ERRORS AND PARADOXES IN QUANTUM MECHANICS

According to one definition, a paradox is a statement that seems self-contradictory or absurd but may be true; according to another, a paradox is a true self-contradiction and therefore false. Let us define paradox to be an apparent contradiction that follows from apparently acceptable assumptions via apparently valid deductions. Since logic admits no contradictions, either the apparent contradiction is not a contradiction, or the apparently acceptable assumptions are not acceptable, or the apparently valid deductions are not valid. A paradox can be useful in developing a physical theory; it can show that something is wrong even when everything appears to be right.

Paradoxes in physics often arise as →thought experiments. For example, to refute Aristotle's statement that a heavy body falls faster than a light one, Galileo [1] invented a paradox: Suppose, with Aristotle, that a large stone falls faster than a small stone. If the stones are tied together, the smaller stone will then retard the large one. But the two stones tied together are heavier than either of them. "Thus you see how, from your assumption that the heavier body moves more rapidly than the lighter one, I infer that the heavier body moves more slowly." Such free invention of paradoxes as thought experiments marks especially the development of twentieth century physics, i.e. of the relativity and quantum theories.

Both relativity theory and quantum theory are well supplied with paradoxes. In relativity theory, however, well known paradoxes such as the twin paradox have accepted resolutions. These paradoxes arise from intuitions, typically about simultaneity, that relativity theory rendered obsolete. By contrast, not all well known paradoxes of quantum theory have accepted resolutions, even today. Below we briefly review seven quantum paradoxes.

In keeping with our definition above, we do not distinguish between "apparent" and "true" paradoxes. But we distinguish between apparent and true contradictions. A true contradiction is a fatal flaw showing that a physical theory is wrong. By contrast, apparent contradictions may arise from errors; they may also arise from a conceptual gap in a theory, i.e. some ambiguity or incompleteness that is not fatal but can be removed by further development of the theory. Thus we can classify [2] physics paradoxes into three classes: Contradictions, Errors and Gaps. The first three paradoxes below are examples of a Contradiction, an Error and a Gap, respectively.

1. By 1911, →Rutherford and his co-workers had presented striking experimental evidence (back-scattering of alpha particles) that neutral atoms of gold have cores of concentrated positive charge. According to classical electrodynamics, an atom made of electrons surrounding a positive nucleus would immediately collapse; but the gold foil in Rutherford's experiment evidently did not collapse. This contradiction between experimental evidence and classical theory was not merely apparent: it showed that atoms do not obey classical electrodynamics. Faced with this evidence, Bohr broke with classical theory and explained the stability of matter by associating →quantum numbers $n = 1, 2, 3, \ldots$ to the allowed orbits of electrons

in atoms. Although →Bohr's model described well only the hydrogen atom, quantum numbers characterize all atoms.

2. Einstein invented thought experiments to challenge Bohr's [3] principle of →complementarity. One thought experiment involved two-slit interference. (See Fig. 1.) Let a wave of (say) electrons of wavelength λ, collimated by a screen with a single slit, impinge on a screen with two slits separated by d. An electron interference pattern—dark lines with separation $D = \lambda L/d$—emerges on a third screen a distance L beyond the second. In Fig. 1, however, the experiment is modified to measure also the transverse recoil of the second screen (the screen with the two slits). Why the modification? According to Bohr, a setup can demonstrate *either* wave behavior (e.g. interference) of electrons *or* particle behavior (e.g. passage through a single slit), but not *simultaneous* wave and particle behavior; these two behaviors are complementary (→"wave-particle duality") and no setup can simultaneously reveal complementary behaviors. Einstein's modified experiment apparently shows electron interference while also revealing through which slit each electron passes (e.g. an electron passing through the right slit makes the screen recoil more strongly to the right) and thus contradicts the principle of complementarity.

