Sufficient conditions for the existence of perfect heterochromatic matchings in colored graphs

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Abstract

Let G = (V, E) be an (edge-)colored graph, i.e., G is assigned a mapping $C: E \to \{1, 2, \cdots, r\}$, the set of colors. A matching of G is called heterochromatic if its any two edges have different colors. Unlike uncolored matchings for which the maximum matching problem is solvable in polynomial time, the maximum heterochromatic matching problem is NP-complete. This means that to find both sufficient and necessary good conditions for the existence of perfect heterochromatic matchings should be not easy. In this paper, we obtain sufficient conditions of Hall-type and Tutte-type for the existence of perfect heterochromatic matchings in colored bipartite graphs and general colored graphs by showing that a colored bipartite graph B = (X, Y) contains a heterochromatic matching that saturates all vertices in X if $|N_c(S)| \ge |S|$ for all $S \subseteq X$, and a general colored graph G contains a perfect heterochromatic matching if $|N_c(S)| \ge |S|$ for all $S \subset V$ such that $0 \leq |S| \leq \frac{|V|}{2}$ and $|N(S) \setminus S| \geq |S|$. We also obtain a sufficient and necessary condition of Berge-type to verify if a heterochromatic matching M of G is maximum, i.e., if and only if there does not exist any heterochromatic M-augmenting path system in G.

Keywords: (edge-)colored graph, heterochromatic matching, *M*-alternating (augmenting) path system.

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1. Introduction

We use Bondy and Murty [2] for terminology and notations not defined here and consider simple graphs only.

Let G = (V, E) be a graph. By an *edge-coloring* of G we mean a function $C : E \to N$, the set of nonnegative integers. If G is assigned such a coloring, then G is called an *edge-colored graph* or, simply *colored graph*. Denote the colored graph by (G, C). We call C(e) the *color* of the edge $e = uv \in E$, and C(e) is also said to *present at* the edge e and its two ends u and v. Note that C is not necessarily a proper edge-coloring, i.e., two adjacent edges may have the same color. For a vertex v of G, the *color* neighborhood CN(v) of v is defined as the set $\{C(e) : e \text{ is incident with } v\}$. Then, $CN(S) = \bigcup_{v \in S} CN(v)$ for $S \subseteq V$. For a subgraph H of G, let $C(H) = \{C(e) : e \in E(H)\}$. A subgraph H of G is called *monochromatic* if its any two edges have the same color, whereas H is called *heterochromatic*, or *rainbow*, or *colorful* if its any two edges have different colors. There are many publications studying monochromatic or heterochromatic subgraphs, done mostly by the Hungarian school, see [1, 5] and the references in [3, 6]. Very often the subgraphs considered are paths, cycles, trees, etc. In this paper we study heterochromatic matchings of colored graphs.

A path P is said to start from (or end at) a color $c \in C(G)$, if c presents at the first (or last) edge of P. A connected graph is called a star if all but at most one (called the *center* of the star) of its vertices are leaves. We call a star T the *tail* of a path P, if the center of T is the last vertex of P. In the following we also regard a cycle as a path whose two ends coincide. A matching M is called *heterochromatic* in a colored graph G, if its any two edges have different colors. The two ends of an edge in M are said to be matched under M. M saturates a vertex u, or u is M-saturated, if there is an edge in M incident with u; otherwise, u is M-unsaturated. If every vertex of G is M-saturated, the heterochromatic matching M is called *perfect*. M is a maximum heterochromatic matching, if G has no heterochromatic matching M' such that |M'| > |M|.

Note that if heterochromatic subgraph problems are concerned, an uncolored graph itself is viewed as heterochromatic, while if monochromatic subgraph problems are concerned, an uncolored graph itself is viewed as monochromatic. Unlike uncolored matchings for which the maximum matching problem is solvable in polynomial time (see [7]), the maximum heterochromatic matching problem is NP-complete (see [4]). This means that to find both sufficient and necessary good conditions for the existence of perfect heterochromatic matchings should be not easy. In the following we will present sufficient conditions of Hall-type and Tutte-type for the existence of perfect heterochromatic matchings in colored bipartite graphs and general colored graphs. We also obtain a sufficient and necessary condition of Berge-type to verify if a heterochromatic matching is maximum.

2. Preliminaries

In order to present our main results, we need some new notations and definitions.

For $S \subseteq V(G)$, denote $N_c(S)$ as one of the minimum set(s) W satisfying $W \subseteq N(S) \setminus S$ and $[CN(S) \setminus C(G[S])] \subseteq CN(W)$. Note that there might be more than one such sets W, however, for any two such sets W and W' we have CN(W) = CN(W') for the given S. In the following $A \triangle B$ denotes the symmetric difference of two sets A and B and $A|_B$ denotes $A \cap B$.

Definition 1. Let M be a heterochromatic matching. $P = P_1 \cup P_2 \cup \cdots \cup P_k$ is called a *heterochromatic M-alternating path system*, if P_1 is an *M*-alternating path (in the common sense) such that any two edges in $M|_{P_1} \triangle P_1$ have different colors, and for every $i \ (2 \le i \le k), P_i$ is a set of paths such that

(1) every path in P_i is an *M*-alternating path p_{i_j} (in the common sense). Remind that a cycle is also regarded as a path (with the same start and end);

- (2) the first edge of every path p_{i_j} of P_i is an edge $v_{i_\omega} u_{i_\omega}$ in M such that $C(v_{i_\omega} u_{i_\omega}) \in C(M|_{P_{i-1}} \bigtriangleup P_{i-1}) \cap C(M \setminus \bigcup_{j=1}^{i-1} P_j);$
- (3) any two edges in $M|_{P_i} \triangle P_i$ have different colors;
- (4) $C(M|_{P_s} \triangle P_s) \cap C(M|_{P_t} \triangle P_t) = \emptyset$ for any $s \neq t \in \{1, 2, \cdots, i\};$

(5)
$$C(M) \cap C(M|_{P_k} \Delta P_k) \subseteq C(M \cap \bigcup_{j=1}^k P_j).$$

From Definition 1, one can see the following facts:

Fact 1. P_1 may be a signal edge e. If $e \in M$, then $P = P_1 = e$; if $e \notin M$, then $M \bigtriangleup P_1 = e$. In particular, if $C(M|_{P_1} \bigtriangleup P_1) \cap C(M) \subseteq C(M \cap P_1)$, then $P = P_1$.

