Polychromatic Colorings of n-dimensional Guillotine-Partitions

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Abstract. A *strong hyperbox-respecting coloring* of an n-dimensional hyperbox partition is a coloring of the corners of its hyperboxes with 2^n colors such that any hyperbox has all the colors appearing on its corners. A *guillotine-partition* is obtained by starting with a single axis-parallel hyperbox and recursively cutting a hyperbox of the partition into two hyperboxes by a hyperplane orthogonal to one of the n axes. We prove that there is a strong hyperbox-respecting coloring of any n-dimensional guillotine-partition. This theorem generalizes the result of Horev et al. [8] who proved the 2-dimensional case. This problem is a special case of the n-dimensional variant of *polychromatic colorings*. The proof gives an efficient coloring algorithm as well.

1 Introduction

A k-coloring of the vertices of a plane graph is *polychromatic* (or *face-respecting*) if on all its faces all k colors appear at least once (with the possible exception of the outer face). The polychromatic number of a plane graph G is the maximum number k such that G admits a polychromatic k-coloring, we denote this number by $\chi_f(G)$. For an introduction about polychromatic colorings see for example the introduction of [2] or [4]. We restrict ourselves to a brief introduction to this topic and list some results. Alon et al. [2] showed that if g is the length of a shortest face of a plane graph G, then $\chi_f(G) \geq \lfloor (3g-5)/4 \rfloor$. (clearly $\chi_f(G) \leq g$), and showed that this bound is sufficiently tight. Mohar and Škrekovski [10] proved using the four-color theorem that every simple plane graph admits a polychromatic 2-coloring, later Bose et al. [3] proved that without using the four-color theorem. Horev and Krakovski [9] proved that every plane graph of degree at most 3, other than K_4 admits a polychromatic 3-coloring. Horev et al. [7] proved that every 2-connected cubic bipartite plane graph admits a polychromatic 4-coloring. This result is tight, since any such graph must contain a face of size four.

We define a *rectangular partition* as a partition of an axis-parallel rectangle into an arbitrary number of non-overlapping axis-parallel rectangles, such that no four rectangles meet at a common point. One may view a rectangular partition as a plane graph whose vertices are the corners of the rectangles and edges are

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the line segments connecting these corners. Dinitz et al. [6] proved that every rectangular partition admits a polychromatic 3-coloring. A *guillotine-partition* is obtained by recursively cutting a rectangle into two subrectangles by either a vertical or a horizontal line. For this subclass of rectangular partitions Horev et al. [8] proved that they admit a polychromatic 4-coloring. Actually, they prove a stronger statement. We define a *strong rectangle-respecting coloring* of a rectangular partition R as a vertex coloring of R with four colors such that every rectangle of R has all four colors among the four corners defining it. This is clearly a polychromatic 4-coloring as well. For examples see Figure 1. They proved that such coloring exists for any guillotine-partition. Recently, Dimitrov et al. [4] proved that any rectangular partition admits a strong rectangle respecting coloring, using a theorem about plane graphs.

Our main result is a generalization of the result for guillotine-partitions for n dimensions. An *n-dimensional hyperbox* is an n-dimensional axis-parallel hyperbox. For us a *partition* of an n-dimensional hypercube or hyperbox is a partition to hyperboxes such that each corner vertex is a corner of 2 hyperboxes, except the corners of the original hypercube. Note that this definition differs a bit from the natural definition, where we would allow a vertex to be the corner of more than 2 hyperboxes. This is needed, as using the more natural definition even in the plane there are simple counterexamples for our main theorem. The hyperboxes of the partition are called the *basic hyperboxes.* A *guillotine-partition* is obtained by starting with a partition containing only one basic hyperbox and recursively cutting a basic hyperbox into two hyperboxes by a hyperplane orthogonal to one of the n axes. The structure of such partitions is widely investigated, used in the area of integrated circuit layouts and other areas. Guillotine-partitions are also the underlying structure of orthogonal *binary space partitions* (BSPs) which are widely used in computer graphics. In [1] Ackerman et al. determine the asymptotic number of structurally different guillotine-partitions, we refer to the introduction of the same paper for more on this topic.