To analyze the modified experiment, let $\mathbf{p}^{(L)}$ and $\mathbf{p}^{(R)}$ denote the momentum of an electron if it arrives at $\mathcal{P}$ via the left and right slits, respectively, and let $p_\perp^{(L)}$ and $p_\perp^{(R)}$ denote the respective transverse components. From a measurement of the change in transverse momentum p_s of the screen with accuracy $\Delta p_s \leq p_\perp^{(R)} - p_\perp^{(L)}$, we can infer through which slit an electron passed. But now apply →Heisenberg's uncertainty principle to the second screen:

$$\Delta x_s \geq h/\Delta p_s \geq h/[p_\perp^{(R)} - p_\perp^{(L)}] \quad ,$$

where x_s is the transverse position of the second screen. Similarity of triangles in Fig. 1(b) implies that $|\mathbf{p}^{(R)} - \mathbf{p}^{(L)}|$ (which equals $|p_\perp^{(R)} - p_\perp^{(L)}|$), divided by the electron's longitudinal momentum $p_\parallel$, equals d/L. The longitudinal momentum $p_\parallel$ is h/λ (assuming $p_\parallel$ large compared to the transverse momentum). Thus

$$\Delta p_s < \frac{d}{L}(h/\lambda) \quad .$$

We obtain $\Delta p_s < h/D$ and thus $\Delta x_s > D$. The uncertainty in the transverse *position* x_s of the screen, arising from an accurate enough measurement of its transverse *momentum* p_s, is the distance D between successive dark bands in the interference pattern, and so the interference pattern is completely washed out. Precisely when Einstein's thought experiment succeeds in showing through which slit each electron passes, it fails to show electron interference; that is, it *obeys* the principle of complementarity after all.

3. In 1931, Landau and Peierls [4] considered the following model measurement of the electric field $\mathbf{E}$ in a region. Send a charged test particle through the region; the electric field deflects the particle, and the change in the momentum $\mathbf{p}$ of the test particle is a measure of $\mathbf{E}$. But an accelerated, charged particle radiates, losing an unknown fraction of its momentum to the electromagnetic field. Reducing the

charge on the test particle reduces radiation losses but then $\mathbf{p}$ changes more slowly and the measurement lasts longer (or is less accurate). On the basis of their model, Landau and Peierls concluded that an instantaneous, accurate measurement of $\mathbf{E}$ is impossible. They obtained a lower bound $\Delta|\mathbf{E}| \geq \sqrt{\hbar c}/(cT)^2$ as the minimum uncertainty in a measurement of $|\mathbf{E}|$ lasting a time T. Their conclusion is paradoxical because it leaves the instantaneous electric field $\mathbf{E}$ with no theoretical or experimental definition. However, the Landau-Peierls model measurement is too restrictive. Bohr and Rosenfeld [5] found it necessary to modify the model in many ways; one modification was to replace the (point) test particles of Landau and Peierls with extended test bodies. In their modified model, they showed how to measure electric (and magnetic) fields instantaneously. Note that the electric field is not a *canonical* variable, i.e. it is not one of the generalized coordinates and momenta appearing in the associated Hamiltonian. (It depends on the time derivative of $\mathbf{A}$, the electromagnetic vector potential, which *is* a canonical variable.) The resolution of this sort of paradox is that quantum measurements of canonical and noncanonical variables differ systematically [6].

4. →*Zeno's paradoxes* are named for the Greek philosopher who tried to understand motion over shorter and shorter time intervals and found himself proving that motion is impossible. The *quantum* Zeno paradox [7] seems to prove that quantum evolution is impossible. Consider the evolution of a simple quantum system: a spin-1/2 atom precesses in a constant magnetic field. If we neglect all but the spin degree of freedom, represented by the →Pauli spin matrices σ_x, σ_y and σ_z, the Hamiltonian is

$$H = \mu B \sigma_z$$

where the direction of the magnetic field defines the z-axis and μ is the Bohr magneton. Suppose that at time $t = 0$ the state is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle + |\downarrow\rangle\right]$$

(where $\sigma_z|\uparrow\rangle = |\uparrow\rangle$ and $\sigma_z|\downarrow\rangle = -|\downarrow\rangle$). Solving →Schrödinger's equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi\rangle \quad ,$$

we obtain the time evolution:

$$\begin{aligned}|\psi(t)\rangle &= e^{-iHt/\hbar}|\psi(0)\rangle \\ &= \frac{1}{\sqrt{2}}\left[e^{-i\mu Bt/\hbar}|\uparrow\rangle + e^{i\mu Bt/\hbar}|\downarrow\rangle\right] \quad .\end{aligned}$$

At $t = 0$, a measurement of σ_x is sure to yield 1; at time $t = T \equiv h/4\mu B$, the σ_x measurement is sure to yield -1; at intermediate times, a measurement may yield either result.