Fact 2. If, for $i \ge 2$, $|\{\omega : \omega \in C(M|_{P_{i-1}} \bigtriangleup P_{i-1}) \cap C(M \setminus \bigcup_{j=1}^{i-1} P_j)\}| = r$, then

 $P_i = p_{i_1} \cup p_{i_2} \cup \cdots \cup p_{i_r}$, where each p_{i_ω} is an *M*-alternating path (in the common sense) such that its first edge $v_{i_\omega} u_{i_\omega}$ is in *M*. We call each p_{i_ω} a *branch* of P_i . Obviously, P_1 has only one branch. If $p_{i_t} \cap p_{i_\omega} \neq \emptyset$ then $p_{i_t} \cap p_{i_\omega} = \{v\}$ for some vertex v, and v is the start of p_{i_t} and the end of p_{i_ω} . So, $p_{i_t} \cup p_{i_\omega}$ is an *M*-alternating path. Hence, different branches must have different ends.

Fact 3. If $P_i \cap P_j \neq \emptyset$ for $i \neq j$, then $P_i \cap P_j = \{v\}$ for some vertex v, and v is the start of a branch of P_i and the end of a branch of P_j , and then $P_i \cup P_j$ is an M-alternating path system. If the start u of P_1 is M-unsaturated and $P_1 \cap P_i = \{v\}$ for $i \geq 2$, then $u \neq v$.

Therefore, for $p_{i_t} \cap p_{j_s} \neq \emptyset$ such that i, j, s, t are pairwise different, if we regard p_{i_t} union p_{j_s} as one path, then $p_{i_t} \cup p_{j_s}$ is again an *M*-alternating path, and so, the heterochromatic *M*-alternating path system *P* can also be equivalently defined as follows:

Definition 2. $P = p_1 \cup p_2 \cup \cdots \cup p_l$ is an *M*-alternating path system, if

- (1) for all $i, j \in \{1, 2, \cdots, l\}$ and $i \neq j$, $p_i \cap p_j = \emptyset$;
- (2) any two edges in $M|_P riangle \bigcup_{i=1}^{l} p_i$ have different colors;

(3)
$$C(M) \cap C(M|_P \bigtriangleup \bigcup_{i=1}^l p_i) \subseteq C(M \cap \bigcup_{i=1}^l p_i).$$

The following two definitions are important in the sequel.

Definition 3. For an *M*-unsaturated vertex *u*, a heterochromatic *M*-augmenting path system starting from *u* is a heterochromatic *M*-alternating path system $P = P_1 \cup P_2 \cup \cdots \cup P_k$ such that

- (1) u is the start of P_1 , and the end of P_1 is M-unsaturated;
- (2) for any P_i $(i \ge 2)$, its every branch p_{i_j} is of even length, and the end of p_{i_j} is M-unsaturated or the start of another branch.

Definition 4. A heterochromatic *M*-augmenting path system of *G* is a heterochromatic *M*-alternating path system $P = p_1 \cup p_2 \cup \cdots \cup p_l$ such that $| M|_P \triangle P | > |M \cap P|$, where $p_i \cap p_j = \emptyset$ for all $i \neq j$ and $i, j \in \{1, 2, \cdots, l\}$.

For an uncolored graph the following results are well-known, which give both sufficient and necessary conditions for the existence of perfect matchings in bipartite graphs and general graphs.

Lemma 2..1 (Hall [2]) Let B be a bipartite graph with bipartition (X, Y). Then B contains a matching that saturates every vertex in X if and only if $|N(S)| \ge |S|$ for all $S \subseteq X$.

Lemma 2..2 (Tutte [2]) A graph G has a perfect matching if and only if

 $o(G-S) \leqslant |S|$

for all $S \subseteq V(G)$, where o(G - S) denotes the number of odd components in the remaining graph G - S.

3. Main results

Although we cannot find both sufficient and necessary conditions for the existence of perfect heterochromatic matchings, in this section we give a sufficient condition of Hall-type for colored bipartite graphs and of Tutte-type for general colored graphs for the existence of perfect heterochromatic matchings.

Theorem 3..1 (Hall-type) Let (B,C) be a colored bipartite graph with bipartition (X,Y). Then, B contains a heterochromatic matching that saturates every vertex in X, if

$$|N_c(S)| \ge |S| \tag{3.1}$$

for all $S \subseteq X$.

Proof. Suppose (B, C) is a colored bipartite graph satisfying condition (3.1), then it is clear that $|N(S)| \ge |S|$ for all $S \subseteq X$ from the definition of $N_c(S)$. So, B contains an uncolored matching that saturates every vertex in X from Lemma 2.1.

By contradiction, suppose B does not contain any heterochromatic matching that saturates every vertex in X. Let $X = \{u_1, u_2, \dots, u_n\}$, $Y = \{v_1, v_2, \dots, v_m\}$, and M^* be a maximum heterochromatic matching of B with $C(M^*) = \{1, 2, \dots, r\}$. So, M^* cannot saturate every vertex in X. Let u be an M^* -unsaturated vertex in X. Without loss of generality, we may assume $u_i v_i \in M^*$ with $C(u_i v_i) = i$ for $i \in \{1, 2, \dots, r\}$. Since $|N(S)| \ge |S|$ for any $S \subseteq X$, there is an M^* -alternating path (in the common sense) starting from u whose end is an M^* -unsaturated vertex.

In the following we call a heterochromatic M^* -alternating path system P a possible M^* -augmenting path system if any two edges of $M^*|_P \triangle P$ have different colors.

Next we will find a heterochromatic M^* -augmenting path system starting from u, which leads to a contradiction to the maximality of M^* . In the following, for a set A of paths and a set of vertices X we simply use $A \cap X$ to denote the set of vertices each of which is both in a path of A and in X, i.e., $V(\bigcup_{p \in A} p) \cap X$. As usual, we use l(p) to denote the length of a path p

denote the length of a path p.