A *strong hyperbox-respecting coloring* of a partition is a coloring of the corners of its basic hyperboxes with 2^n colors such that any basic hyperbox has all the colors appearing on its corners. Note that a corner belongs to two basic hyperboxes except the 2^n corners of the partitioned big hyperbox, which belong to only one basic hyperbox. The natural extension to n dimensions of a polychromatic coloring would be a coloring of the corners of its basic hyperboxes with 2^n colors such that any basic hyperbox has all the colors appearing on its boundary. Clearly, every strong hyperbox-respecting coloring has this property.

Theorem 1. *There is a strong hyperbox-respecting coloring of any n-dimensional guillotine-partition.*

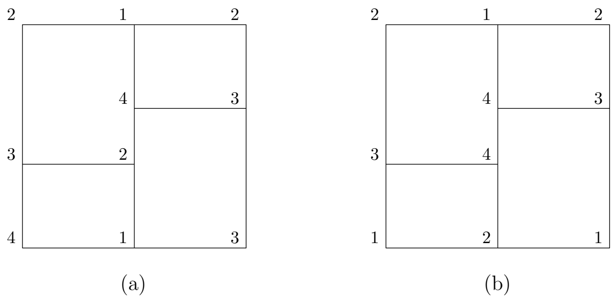

Fig. 1. a (a) polychromatic 4-coloring and (b) strong rectangle-respecting coloring of a guillotine partition

2 Proof of the main theorem

First we start with some definitions to be able to phrase the theorem we will actually prove, implying Theorem 1. We can assume w.l.o.g. that every hyperbox is a hypercube. Let us formulate this more precisely. We begin by introducing some notations. From now on $x = (x_1, x_2, \ldots, x_n)$, y, a, b, etc. always refer to some n-long 0-1 vector. We define the sum of two such vectors (denoted simply by $+$) as summing independently all coordinates mod 2. The $(0, 0, \ldots, 0)$ vector is denoted by $\mathbf{0}$ and the vector $(1, 0, 0, \ldots, 0)$ by e_1. A *face* is always an $(n-1)$-dimensional face of a hyperbox. Any axis-parallel n-dimensional hyperbox can be uniquely scaled and translated to be the hypercube with the two opposite corners being $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$. We refer to the corner of the hyperbox which maps into the point with coordinates $x = (x_1, x_2, \ldots, x_n)$ by $C(x)$. For some fixed $x \neq \mathbf{0}$ we define the *reflection* R_x being the function on the set of corners for which $R_x(C(y)) = C(x + y)$ for all y. Observe that $R_x(R_x(C(y))) = C(y)$ for any y.

From now on when we speak about a coloring of some hyperbox then it is always a hyperbox-respecting coloring. We say that a coloring of the corners of a hyperbox is an R_x-*coloring* ($x \neq \mathbf{0}$) if two corners $C(y)$ and $C(z)$ have the same colors if and only if $R_x(C(y)) = C(z)$ (or equivalently $R_x(C(z)) = C(y)$). Observe that such a coloring will have 2^{n-1} different colors appearing on the corners of the hyperbox, each occuring twice. Further, we say that the coloring of the corners is an $R_{\mathbf{0}}$-*coloring* if all corners are colored differently. Note that permuting the colors of an R_x-coloring gives another R_x-coloring for any x. From now on we will always restrict ourselves to these kinds of colorings. If any pair of such colorings could be put together along any axes to form another such coloring then it would already imply a recursive proof for the main theorem. As this is not the case we have to be more precise about our freedom of how to color a partition, making necessary to define sets of such colorings.

For any $x \neq \mathbf{0}$ S_x is defined as the union of all R_y for which $x \cdot y = 1$ (the scalar product of x and y mod 2). $S_{\mathbf{0}}$ is the one element set of R_0. If for some $x \neq \mathbf{0}$ for all $y \in S_x$ the hyperbox partition has a strong hyperbox-respecting coloring which is an R_y-coloring on its corners, we say that the hyperbox partition can be colored by the *color-range* S_x. If it has a strong hyperbox-respecting coloring which is an R_0-coloring on its corners, we say that the hyperbox partiton can be colored by the (one element) *color-range* S_0.