At no time does a measurement of σ_x yield a value other than 1 and -1; the spin component σ_x apparently →jumps discontinuously from 1 to -1, defining a moment in time by jumping. *When* does the spin jump? We cannot predict when it will jump, but we can make many measurements of σ_x between $t = 0$ and $t = T$. The jump in σ_x must occur between two successive measurements. When it does, we will know when the jump occurred, to an accuracy Δt equal to the time between the measurements. But now we apparently violate the uncertainty relation for energy and time:

$$\Delta E \Delta t \geq \hbar/2 \quad .$$

Here E is the energy of the measured system and t is time as defined *by* the system. (Although t is not an operator, we can define t via an operator that changes smoothly in time, and then derive $\Delta E \Delta t \geq \hbar/2$ indirectly [8].) The problem is that the uncertainty ΔE in the energy cannot be greater than the difference $2\mu B$ between the two eigenvalues of H; but the measurements can be arbitrarily dense, i.e. Δt can be arbitrarily small.

Since quantum mechanics will not allow a violation of the uncertainty principle, we may guess that the atomic spin will simply refuse to jump! A short calculation verifies this guess. Consider N measurements of σ_x, at equal time intervals, over a period of time T. The interval between measurements is T/N. What is the probability of finding the spin unchanged after the first measurement? The state at time $t = T/N$ is

$$\frac{1}{\sqrt{2}} \left[e^{-i\mu BT/N\hbar} | \uparrow\rangle + e^{i\mu BT/N\hbar} | \downarrow\rangle \right] \quad ,$$

so the probability of finding the spin unchanged is $\cos^2(\mu BT/N\hbar)$. Hence the probability of finding the spin unchanged at time T, after N measurements, is $\cos^{2N}(\mu BT/N\hbar)$. As N approaches infinity, $\cos^{2N}(\mu BT/N\hbar)$ approaches 1: the spin never jumps. Here quantum evolution is impossible. But consider a dual experiment: instead of N measurements of σ_x on an atom in a magnetic field, consider N measurements of $\sigma_x \cos(2\mu Bt/\hbar) + \sigma_y \sin(2\mu Bt/\hbar)$, at equal time intervals, on an atom in no magnetic field ($H = 0$). In the limit $N \to \infty$, the atom precesses: each measurement of $\sigma_x \cos(2\mu Bt/\hbar) + \sigma_y \sin(2\mu Bt/\hbar)$ yields 1. Experiments from 1990 on have progressively demonstrated such quantum Zeno effects.

5. A thought experiment due to $\to$Einstein, Podolsky and Rosen [9] (EPR) shows how to measure precisely the position $\mathbf{x}_A(T)$ *or* the momentum $\mathbf{p}_A(T)$ of a particle A at a given time T, *indirectly* via a measurement on a particle B that once interacted with A. The measurement on B is spacelike separated from $\mathbf{x}_A(T)$, and so it cannot have any measurable effect on $\mathbf{x}_A(T)$ or $\mathbf{p}_A(T)$ (no superluminal signalling). It is indeed reasonable to assume ($\to$ "Einstein locality") that the measurement on B has no effect whatsoever on $\mathbf{x}_A(T)$ or $\mathbf{p}_A(T)$; thus $\mathbf{x}_A(T)$ and $\mathbf{p}_A(T)$ are simultaneously defined (in the sense that either is measurable without any effect on the other) and a particle has a precise position and momentum simultaneously. Since quantum mechanics does not define the precise position and momentum of a particle simultaneously, quantum mechanics does not completely describe particles. EPR envisioned a theory that would be *consistent* with quantum mechanics but more complete, just as statistical mechanics is consistent with thermodynamics but more complete.

Almost 30 years after the EPR paper, →Bell [10] proved a startling, and—to Bell himself—disappointing theorem: Any more complete theory of the sort envisioned by EPR would contradict quantum mechanics! Namely, the correlations of any such theory must obey →Bell'a inequality; but according to quantum mechanics, some correlations of →entangled states of particles A and B violate Bell's inequality. If quantum mechanics is correct, then there can be no theory of the sort envisioned by EPR. →Experiments have, with increasing precision and rigor, demonstrated violations of Bell's inequality and ruled out any theory of the sort envisioned by EPR.