Step 1. Denote by A_1 the set of all possible M^* -augmenting paths starting from u (see Definition 3) such that each such path is extended with a length as large as possible. Then, every $P_1 \in A_1$ has one of the following properties:

- 1) $l(P_1) \equiv 0 \pmod{2}$, and any two edges in $M^*|_{P_1} \bigtriangleup P_1$ have different colors;
- 2) $l(P_1) \equiv 1 \pmod{2}$, and the end of P_1 is M^* -saturated. Moreover, any two edges in $(M^*|_{P_1} \Delta P_1) \setminus e$ have different colors, and $C(e) \in C((M^*|_{P_1} \Delta P_1) \setminus e)$, where e is the last edge of P_1 ;
- 3) $l(P_1) \equiv 1 \pmod{2}$, and the end of P_1 is M^* -unsaturated. Moreover, any two edges in $M^*|_{P_1} \vartriangle P_1$ have different colors.

Define $\wp_1 = \{P_1 : P_1 \in A_1 \text{ has property } 3)\}$, then $\wp_1 \neq \emptyset$. Otherwise, every path in A_1 has either property 1) or 2), and so, every this kind of path is not augmenting. Let

$$S'_1 = A_1 \cap X, \ S''_1 = (M^* \cap A_1) \cap Y.$$

Clearly, $|S_1''| = |S_1' \setminus u|$ and $CN(S_1') \subseteq CN(S_1'')$. So, $|N_c(S_1')| \leq |S_1''| = |S_1' \setminus u| = |S_1'| - 1$, a contradiction to condition (3.1).

If some $P_1 \in \wp_1$ satisfies $C(M^*) \cap C(M^*|_{P_1} \triangle P_1) \subseteq C(M^* \cap P_1)$, then $P = P_1$ is a heterochromatic M^* -augmenting path starting from u. So, the matching $M' = (M^* \backslash P_1) \cup (M^*|_{P_1} \triangle P_1)$ is heterochromatic that saturates u, and hence, $|M'| > |M^*|$, which contradicts the maximality of M^* . So, for any $P_1 \in \wp_1$ we must have $C(M^*|_{P_1} \triangle P_1) \cap C(M^* \backslash P_1) \neq \emptyset$. Then, set $S_1^* = \wp_1 \cap X$, and go to the next step.

Step 2. For every $P_1 \in \wp_1$ and every $\omega \in C(M^*|_{P_1} \Delta P_1) \cap C(M^* \backslash P_1)$, let P_2 denote the set of all possible heterochromatic M^* -augmenting paths starting from v_{ω} with first edge $u_{\omega}v_{\omega}$, such that each such path is extended with a length as large as possible. Denote by A_2 the set of all these P_2 's corresponding to all P_1 's. Then, any branch $p_{2_{\omega}}$ of every $P_2 \in A_2$ has one of the following properties:

- 1) $l(p_{2_{\omega}}) \equiv 1 \pmod{2}$, and any two edges in $M^*|_{p_{2_{\omega}}} \bigtriangleup p_{2_{\omega}}$ have different colors;
- 2) $l(p_{2_{\omega}}) \equiv 0 \pmod{2}$, and any two edges in $(M^*|_{p_{2_{\omega}}} \bigtriangleup p_{2_{\omega}}) \setminus e$ have different colors, and $C(e) \in C(M^*|_{p_{2_{\omega}} \cup P_1} \bigtriangleup (P_1 \cup (p_{2_{\omega}} \setminus e)))$, where e is the last edge of $p_{2_{\omega}}$;
- 3) $l(p_{2_{\omega}}) \equiv 0 \pmod{2}$, and any two edges in $M^*|_{p_{2_{\omega}}} \Delta p_{2_{\omega}}$ have different colors.

Define

$$\wp_2 = \{ P_2 : P_2 = \bigcup_{\omega \in C(M^* \triangle P_1) \cap C(M^* \backslash P_1)} p_{2_\omega} \},$$

where every $p_{2\omega}$ is a path with property 3), and any two edges in $M^*|_{P_2} \Delta P_2$ have different colors. We claim $\wp_2 \neq \emptyset$. Otherwise, for every $P_1 \in \wp_1$, there is a corresponding set P_2 of paths in Step 2, and among the branches of P_2 , there exists a branch with property 1) or 2), or there are at least two branches each of which has an even numbered edge with the same color c. So, these branches are not augmenting. For the latter case, we take a subbranch of each such branch to end at the color c. Particularly, if p_1 , p_2 and p_3 are branches of a P_2 such that p_1 and p_2 have a common color c_1 and p_2 and p_3 have a common color c_2 , and if the color of the first edge, respectively, in the three paths is ω_1 , ω_2 and ω_3 , and the three colors appear in P_1 in the same ordering, then the subbranches of p_1 and p_2 are taken to end at the color c_1 , and the subbranch of p_3 is taken to end at the color c_2 . Let A'_2 denote the set of all these not augmenting branches and subbranches corresponding to the P_2 's. Denote $S'_2 = A'_2 \cap X$.

Note that if some $P_1 \in \wp_1$ has a tail, then the structure of the P_2 corresponding to P_1 in Step 2 is complicated, for which we analysis as follows:

(i) $P_1 \in \wp_1$ has a tail T such that $C(T) \cap C(M^*) = \emptyset$. If the P_2 corresponding to this P_1 has a branch $p_{2_{\omega}}$ with property 2), and $C(e) \in C(M^*|_{P_1} \land (P_1 \setminus T))$ for the last edge e of $p_{2_{\omega}}$, then we have $p_{2_{\omega}} \in A'_2$. Denote $\wp'_1 = \{P_1 : P_1 \in \wp_1$ has a tail $T_{1,1}\}$, where $C(T_{1,1}) \cap C(M^*) = \emptyset$ and $|C(T_{1,1})| = |C(M^*|_{P_1} \land P_1) \cap C(M^* \setminus P_1)|$, and all the branches of the P_2 are in A'_2 and have property 1) or 2). Then, denote

$$T'_{1} = \{T_{1,1} : T_{1,1} \text{ is the tail of a } P_{1} \in \wp'_{1}\}$$
$$X'_{1} = \{x : x \text{ is the center of a } T_{1,1} \in T'_{1}\}.$$