We will prove the following theorem, which implies Theorem 1.

Theorem 2. *Any n-dimensional guillotine-partition can be colored by some color-range.*

Proof. We proceed by induction on the number of guillotine-cuts of the partiton. The corners of a hyperbox containing only one basic hyperbox (i.e. the partition has 0 cuts) can be colored trivially with all different colors, thus colorable by color-range $S_{\mathbf{0}}$. In the general step we take a cut of the hyperbox B splitting it into two hyperboxes B_1 and B_2 with smaller number of cuts in them. Thus, by induction they can be colored by some color-ranges S_x and S_y for some x and y. We need to prove that there exists a z for which our hyperbox partition can be colored by S_z. First we prove this for the case when the cut is orthogonal to the first axis. Finally, we will prove that as the definition of R's and S's is symmetrical on every pair of axes, the claim follows for any kind of cut.
We regard the first axis (the one which corresponds to the first coordinate of points) as the usual x-axis, and so we can say that an object (corner, face, hyperbox etc.) is *left* from another if its first coordinates are smaller or equal than the other's (B_1 is left from B_2 for example). Similarly we can say *right* when its coordinates are bigger or equal than the other's.
We always do the following. Take an $R_a \in S_x$ and $R_b \in S_y$ and take a coloring of B_1 which is an R_a-coloring on its corners and a coloring of B_2 which is an R_b-coloring on its corners by induction such that the colors of the corners which should fit together (the right face of B_1 and the left face of B_2) have the same colors at the corners which will be identified. This is not always possible but when it is, it gives a coloring of B (the corners on the left face of B_1 and on the right face of B_2 are the corners of B). Note that we can permute the colors on the two hyperboxes in order to achieve such a fit of the colors. Clearly, the resulting coloring of B is a hyperbox-respecting coloring by induction. If the resulting coloring can be an R_c-coloring on the corners for some c then we write $R_a \cdot R_b \to R_c$. See Figure 2 and 3 for examples for 3 dimensions. The definition of $\to$ is good as the existence of such a fit depends only on the color of the corners. Observe that this operation is not commutative by definition and can hold for more than one c and has the hidden parameter that we put them together along the first axis (i.e. the two partitions are put together by the face which is orthogonal to the first axis). As we remarked earlier, if for any a and b there would be a c with $R_a \cdot R_b \to R_c$ then it would be enough to prove the main theorem by induction without defining color-ranges. As this is not the case we need to deal with color-ranges and define the function $\to$ on them as well.
We write $S_x \cdot S_y \to S_z$ if $\forall R_c \in S_z$ $\exists R_a \in S_x$ and $R_b \in S_y$ such that $R_a \cdot R_b \to R_c$. Clearly, we need to prove that there exists such a z for any choice of x and y that $S_x \cdot S_y \to S_z$. Lemma 2 states this. For the proof of this lemma we will first need to prove Lemma 1 about the behaviour of $\to$ for R's.

Finally, we need to prove that we can put together color-ranges along any axis. One can argue that we can obviously do that as the definition of color-ranges is symmetrical on any pair of coordinates and because of that analogs of Lemma 1 and Lemma 2 are true for an arbitrary axis. For a more rigorous argument see Lemma 3 and its proof in the Appendix.