6. In 1927, at the fifth Solvay congress, Einstein presented "a very simple objection" to the probability interpretation of quantum mechanics. According to quantum mechanics, the state of an electron approaching a photographic plate is an extended object; the probability density for the electron to hit varies smoothly over the plate. Once the electron hits somewhere on the plate, however, the probability for the electron to hit anywhere else drops to zero, and the state of the electron →collapses instantaneously. But instantaneous collapse of an extended object is not compatible with relativity. A related paradox is the following. Fig. 2 shows two atoms, prepared in an entangled state at O, flying off in different directions. (For simplicity, assume that they separate at nonrelativistic speeds.) One atom enters the laboratory of Alice, who measures a component of its spin at a; the other enters the laboratory of Bob, who measures a component of its spin at b. After Alice's measurement, the atoms are not in an entangled state anymore, hence collapse cannot occur anywhere outside the past light cone of a. Likewise, collapse cannot occur anywhere outside the past light cone of b. Hence collapse cannot occur anywhere outside the *intersection* of the past light cones of a and b. But then, in the inertial reference frame of Fig. 2, the state of the atoms just before either measurement is a product (collapsed) state, not an entangled state. Now this conclusion contradicts the fact that, by repeating this experiment on many pairs of atoms, Alice and Bob can obtain violations of Bell's inequality, i.e. can demonstrate that the atomic spins were in an entangled state until Bob's measurement. This paradox shows that there can be no Lorentz-invariant account of the collapse. In general, observers in different inertial reference frames will disagree about collapse. They will not disagree about the results of local measurements, because local measurements are spacetime events, hence Lorentz invariant; but they will have different accounts of the collapse of nonlocal states. Collapse is Lorentz *co*variant [11].

7. →Schrödinger's Cat is a paradox of quantum evolution and measurement. For simplicity, let us consider just the σ_z degree of freedom of spin-1/2 atoms and define a superposition of the two normalized eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$ of σ_z:

$$|\Psi_{\alpha\beta}\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle \quad ;$$

we assume $|\alpha|^2 + |\beta|^2 = 1$. The Born probability rule states that a measurement of σ_z on many atoms prepared in the state $|\Psi_{\alpha\beta}\rangle$ will yield a fraction approaching $|\alpha|^2$ of atoms in the state $|\uparrow\rangle$ and a fraction approaching $|\beta|^2$ of atoms in the state $|\downarrow\rangle$. If quantum mechanics is a complete theory, it should be possible to describe these measurements themselves using Schrödinger's equation. We can describe a measurement on an atom abstractly by letting $|\Phi_0\rangle$ represent the initial state of a measuring device, and letting $|\Phi_\uparrow\rangle$ or $|\Phi_\downarrow\rangle$ represent the final state of the measuring device if the state of the atom was $|\uparrow\rangle$ or $|\downarrow\rangle$, respectively. If the Hamiltonian for the measuring device and atom together is H, during a time interval $0 \leq t \leq T$ that includes the measurement, then the Schrödinger equation implies

$$e^{-i\int_0^T Hdt/\hbar}|\uparrow\rangle \otimes |\Phi_0\rangle = |\uparrow\rangle \otimes |\Phi_\uparrow\rangle \quad ,$$
$$e^{-i\int_0^T Hdt/\hbar}|\downarrow\rangle \otimes |\Phi_0\rangle = |\downarrow\rangle \otimes |\Phi_\downarrow\rangle \quad .$$

(The spin states do not change as they are eigenstates of the measured observable σ_z.) If the initial spin state is neither $|\uparrow\rangle$ nor $|\downarrow\rangle$ but the superposition $|\Psi_{\alpha\beta}\rangle$, the evolution of the superposition is the superposition of the evolutions:

$$e^{-i\int_0^T Hdt/\hbar}|\Phi_{\alpha\beta}\rangle \otimes |\Phi_0\rangle = \alpha|\uparrow\rangle \otimes |\Phi_\uparrow\rangle + \beta|\downarrow\rangle \otimes |\Phi_\downarrow\rangle \quad .$$

The right side of this equation, however, does not describe a completed measurement at all: the measuring device remains entangled with the atom in a superposition of incompatible measurement results. It does not help to couple additional measuring devices to this device or to the atom; since the Schrödinger equation dictates linear, unitary evolution, additional devices will simply participate in the superposition rather than collapse it. Even a cat coupled to the measurement will participate in the superposition. Suppose the measuring device is triggered to release poison gas into a chamber containing a cat, *only* if the spin state of the measured atom is $|\uparrow\rangle$. The state of the atom, measuring device and cat at time $t = T$ will be a superposition of $|\uparrow\rangle \otimes |\Phi_\uparrow\rangle \otimes |\text{dead}\rangle$ and $|\downarrow\rangle \otimes |\Phi_\downarrow\rangle \otimes |\text{live}\rangle$ with coefficients α and β, respectively. So we do not know how to describe even one measurement using Schrödinger's equation.