(*ii*) $P_1 \in \wp_1$ has a tail $T_{1,2}$ such that $C(T_{1,2}) \subseteq C(M^* \setminus P_1)$. Denote $\wp''_1 = \{P_1 : T_{1,2} \text{ is the tail of } P_1 \in \wp_1\}$, where for any P_1 with tail $T_{1,2}$, all branches of P_2 corresponding to the $T_{1,2}$ are in A'_2 . That is, all the branches of P_2 such that the color of the first edge is in $C(T_{1,2})$ belong to A'_2 . Then, denote

$$T_1'' = \{T_{1,2} : T_{1,2} \text{ is the tail of a } P_1 \in \wp_1''\}$$
$$X_1'' = \{x : x \text{ is the center of a } T_{1,2} \in T_1''\}.$$

(*iii*) $P_1 \in \wp_1$ has a tail $T_{1,3}$ such that $C(T_{1,3}) \cap C(M^*) \neq \emptyset$ and $C(T_{1,3}) \setminus C(M^*) \neq \emptyset$, and

$$G[\{e : e \in E(T_{1,3}) \text{ and } C(e) \in C(M^*)\}] \subseteq T_1''$$

$$G[\{e : e \in E(T_{1,3}) \text{ and } C(e) \notin C(M^*)\}] \subseteq T_1'.$$

Then, denote

$$\wp_1''' = \{P_1 : P_1 \in \wp_1 \text{ has a tail } T_{1,3}\}$$
$$T_1''' = \{T_{1,3} : T_{1,3} \text{ is the tail of a } P_1 \in \wp_1'''\}$$
$$X_1''' = \{x : x \text{ is the center of a } T_{1,3} \in T_1'''\}.$$

Now we consider the colors of all paths of A'_2 . If we do not consider the colors presenting in the path P_1 , then we have the following two cases:

Case 1. There is a path $p \in A'_2$ with property 1) or 2).

Case 2. A'_2 has at least two branches in P_2 each of which has an even numbered edge with the same color c.

Then, $CN(A'_2 \cap Y) \supseteq CN(A'_2 \cap X)$, and the vertices of $(A'_2 \cap X) \setminus \{u_{\omega}\}$ are perfectly matched with the vertices of $(A'_2 \cap Y) \setminus \{v', v'', v_{\omega}\}$ under M^* , where $v'' \in Y$ is the end

of a subbranch constructed in Case 2, $v' \in Y$ is the end of a path with property 2). If only Case 1 happens, then the vertices of $(A'_2 \cap X) \setminus \{u_\omega\}$ are perfectly matched with the vertices of $(A'_2 \cap Y) \setminus \{v', v_\omega\}$ under M^* . Since $|\{u_\omega\}| \ge 1$, we have $|N_c(S'_2)| < |S'_2|$. Otherwise, the vertices of $(A'_2 \cap X) \setminus \{u_\omega\}$ are perfectly matched with the vertices of $(A'_2 \cap Y) \setminus \{v', v'', v_\omega\}$ under M^* . Among the branches of Case 2, consider those ending at a same color, and take the end v of one of them and put it into $(A'_2 \cap Y) \setminus \{v', v'', v_\omega\}$. Then, $CN((A'_2 \cap X) \setminus \{u_\omega\}) \subseteq CN(((A'_2 \cap Y) \setminus \{v', v'', v_\omega\}) \cup \{v\})$ and $|(A'_2 \cap X) \setminus \{u_\omega\}| +$ $1 \ge |((A'_2 \cap Y) \setminus \{v', v'', v_\omega\}) \cup \{v\}|$. Since $S'_2 = A'_2 \cap X$ and $|\{u_\omega\}| \ge 2$, we have $|N_c(S'_2)| < |S'_2|$, which always holds provided we do not consider the colors presenting in P_1 .

Therefore, if some P_1 has a tail $T \notin T'_1 \cup T''_1 \cup T''_1$ such that $C(T) \cap C(M^*) \neq \emptyset$ and $C(T) \setminus C(M^*) \neq \emptyset$, then let $A''_2 = A'_2 \setminus \{p : p \text{ is a branch constructed in Step 2}$, corresponding to the edge $e \in E(T)$ such that $C(e) \in C(M^*)\}$ and let $S''_2 = A''_2 \cap X$. Since $|N_c(\{p\} \cap X)| \ge |\{p\} \cap X|$, we again have $|N_c(S''_2)| < |S''_2|$, provided we do not consider the colors presenting in P_1 of A''_2 . Set

$$S_2 = S_2'' \cup ((S_1^* \setminus T_1) \cup X_1' \cup X_1'' \cup X_1'''),$$

where $S_1^* = \wp_1 \cap X$ and $T_1 = \{T : T \text{ is the tail of a } P_1 \in \wp_1\}.$

If $|N_c((S_1^*\backslash T_1) \cup X_1' \cup X_1'' \cup X_1''')| \leq |(S_1^*\backslash T_1) \cup X_1' \cup X_1'' \cup X_1'''|$, then $|N_c(S_2)| < |S_2|$. Otherwise, $|N_c((S_1^*\backslash T_1) \cup X_1' \cup X_1'' \cup X_1''')| > |(S_1^*\backslash T_1) \cup X_1' \cup X_1'' \cup X_1'''|$, and hence, their difference comes from the numbers of colors of T_1', T_1'' and T_1''' . If $P_1 \in \wp_1'$, then $|N_c(P_1 \cap X)| - |P_1 \cap X| \leq |C(T_{1,1})| - 1$; if $P_1 \in \wp_1''$, then $|N_c(P_1 \cap X)| - |P_1 \cap X| \leq |C(T_{1,2})| - 1$; and if $P_1 \in \wp_1'''$, then $|N_c(P_1 \cap X)| - |P_1 \cap X| \leq |C(T_{1,2})| - 1$. Since A_2'' has $|C(T_{1,1}) \cup C(T_{1,2}) \cup C(T_{1,3})|$ branches such that their vertices belong to S_2'' whenever they also belong to X, we have $|N_c(S_2)| < |S_2|$, a contradiction to condition (3.1), and hence, $\wp_2 \neq \emptyset$.