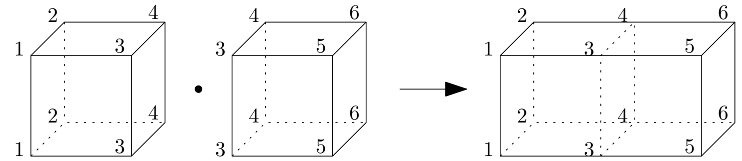

Fig. 2. Example to Lemma 1(c): $R_{010} \cdot R_{010} \rightarrow R_{010}$

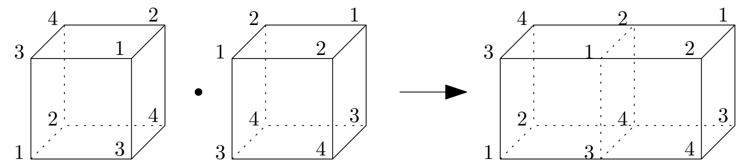

Fig. 3. Example to Lemma 1(d): $R_{110} \cdot R_{101} \rightarrow R_{111}$

Lemma 1 (fitting together colorings). *For $a, b, c \neq \mathbf{0}$ we have*
(a) $R_{\mathbf{0}} \cdot R_{\mathbf{0}} \rightarrow R_c$, if the first coordinate of c is 1,
(b) $R_a \cdot R_{\mathbf{0}} \rightarrow R_{\mathbf{0}}$ and $R_{\mathbf{0}} \cdot R_a \rightarrow R_{\mathbf{0}}$, if the first coordinate of a is 1,
(c) $R_a \cdot R_a \rightarrow R_a$, if the first coordinate of a is 0,
(d) $R_a \cdot R_b \rightarrow R_c$, if the first coordinate of a and b is 1 and $c = a + b + e_1$.

Proof. (a) Take an arbitrary c with its first coordinate being 1. For an R_c-coloring each color appearing on the corners of the hyperbox appears once on its left and once on its right face. We want to fit together two $R_{\mathbf{0}}$-colorings to have an R_c-coloring. Take an arbitrary $R_{\mathbf{0}}$-coloring of B_1. Take an R_0-coloring of B_2 and permute its colors such that the corners on its left face fit together with the corners on the right face of B_1. Now the set of colors on the right face of B_2 is the same set of colors as on the left face of B_1. After a possible permutation of these colors on B_2 we can get an R_c-coloring on B.
(b) Take an R_a-coloring of B_1 and an $R_{\mathbf{0}}$-coloring of B_2, permute the colors on B_2 such that the needed faces fit together. This can be done as R_a does not have a color appearing twice on its right face. As on B_1's left face the same set of colors appear as on its right face, B has all the colors of B_2's coloring appearing on its corners, thus it is an R_0-coloring of B. The proof for the other claim is

similar.
(c) Take an R_a-coloring of B_1. Take an R_a-coloring of B_2 and permute the colors on it such that the corners on its left face fit together with the corners on the right face of B_1 and on its right face all colors are different from the ones we used to color the corners of B_1. This can be done as both are R_a-colorings where the first coordinate of a is 0, so on the common face the same pair of corners need to have the same color. Similarly, we see that these fit together to form an R_a-coloring of B. For an illustration for 3 dimensions see Figure 2.
(d) Take an R_a-coloring of B_1. Take an R_b-coloring of B_2 and permute again the colors such that the corners on its left face fit together with the corners on the right face of B_1. This can be done as the corners on the right face of B_1 all have different colors and this is what we need on the left face of B_2 to make an R_b coloring (as the first coordinate of a and b is 1). Now it is enough to see that the resulting coloring of B is an R_c coloring with $c = a + b + e_1$ (recall that e_1 is the vector with all-0 coordinates except the first coordinate one being 1). Take an arbitrary corner on its left face, $C(d)$ (thus d has first coordinate 0). In the coloring of B_1 its pair (the corner with the same color) is $C(d + a)$. This is on the right face of B_1, and so it is fitted together with the corner $C(d + a + e_1)$ of B_2 on B_2's left face. By the R_b-coloring of B_2 the corner $C(d + a + e_1 + b)$ has the same color. This is also the $C(d + a + e_1 + b)$ corner of B. This holds for any corner of B on its left side and symmetrically on its right side as well, and so this is indeed an R_c-coloring of B. For an illustration for 3 dimensions see Figure 3. $\square$

