $p_Y \tilde{f} = f$. Finally, if $h : X \to Y^C$ is closed and $p_Y h = f$ then $h = \tilde{f}$.*

A *calming map* is a chaining map f for which $\tilde{f}$ is a closed map.

Compact Hausdorff spaces are also ofen approximated using polyhedra (see the survey [17]; the following relates our approach to this:

Theorem 9. *Suppose (X_n, f_n) is an inverse sequence of finite T_0-spaces and calming maps. Then the limit of the $(|X_n^C|, |f_n^C|)$ is homeomorphic to the space of minimal points of the limit of the (X_n, f_n).*

Corollary 1. *(a) A metrizable space is compact if and only if it is the Hausdorff reflection of the limit of an inverse sequence of finite T_0-spaces and calming maps.*

(b) Under these conditions, our space is $\leq k$-dimensional if and only if these finite spaces can be assumed $\leq k$-dimensional, and is connected and only the finite spaces can be assumed connected.

6 Summary and Further Indicated Work

Of course, the topological spaces that can be completely stored and studied in a computer are finite. These spaces can be completely analyzed using the specialization order, $x \leq_X y \Leftrightarrow x \in \mathsf{cl}(y)$, and this “asymmetric adjacency gives rise to an adjacency, defined by: for $x \neq y$, $(x, y) \in A_X \Leftrightarrow x \leq_\tau y$ or $y \leq_\tau x$.

The traditional adjacencies, $(4, 8)$, $(8, 4)$, and $(6, 6)$, and their n-dimensional analogues can be used to study $\mathbb{Z}^n$, and have been shown to capture the notions of connectedness and boundary quite well. But by their definitions, $\leq_{\mathbb{Z}^n}$ perfectly captures all of the properties of these spaces, and determines the adjacency $A_{\mathbb{Z}^n}$, with which boundary tracking and other traditional algorithms (typically written in terms of the traditional adjacencies) can be written. Further, the latter need not be adjusted to take into account the background and foreground. It should be repeated that there are “sparse” (nontopological, and typically nonsymmetric) adjacencies which limit the number of choices available and thus can result in faster execution times.

However, all compact Hausdorff spaces arise by approximation using finite T_0-spaces. These finite spaces can be completely analyzed as partially ordered sets, using their specializations $\leq_X$ and, algorithms in terms of this relation work well for them as they do for the traditional digital n-spaces that arise in image processing. Note that it is easy to find spaces for which “boundary tracking” is a useless idea; for example, in the two-dimensional space on the left hand side of Figure 3, imagine that none of the points both of whose coordinates are odd are in the space (so it represents a "graph paper" grid). Then almost no boundaries are connected, and none can be tracked. On the other hand, they can still be found, and can be useful in storing sets.

More must be learned about this approximation; we know, for example that dimension is preserved in the approximation of spaces, and are presently working to find how homotopy and homology are preserved. We are also studying how to best represent functions between spaces in terms of finite approximation.

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