Paradoxes 1-4 and 6 and their resolutions are not controversial. Paradoxes 5 and 7, however, do excite controversy. For many physicists, the EPR paradox and Bell's theorem remain unresolved because, for them, renouncing the "reasonable" assumption of EPR is just not a resolution. As one distinguished physicist put it [12], "Anybody who's not bothered by Bell's theorem has to have rocks in his head." (No such statement would apply to any well known paradox in relativity theory.)

The Schrödinger Cat paradox has been resolved several times over—with spontaneous $\rightarrow$"collapse" of quantum states [13], nonlocal $\rightarrow$"hidden variables" [14], $\rightarrow$"many (parallel) worlds" [15] and future boundary conditions [16] (conditions on the future state in a $\rightarrow$"two-state" formalism [17])—but since experiments are consistent with all these resolutions, there is no one accepted resolution, at least within nonrelativistic quantum mechanics. The predictions of quantum mechanics with and without collapse differ, but the differences are (so far) not accessible to experiment. (There is even a proof [18] that if quantum mechanics is correct and an experiment could verify that a cat is in the superposition $\alpha|\text{dead}\rangle + \beta|\text{live}\rangle$, i.e. if it could verify that collapse has not occurred, the same experiment could transform the state $|\text{dead}\rangle$ into the state $|\text{live}\rangle$, i.e. it could revive a dead cat.) However, it is doubtful whether collapse or hidden-variable theories can be made relativistic; hence resolutions via many worlds or future boundary conditions, which require neither collapse nor "hidden" superluminal signalling, seem preferable so far.