Lemma 2 (fitting together color-ranges). *For $x, x', y \neq \mathbf{0}$ we have*
(a) $S_\mathbf{0} \cdot S_\mathbf{0} \rightarrow S_{e_1}$,
(b) $S_x \cdot S_\mathbf{0} \rightarrow S_\mathbf{0}$ and $S_\mathbf{0} \cdot S_x \rightarrow S_\mathbf{0}$,
(c) $S_x \cdot S_y \rightarrow S_{e_1}$, if x and y differ somewhere which is not the first coordinate,
(d) $S_x \cdot S_x \rightarrow S_{x'}$, if x' is the same as x with the possible exception at the first coordinate, which is 1 in x'.
(e) $S_x \cdot S_{x'} \rightarrow S_x$ and $S_{x'} \cdot S_x \rightarrow S_x$, if x' is the same as x except at the first coordinate, which is 0 in x and 1 in x'.

Proof. Let us recall first that $S_\mathbf{0}$ is the one element set of R_0 and for any $x \neq \mathbf{0}$ S_x is defined as the union of all R_y for which $x \cdot y = 1$.
(a) by Lemma 1(a) $R_\mathbf{0} \cdot R_\mathbf{0} \rightarrow R_c$ for any $c \cdot e_1 = 1$.
(b) In S_x ($x \neq \mathbf{0}$) there is always an R_a where the first coordinate of a is 1. By Lemma 1(b) $R_a \cdot R_\mathbf{0} \rightarrow R_\mathbf{0}$. The proof for the other claim is similar.
(c) We need to prove that for any $R_c \in S_{e_1}$ ($c \cdot e_1 = 1$) there is an $R_a \in S_x$ and $R_b \in S_y$ such that $R_a \cdot R_b \rightarrow R_c$.
Suppose x and y differ in the kth coordinate ($k \neq 1$). Define X as the set of coordinates l where $x_l = 1$, and Y the set of coordinates l where $y_l = 1$. We want to apply Lemma 1(d) which is symmetrical on a and b and so we can suppose that $k \notin X$ and $k \in Y$. The first coordinate of c is 1, so we choose a and b having the first coordinate 1 as well. We need that $a + b + e_1 = c$ to be able to apply Lemma 1(d). First define the coordinates of a being in X all zero except

one (this is the first if $1 \in X$, some other otherwise), thus by any choice of the other coordinates we will have $R_a \in S_x$. Now define the coordinates of b being in $X \setminus \{1\}$ such that $a_l + b_l = c_l$ for all $l \in X$. Define the rest of the coordinates of b such that $b \cdot y = 1$, this can be done as we can choose the kth coordinate as we want. Thus, $R_b \in S_y$ as well. Finally, choose the coordinates of a not in X such that $a_l + b_l = c_l$ for all $l \notin X$. This way $a + b + e_1 = c$ as needed.
(d) We need to prove that for any $R_c \in S_{x'}$ there is an $R_a \in S_x$ and $R_b \in S_x$ such that $R_a \cdot R_b {\rightarrow} R_c$.
First we prove the case when the first coordinate of x is 1 and so $x' = x$. For a c with first coordinate 0 by Lemma 1(c) we have $R_c \cdot R_c {\rightarrow} R_c$, all in S_x as needed. For a c with first coordinate 1 take an arbitrary a with first coordinate 1 and $R_a \in S_x$. Choose b such that $a + b + e_1 = c$ and so by Lemma 1(d) $R_a \cdot R_b {\rightarrow} R_c$ holds. We need that R_b is in S_x, which is true as $b \cdot x = (a+c+e_1) \cdot x = 1+1+1 = 1$.
Now we prove the case when the first coordinate of x is 0 and so $x' = x + e_1$. For a c with first coordinate 0 by Lemma 1(c) we have $R_c \cdot R_c {\rightarrow} R_c$, all in S_x and in S'_x too (as for such a c we have $c \cdot x = c \cdot x' = 1$). For a c with first coordinate 1 take an arbitrary a with first coordinate 1 and $R_a \in S_x$. Choose b such that $a+b+e_1 = c$ and so by Lemma 1(d) $R_a \cdot R_b {\rightarrow} R_c$ holds. We need that R_b is in S_x, which is true as $c \cdot x' = 1$, $c \cdot e_1 = 1$ and so $b \cdot x = (a+c+e_1) \cdot x = 1 + c \cdot (x' + e_1) + 0 = 1$.
(e) For $S_x \cdot S_{x'} {\rightarrow} S_x$ we need to prove that for any $R_c \in S_x$ there is an $R_a \in S_x$ and $R_b \in S_{x'}$ such that $R_a \cdot R_b {\rightarrow} R_c$.
For a c with first coordinate 0 by Lemma 1(c) we have $R_c \cdot R_c {\rightarrow} R_c$, all in S_x and in S'_x too (as for such a c we have $c \cdot x = c \cdot x' = 1$). For a c with first coordinate 1 take an arbitrary a with first coordinate 1 and $R_a \in S_x$. Again, choose b such that $a + b + e_1 = c$ and so by Lemma 1(d) $R_a \cdot R_b {\rightarrow} R_c$ holds. We need that R_b is in S'_x, which is true as $b \cdot x' = (a + c + e_1) \cdot x' = (a + c + e_1) \cdot (x + e_1) = a \cdot x + c \cdot x + e_1 \cdot x + a \cdot e_1 + c \cdot e_1 + e_1 \cdot e_1 = 1 + 1 + 0 + 1 + 1 + 1 = 1$.
As Lemma 1(d) is symmetrical on a and b, $S_{x'} \cdot S_x {\rightarrow} S_x$ follows the same way. □