Therefore, if there is a $P_2 \in \wp_2$ corresponding to $P_1 \in \wp_1$ such that $C(M^*) \cap C(M^*|_{P_2} \triangle P_2) \subseteq C(M^* \cap (P_2 \cup P_1))$, then $P = P_1 \cup P_2$ is a heterochromatic M^* -augmenting path starting from u, and then, the matching $M' = (M^* \setminus (P_2 \cup P_1)) \cup (M^*|_{P_1 \cup P_2} \triangle (P_2 \cup P_1))$ is heterochromatic that saturates u, and hence, $|M'| > |M^*|$, a contradiction to the maximality of M^* . So, for any $P_2 \in \wp_2$, we have $C(M^*|_{P_2} \triangle P_2) \cap C(M^* \setminus (P_2 \cup P_1)) \neq \emptyset$. Then, set $S_2^* = (\wp_1 \cup \wp_2) \cap X$, and proceed as above. Similarly, we can obtain $\wp_3, \wp_4, \cdots, \wp_{k-1}$, and arrive at the next step.

Step k. For every
$$P_{k-1} \in \wp_{k-1}$$
 and every $\omega \in C(M^*|_{P_{k-1}} \triangle P_{k-1}) \cap C(M^* \setminus \bigcup_{i=1}^{k-1} P_i)$

let P_k be the set of all possible M^* -augmenting paths starting from v_{ω} with first edge $u_{\omega}v_{\omega}$, such that each such path is extended with a length as large as possible. Denote by A_k the set of all these P_k 's corresponding to all P_{k-1} 's. Then, any branch $p_{k\omega}$ of every $P_k \in A_k$ has one of the following properties:

- 1) $l(p_{k_{\omega}}) \equiv 1 \pmod{2}$, and any two edges in $M^*|_{p_{k_{\omega}}} \bigtriangleup p_{k_{\omega}}$ have different colors;
- 2) $l(p_{k_{\omega}}) \equiv 0 \pmod{2}$, and any two edges in $(M^*|_{p_{k_{\omega}}} \Delta p_{k_{\omega}}) \setminus e$ have different colors, where e is the last edge e of $p_{k_{\omega}}$ such that $C(e) \in C((M^* \cap (\bigcup_{1}^{k-1} P_i \cup p_{k_{\omega}})) \Delta (\bigcup_{1}^{k-1} P_i \cup (p_{k_{\omega}} \setminus e)));$

3) $l(p_{k_{\omega}}) \equiv 0 \pmod{2}$, and any two edges in $M^*|_{p_{k_{\omega}}} \bigtriangleup p_{k_{\omega}}$ have different colors.

Define

$$\wp_k = \{ P_k : P_k = \bigcup_{\omega \in C(M^* \triangle P_{k-1}) \cap C(M^* \setminus (P_1 \cup P_2 \cdots P_{k-1}))} p_{k\omega} \},$$

where every path $p_{k_{\omega}}$ has property 3), and any two edges in $M^*|_{P_k} \Delta P_k$ have different colors. We claim $\wp_k \neq \emptyset$. Otherwise, for every $P_{k-1} \in \wp_{k-1}$, there is a set P_k of paths in Step k, and among the branches of P_k , there exists a branch with property 1) or 2), or there are at least two branches each of which has an even numbered edge with the same color c. So, such branches are not augmenting. For the latter case, similar to Step 2 we construct their subbranches by cutting at the same color c. Let A'_k denote the set of all these not augmenting branches and subbranches of all P_k 's, and let $S'_k = A'_k \cap X$.

Similar to Step 2, if a branch of P_i $(1 \le i \le k-1)$ has a tail, then the structure of P_k corresponding to P_i in Step k is complicated, for which we analysis as follows:

(i) A branch of $P_i \in \wp_i$ $(1 \leq i \leq k-1)$ has a tail T such that $C(T) \cap C(M^*) = \emptyset$. If a branch $p_{k_{\omega}}$ of P_k corresponding to the P_i is a path with property 2) and $C(e) \in C(M^*|_{P_i} \Delta(P_i \setminus T))$, where e is the last edge of $p_{k_{\omega}}$, then $p_{k_{\omega}} \in A'_k$.

Then, let $\wp'_i = \{P_i : P_i \in \wp_i \text{ has a tail } T_{i,1}\}$, where $C(T_{i,1}) \cap C(M^*) = \emptyset$, $|C(T_{i,1})| = |C(M^*|_{P_{k-1}} \bigtriangleup P_{k-1}) \cap C(M^* \setminus \bigcup_{i=1}^{k-1} P_i)|$, and all branches of P_k corresponding to the P_i are in A'_k and have property 1) or 2). Define

$$T'_i = \{T_{i,1} : T_{i,1} \text{ is the tail of a } P_i \in \wp'_i\}$$

and

$$X'_i = \{x : x \text{ is center of a } T_{i,1} \in T'_i\}.$$

(*ii*) $T_{k-1,2}$ is the tail of a $P_{k-1} \in \wp_{k-1}$ and $C(T_{k-1,2}) \subseteq C(M^* \setminus \bigcup_{i=1}^{k-1} P_i)$. Define $\wp''_{k-1} = \{P_{k-1} : P_{k-1} \in \wp_{k-1} \text{ has a tail } T_{k-1,2}\}$, where all branches of all P_k 's corresponding to all P_{k-1} 's with the tail $T_{k-1,2}$ belong to A'_k . Let

$$T_{k-1}'' = \{T_{k-1,2} : T_{k-1,2} \text{ is the tail of a } P_{k-1} \in \wp_{k-1}''\}$$
$$X_{k-1}'' = \{x : x \text{ is the center of a } T_{k-1,2} \in T_{k-1}''\}.$$

(*iii*) $T_{k-1,3}$ is the tail of a $P_{k-1} \in \wp_1$ such that $C(T_{k-1,3}) \cap C(M^*) \neq \emptyset$, $C(T_{1,3}) \setminus C(M^*) \neq \emptyset$, and \emptyset , and

$$G[\{e: e \in E(T_{k-1,3}) \text{ and } C(e) \in C(M^*)\}] \subseteq T''_{k-1}$$
$$G[\{e: e \in E(T_{k-1,3}) \text{ and } C(e) \notin C(M^*)\}] \subseteq T'_i$$

Then, define

$$\wp_{k-1}^{\prime\prime\prime} = \{P_{k-1} : P_{k-1} \in \wp_{k-1} \text{ has a tail } T_{k-1,3}\}$$
$$T_{k-1}^{\prime\prime\prime} = \{T_{k-1,3} : T_{k-1,3} \text{ is the tail of a } P_{k-1} \in \wp_{k-1}^{\prime\prime\prime}\}$$
$$X_{k-1}^{\prime\prime\prime} = \{x : x \text{ is the center of a } T_{k-1,3} \in T_{k-1}^{\prime\prime\prime}\}.$$

Now we consider the colors of all paths of A'_k . If we do not consider the colors presenting in $\bigcup_{1}^{k-1} P_t$, then we have the following two cases:

Case 1. $p \in A'_k$ is a path with property 1) or 2).