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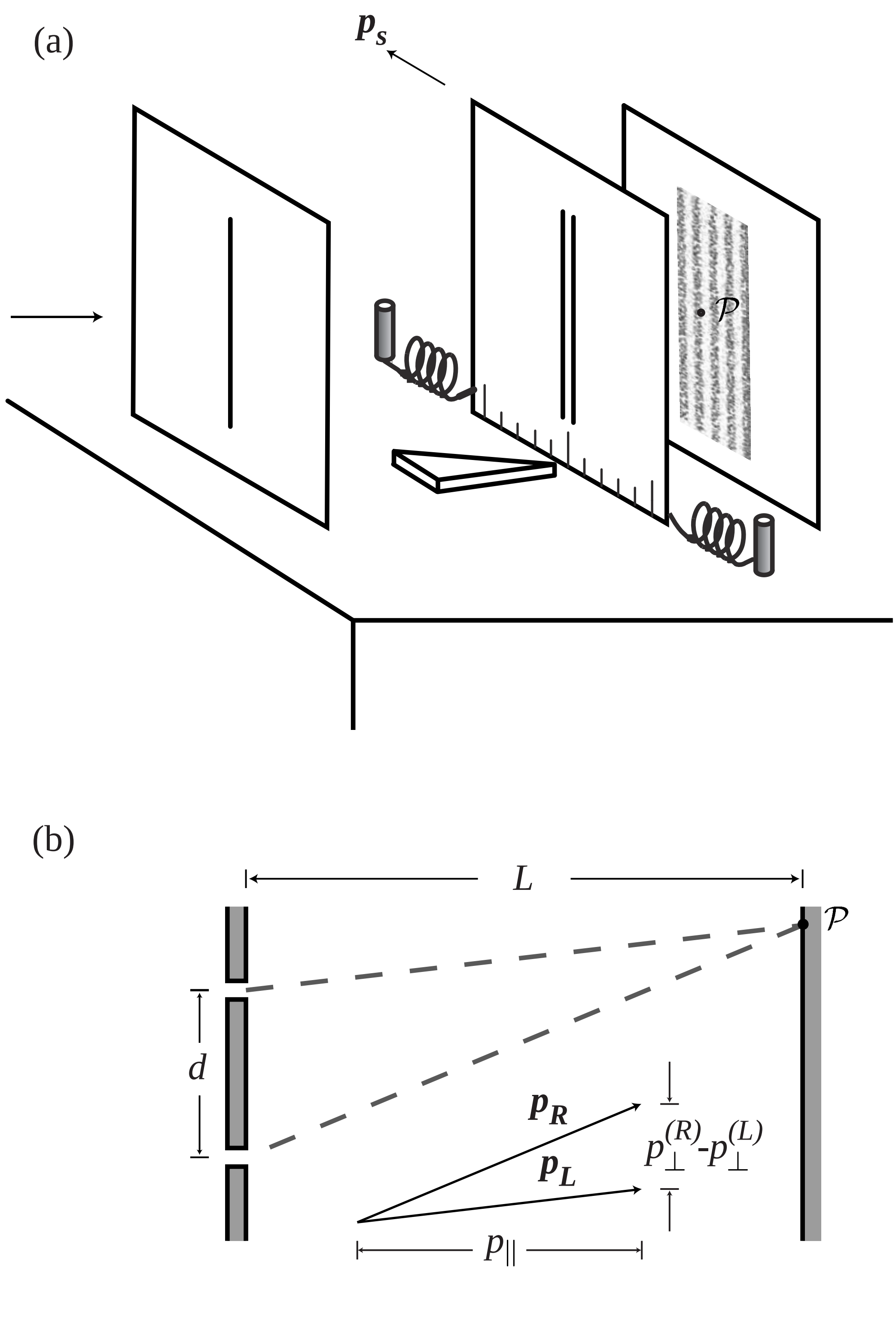
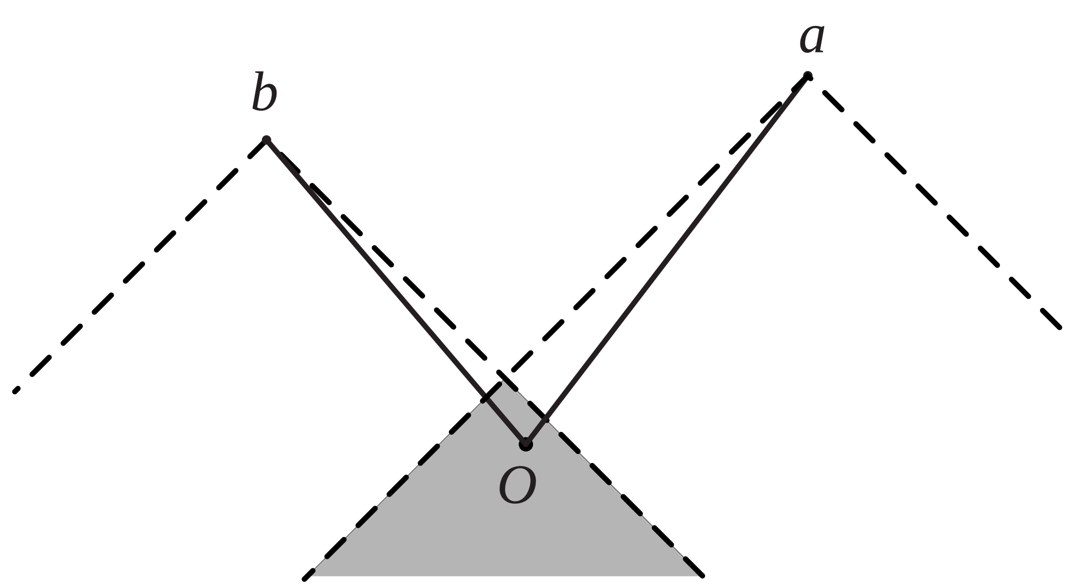
Figure Captions

Fig. 1. (a) A two-slit interference experiment adapted for measuring the transverse momentum of the middle screen. (b) The second and third screens seen from above, with interfering electron paths and corresponding momenta.

Fig. 2. Two atoms, produced in an entangled state at O, fly off in opposite directions (solid lines) in this spacetime figure. Alice measures a spin component of one atom at a; Bob measures a spin component of the other atom at b. Collapse cannot occur anywhere outside the past light cones of a and b (dotted lines), hence it cannot occur anywhere outside the intersection of their past light cones (shaded region).



(a)
$\boldsymbol{p_s}$
$\mathcal{P}$
(b)
L
$\mathcal{P}$
d
$\boldsymbol{p_R}$
$\boldsymbol{p_L}$
$p_\perp^{(R)}$-$p_\perp^{(L)}$
$p_{||}$



b
a
O