The Lemmas above conclude the proof of Theorem 2. □

3 Algorithm and remarks

Assuming we know the cut-structure of the partition, the proof yields a simple linear time algorithm (in the number of cuts, regarding the dimension n as a fixed constant) to give a strong hyperbox-respecting coloring. First we determine the color-ranges and then the colorings of the hyperboxes using the lemmas. We will sketch how to do that.
First we construct the rooted binary tree with its root on the top representing our guillotine-cuts (each node corresponds to a hyperbox, the leaves are the basic boxes, the root is the original hyperbox). From bottom to top we can determine for each node v the unique $s(v)$ for which the corresponding hyperbox will have color-range $S_{s(v)}$ (leaves have color-range $S_{\mathbf{0}}$, then it is easy to determine the rest going upwards using Lemma 2). Now from top to bottom we can give appropriate R_y-colorings to the hyperboxes. For the root w give arbitrary $R_{r(w)}$-coloring

with $r(w) \in S_{s(w)}$. Then by induction if we gave an $R_{r(w)}$-coloring ($r_w \in S_{s(w)}$) to some hyperbox corresponding to the node w with children u and v then by Lemma 2 there exists $r(u) \in S_{s(u)}$ and $r(v) \in S_{s(v)}$ such that an $R_{r(u)}$ and an $R_{r(v)}$ can be put together (at the appropriate face) to form an $R_{r(w)}$-coloring. Such colorings can be found in the same way as in the proof of Lemma 2. Thus, we can give such colorings to the hyperboxes corresponding to u and v. Finishing the coloring this way the basic boxes will have $R_{\mathbf{0}}$-colorings, i.e. the coloring will be a strong hyperbox-respecting coloring.
It is easy to see that using this algorithm any S_x color-range can appear with appropriate cuts.
It was observed by D. Dimitrov and R. Skrekovski [5] using a double-counting argument that when a (not necessary guillotine) partition contains an odd number of basic hyperboxes then a coloring of it must have all the corners colored differently. From Lemma 1 one can easily deduce that when the partition contains an odd number of basic hyperboxes then our algorithm will give an $R_{\mathbf{0}}$-coloring thus having all corners colored differently indeed. Further it was also observed that when a partition contains an even number of basic hyperboxes then all the colors appear pair times on the corners of the hyperbox. In the even case our algorithm will give an R_a-coloring with $a \neq \mathbf{0}$ thus having all colors appearing zero times or twice on the corners.
As mentioned in the Introduction, the general case is solved in 2-dimensions, but it is still unknown for which other dimensions can it hold.