Case 2. A'_k has at least two branches in P_k each of which has an even numbered edge with the same color c.

Then, $CN(A'_k \cap Y) \supseteq CN(A'_k \cap X)$ and the vertices of $(A'_k \cap X) \setminus \{u_\omega\}$ are perfectly matched with the vertices of $(A'_k \cap Y) \setminus \{v', v'', v_\omega\}$ under M^* , where $v'' \in Y$ is the end of a subbranch constructed in Case 2, $v' \in Y$ is the end of a path with property 2). If only Case 1 happens, then the vertices of $(A'_k \cap X) \setminus \{u_\omega\}$ are perfectly matched with the vertices of $(A'_k \cap Y) \setminus \{v', v_\omega\}$ under M^* . Since $|\{u_\omega\}| \ge 1$, we have $|N_c(S'_k)| < |S'_k|$. Otherwise, the vertices of $(A'_k \cap X) \setminus \{u_\omega\}$ are perfectly matched with the vertices of $(A'_k \cap Y) \setminus \{v', v'', v_\omega\}$ under M^* . Among the branches of Case 2, consider those ending at a same color, and take the end v of one of them and put it into $(A'_k \cap Y) \setminus \{v', v'', v_\omega\}$. Then, $CN((A'_k \cap X) \setminus \{u_\omega\}) \subseteq CN(((A'_k \cap Y) \setminus \{v', v'', v_\omega\}) \cup \{v\})$, and $|A'_k \cap X) \setminus \{u_\omega\}| + 1 \ge |((A'_k \cap Y) \setminus \{v', v'', v_\omega\}) \cup \{v\}|$. Since $|\{u_\omega\}| \ge 2$, we have $|N_c(S'_k)| < |S'_k|$, which always holds provided we do not consider the colors presenting in $\bigcup^{k-1} P_t$.

Therefore, if some P_{k-1} has a tail $T \notin T'_{k-1} \cup T''_{k-1} \cup T''_{k-1}$ such that $C(T) \cap C(M^*) \neq \emptyset$ and $C(T) \setminus C(M^*) \neq \emptyset$, then let $A''_k = A'_k \setminus \{p : p \text{ is a branch constructed in Step } k$, corresponding to the edge $e \in E(T)$ such that $C(e) \in C(M^*)\}$, and let $S''_k = A''_k \cap X$. Since $|N_c(\{p\} \cap X)| \ge |\{p\} \cap X|$, we again have $|N_c(S''_k)| < |S''_k|$, provided we do not consider the colors presenting in $\bigcup_{i=1}^{k-1} P_t$ of A''_k . Define

$$S_{k} = S_{k}'' \cup ((S_{k-1}^{*} \setminus \bigcup_{1}^{k-1} T_{t}) \cup X_{i}' \cup X_{k-1}'' \cup X_{k-1}'''),$$

where $S_{k-1}^* = (\wp_1 \cup \wp_2 \cup \cdots \cup \wp_{k-1}) \cap X$ and $T_t = \{T : T \text{ is the tail of a } P_k \in \wp_k\}$ for $t \in \{1, 2, \cdots, k-1\}.$

If
$$|N_c((S_{k-1}^* \setminus \bigcup_{1}^{k-1} T_t) \cup X_i' \cup X_{k-1}'' \cup X_{k-1}''')| \leq |(S_{k-1}^* \setminus \bigcup_{1}^{k-1} T_t) \cup X_i' \cup X_{k-1}'' \cup X_{k-1}''|$$
, then $|N_c(S_k)| < |S_k|$. Otherwise, $|N_c((S_{k-1}^* \setminus \bigcup_{1}^{k-1} T_t) \cup X_i' \cup X_{k-1}'' \cup X_{k-1}'')| > |(S_{k-1}^* \setminus \bigcup_{1}^{k-1} T_t) \cup X_i' \cup X_{k-1}'' \cup X_{k-1}'')| > |(S_{k-1}^* \setminus \bigcup_{1}^{k-1} T_t) \cup X_i' \cup X_{k-1}'' \cup X_{k-1}'')| > |(S_{k-1}^* \setminus \bigcup_{1}^{k-1} T_t) \cup X_i' \cup X_{k-1}'' \cup X_{k-1}'')| > |(S_{k-1}^* \setminus \bigcup_{1}^{k-1} T_t) \cup X_i' \cup X_{k-1}'' \cup X_{k-1}'')| > |(S_{k-1}^* \setminus \bigcup_{1}^{k-1} T_t) \cup X_i' \cup X_{k-1}'' \cup X_{k-1}'')| > |(S_{k-1}^* \setminus \bigcup_{1}^{k-1} T_t) \cup X_i' \cup X_{k-1}'' \cup X_{k-1}'')| > |(S_{k-1}^* \setminus \bigcup_{1}^{k-1} T_t) \cup X_i' \cup X_{k-1}'' \cup X_{k-1}'')| > |(S_{k-1}^* \setminus \bigcup_{1}^{k-1} T_t) \cup X_i' \cup X_{k-1}'' \cup X_{k-1}'')| > |(S_{k-1}^* \setminus \bigcup_{1}^{k-1} T_t) \cup X_i' \cup X_{k-1}'' \cup X_{k-1}'' \cup X_{k-1}'' \cup X_{k-1}'')| > |(S_{k-1}^* \setminus \bigcup_{1}^{k-1} T_t) \cup X_i' \cup X_{k-1}'' \cup X_{k-1}'' \cup X_{k-1}'' \cup X_i' \cup X_{k-1}'' \cup X_{k-1}'' \cup X_i' \cup X_{k-1}'' \cup X_{k-1}'' \cup X_{k-1}'' \cup X_i' \cup X_{k-1}'' \cup X$

Therefore, there exists a $P_k \in \wp_k$ corresponding to $P_{k-1} \in \wp_{k-1}, P_{k-2} \in \wp_{k-2}, \cdots, P_2 \in \wp_2, P_1 \in \wp_1$ such that $C(M^*) \cap C(M^*|_{P_k} \triangle P_k) \subseteq C(M^* \cap (P_k \cup P_{k-1} \cup \cdots \cup P_1))$, and so, $P = P_1 \cup P_2 \cup \cdots \cup P_k$ is a heterochromatic M^* -augmenting path system starting from u, and then, the matching $M' = (M^* \setminus (P_k \cup P_{k-1} \cup \cdots \cup P_1)) \cup ((M^* \cap (P_k \cup P_{k-1} \cup \cdots \cup P_1)) \land (P_k \cup P_{k-1} \cup \cdots \cup P_1))$ is heterochromatic that saturates u, and hence,

 $|M'| > |M^*|$, a contradiction to the maximality of M^* . The proof is now complete.

As one can see, Step k is almost the same as Step 2, and so we should have omitted its details. But, for easy of understanding we prefer to repeat them. Also, note that our proof gives an algorithm to find a larger heterochromatic matching from a given one by extending from a series of constructible heterochromatic augmenting path systems, or find a subset S of X such that $|N_c(S)| < |S|$.

In the following we will give an analogue of Tutte's Theorem for general colored graphs. Before proceeding, we need more notations. Let M be a heterochromatic matching of a general colored graph G. For $\omega \in C(M)$, define $E_{\omega}(M) = \{e : e \in E(M) \text{ and } e \text{ has the color } \omega\}$. We say that two edges e_1 and e_2 of G are connected with respect to the color ω if e_1 , $e_2 \in E_{\omega}(M)$ and there is an M-alternating path from e_1 to e_2 , or there is an $e_3 \in E_{\omega}(M)$ such that there is an M-alternating path from e_1 to e_3 and another M-alternating path from e_3 to e_2 . This sets up a relation between the edges of G, and it is easy to see that it is an equivalent relation. Let $E_{\omega}^s = \{e_1, e_2, \cdots, e_s\} \subseteq E_{\omega}(M)$ be the set of all connected edges with respect the color ω . Denote by A the set of all M-alternating paths starting from an edge $e \in E_{\omega}^s$. Then, we say that G[V(A)] is a connected components with respect to the color ω .

Theorem 3..2 A colored graph (G, C) has a perfect heterochromatic matching, if

- 1) $o(G-S) \leq |S|$, where o(G-S) denotes the number of odd components in the remaining graph G-S, and
- 2) $|N_c(S)| \ge |S|$

for all $S \subseteq V$ such that $0 \leq |S| \leq \frac{|G|}{2}$ and $|N(S) \setminus S| \geq |S|$.

Proof. Suppose (G, C) is a colored graph satisfying the conditions. Then, condition 1) guarantees that G contains an uncolored perfect matching from Lemma 2.2. In the following we will prove that G also contains a perfect heterochromatic matching by deducing contradictions.

Suppose G does not contain any perfect heterochromatic matching. Let M^* be an uncolored perfect matching of G such that there is a $\omega \in C(M^*)$ satisfying $|E_{\omega}(M^*)| = \max_{M} \max_{c \in C(M)} |E_c(M)|$, where M runs over all the uncolored perfect matchings of G. Let $C^1_{\omega}, C^2_{\omega}, \cdots, C^r_{\omega}$ be the connected components with respect to the color ω .

Next, we will find a set $S \subseteq V(G)$ such that for any $e \in E(M^*)$ we have $|V(e) \cap S| \leq 1$, discussed in the following cases:

Case 1. There is a C^i_{ω} $(1 \le i \le r)$ that has at least two edges in $E_{\omega}(M^*)$.

Then, there are two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2 \in E_{\omega}(M^*)$ such that there is an M^* -alternating path P between them. Assume that the path P is oriented along with the direction of $\overline{u_1v_1}$. Then, let $S_1 = \{v_1\} \cup \{u : u \in N(v_1)\}$, such that if $e \in E(M^*) \cap E(G[N(v_1)])$ then $|V(e) \cap S_1| = 1$. Find all M^* -alternating paths along with the direction of $\overline{u_1v_1}$, and then proceed as the following steps: **Step 1.** $\overrightarrow{u_1v_1}$ is in an M^* -alternating cycle L.

Then, define $S_2 = S_1 \cup \{u : \text{there is a } v \text{ such that } uv \in E(M^* \cap L) \text{ and } u, v \notin N(u_1)\} \cup \{u : \text{there is a } v \text{ such that } uv \in E(M^* \cap L) \text{ and } u \notin N(u_1), v \in N(u_1)\},$ where S_2 is obtained by letting L run over all the M^* -alternating cycles that contain the oriented edge $\overline{u_1v_1}$.

Step 2. $\overrightarrow{u_1v_1}$ is in an M^* -alternating path.

Then, check the even numbered vertices on this path. Similarly, by running over all the M^* -alternating paths that contain the oriented edge $\overrightarrow{u_1v_1}$, we define $S'_3 = \{u : uv \in E(M^*), v \text{ is an even numbered vertex on an } M^*$ -alternating path, and $v \in N(u_1)\}$, and $S''_3 = \{u : u \text{ is an even numbered vertex on an } M^*$ -alternating path, and there is a $v \notin S_2$ such that $uv \in E(M^*)\}$.

Let $S_3 = S'_3 \cup S''_3$ and $S = S_2 \cup S_3$. Then, from the definitions of S_2 and S_3 and the fact that M^* is an uncolored perfect matching, we have $0 \leq |S| \leq \frac{|G|}{2}$ and $|N(S)\backslash S| = |S|$. Since $CN(S) \subseteq CN((N(S)\backslash S)\backslash \{u_1\})$ and $|(N(S)\backslash S)\backslash \{u_1\}| < |S|$, we have $|N_c(S)| < |S|$, a contradiction to condition 2).