Problem 1. For which $n > 2$ do exist a strong hyperbox-respecting coloring of any n-dimensional partition.

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Appendix

Define $\circ^i$ as the function on the 0-1 vectors which exchanges the first and the ith coordinates, i.e. for a vector x the vector x^i has the same coordinates except that $x^i_1 = x_i$ and $x^i_i = x_1$ (thus x^{ii} is the identity and x^i is a bijection). For vectors corresponding to corners of a hyperbox this is a reflection on a hyper-plane going through the corners having the same first and ith coordinate. Clearly, applying $\circ^i$ on an R_x-coloring of the corners we get an R_{x^i}-coloring of the corners. Lemma 3 states that the color-ranges S_x and S_y can be put together along the ith axis to give the color-range S_z if the color-ranges S_{x^i} and S_{y^i} can be put together along the first axis to give the color range S_{z^i}. We have seen this can be done for any x^i and y^i with some z^i, thus fitting along any other axis is also possible.

Lemma 3 (fitting together along a general axis). *If the color-ranges S_{x^i} and S_{y^i} can be put together along the first axis to give the color range S_{z^i} then the color-ranges S_x and S_y can be put together along the ith axis to give the color-range S_x.*

Proof. First we prove that if $R_{a^i} \cdot R_{b^i} \rightarrow R_{c^i}$ for some c then an appropriate R_a-coloring and R_b-coloring can be put together by the ith axis to form an R_c-coloring. For that take an R_{a^i}-coloring and an R_{b^i}-coloring which fit together along the first axis to form an R_{c^i}-coloring. Apply $\circ^i$ on these colorings. The original ones had the same colors on the pair of corners $C(v)$ on the first one and $C(v + e_1)$ on the second one for arbitrary v having first coordinate 1. Thus after applying $\circ^i$ their images, the pair of corners $C(w)$ and $C(w + e_i)$ (e_i is the vector with all-0 coordinates except the ith coordinate being 1), will have the same colors for arbitrary w with ith coordinate 1 and so we can put together the two colorings along the ith axis.
By assumption when putting together along the first axis, the result was an R_{c^i}-coloring. If $c = c^i = \mathbf{0}$ then it had all different colors on its corners, thus the same is true after applying $\circ^i$ and putting together along the ith axis, so the result is indeed an R_c-coloring.
Otherwise if c^i has first coordinate 0 then on the R_{a^i}-coloring the corners $C(v)$ and $C(v + c^i)$ had the same colors for any v with first coordinate 0 and on the R_{b^i}-coloring the corners $C(w)$ and $C(w + c^i)$ had the same colors for any w with

first coordinate 1. Thus after applying $\circ^i$, the corners $C(v)$ and $C(v+c)$ of the R_a-coloring have the same colors for any v with ith coordinate 0 and the corners $C(w)$ and $C(w+c)$ of the R_b-coloring have the same colors for any w with ith coordinate 1. As in this case the ith coordinate of c is 0, the resulting coloring after fitting these two together along the ith axis is indeed an R_c-coloring.
If c^i has first coordinate 1 then the corner $C(v)$ of the $R(a^i)$-coloring and the corner $C(v+c^i)$ of the $R(b^i)$-coloring had the same color for any v with first coordinate 0. Thus after applying $\circ^i$, the corners $C(v)$ of the $R(a)$-coloring and the corner $C(v+c)$ of the $R(b)$-coloring have the same colors for any v with ith coordinate 0. Putting these together along the ith axis gives indeed an R_c-coloring.

Finally, back to the hyperboxes colorable with color-ranges S_x and S_y which need to be put together along the ith axis, applying x^i on all the colorings of S_x we get S_{x^i} and similarly from S_y we get the color-range S_{y^i} and we can put these together by the first axis to get the color-range S_{z^i} for some z and so S_x and S_y can be put together by the ith axis to get the color-range S_z. $\square$