Case 2. All C^i_{ω} $(1 \leq i \leq r)$ contain only one edge with the color ω .

Consider a fixed C_{ω}^{i} , and let $e_{i} = u_{i}v_{i}$ is the only edge with the color ω . Then, similar to Case 1, along with the direction of $\overrightarrow{u_{i}v_{i}}$, we can define sets S_{2} and S_{3} . Let $S_{4} = S_{2} \cup S_{3}$, then $|N_{c}(S_{4})| = |S_{4}|$. For another C_{ω}^{j} $(i \neq j)$, similar to C_{ω}^{i} we can define the last set S_{5} such that $|N_{c}(S_{5})| = |S_{5}|$. Let $S = S_{4} \cup S_{5}$, then $CN(S) \subseteq CN((N(S) \setminus S) \setminus \{u_{i}\})$. Since $|(N(S) \setminus S) \setminus \{u_{i}\}| < |S|$, we have $|N_{c}(S)| < |S|$, again a contradiction to condition 2). The proof is thus complete.

To conclude this paper, we give a sufficient and necessary condition to verify if a heterochromatic matching of a general colored graph is maximum, which is an analogue of the well-known Berge's Theorem [2].

Theorem 3..3 A heterochromatic matching M of a colored graph (G,C) is maximum if and only if there is no heterochromatic M-augmenting path system in G.

Proof. Let M be a heterochromatic matching of G. Suppose there is a heterochromatic M-augmenting path system P in G. Then, define $M' = (M \setminus P) \cup (M|_P \triangle P)$, and M' is a heterochromatic matching of G such that |M'| > |M|, and hence, M is not a maximum heterochromatic matching of G.

Conversely, suppose M is not a maximum heterochromatic matching of G. Let M' be a maximum heterochromatic matching of G. Then, |M'| > |M|. Denote $H = G[M \triangle M']$, then every component p of H has one of the following properties:

- 1) the edges of p appear alternately in M and M'. Note that a cycle is also regarded as this kind of path;
- 2) the length of p is odd and at least 3, and the edges of p appear alternately in M and M';
- 3) the length of p is 1, and the signal edge of p is in M or M'.

Since |M'| > |M| and both M' and M are heterochromatic, H contains more edges of M' than of M, and then, $|M \triangle H| > |M' \triangle H|$ and $C(M \triangle H) \cap C(M) \subseteq$ $C(M \cap H)$, and hence, we have $|M \triangle H| > |M \cap H|$. It is not difficult to see that $H = \bigcup_{p \text{ is a component of } H} p$ is a heterochromatic M-augmenting path system of G.

To our knowledge there are few published results on perfect heterochromatic matchings. Suzuki [8] did something only for colored complete graphs, but not for general colored graphs. Our conditions are only sufficient, but not necessary. The following are some simple examples:

Example 1. Let B = (X, Y) with $X = \{v_1, v_2, v_3, v_4\}$ and $Y = \{v_5, v_6, v_7, v_8, v_9, v_{10}\}$ be a bipartite graph such that $E = \{v_1v_5, v_1v_6, v_2v_6, v_2v_7, v_2v_9, v_3v_5, v_3v_8, v_4v_8, v_4v_{10}\}$ and $C(v_1v_5) = C(v_4v_8) = 1$, $C(v_1v_6) = C(v_2v_9) = 2$, $C(v_2v_6) = C(v_3v_5) = C(v_4v_{10}) = 3$, $C(v_2v_7) = 4$, $C(v_3v_8) = 5$. Then, $\{v_1v_6, v_2v_7, v_3v_8, v_4v_{10}\}$ is a heterochromatic matching that saturates every vertex of X. But, if we take $S = \{v_1, v_3, v_4\}$, then $N_c(S) = \{v_6, v_8\}$, and so, $|N_c(S)| < |S|$, which tells us that the condition in our Theorem 3.1 is not necessary. This very example is also valid for showing the un-necessity of the conditions in our Theorem 3.1.

Example 2. For non-bipartite graph, let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{v_1v_2, v_1v_4, v_1v_5, v_2v_3, v_2v_6, v_3v_4, v_3v_6, v_4v_5, v_5v_6\}$ such that $C(v_1v_2) = C(v_2v_6) = C(v_3v_4) = C(v_4v_5) = C(v_1v_4) = 1$, $C(v_2v_3) = C(v_3v_6) = 2$, $C(v_1v_5) = C(v_5v_6) = 3$. Take $S = \{v_1, v_4, v_5\}$. Then $N_c(S) = \{v_2, v_6\}$, and so the conditions in our Theorem 3.2 are not necessary.

From these simple examples, we can see that our conditions of both theorems need great improvement. Moreover, how to find both sufficient and necessary conditions for the existence of perfect heterochromatic matchings in general colored graphs is a more interesting question.

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References

- N. Alon, T. Jiang, Z. Miller and D. Pritikin, Properly colored subgraphs and rainbow subgraphs in edge-colored graphs with local constraints, Random Struct. Algorithoms 23(2003), No. 4, 409-433.
- [2] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, Macmillan, London, Elsevier, New York, 1976.
- [3] H.J. Broersma, X. Li, G. Wöginger and S. Zhang, Paths and cycles in colored graphs, Australasian J. Combin. 31(2005), 299-312.
- [4] M.R. Garey and D.S. Johnson, Computers and Intractability, Freeman, New York, 1979, Page 203. GT55: Multiple Choice Matching Problem.

- [5] R. Jamison, T. Jiang and A. Ling, Constrained Ramsey numbers of graphs, J. Graph Theory 42(1)(2003), 1-16.
- [6] Z. Jin and X. Li, The complexity for partitioning graphs by monochromatic trees, cycles and paths, Intern. J. Computer Math. 81(11)(2004), 1357-1362.
- [7] E.L. Lawler, Combinatorial Optimization: Networks and Matroids, Holt, Rinehart and Winston, New York, 1976.
- [8] K. Suzuki, Heterochromatic matchings in colored complete graphs, Private communication, 2004.