The Compositional Rule of Inference and Zadeh's Extension Principle for Non-normal Fuzzy Sets

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Abstract. Defining the standard Boolean operations on fuzzy Booleans with the compositional rule of inference (CRI) or Zadeh's extension principle gives counter-intuitive results. We introduce and motivate a slight adaptation of the CRI, which only effects the results for non-normal fuzzy sets. It is shown that the adapted CRI gives the expected results for the standard Boolean operations on fuzzy Booleans. As a second application, we show that the adapted CRI enables a don't-care value in approximate reasoning. From the close connection between the CRI and Zadeh's extension principle, we derive an adaptation of the extension principle, which, like the modified CRI, also gives the expected Boolean operations on fuzzy Booleans.

Keywords: compositional rule of inference, extension principle, fuzzy Booleans

1 Introduction

Fuzzy Booleans are introduced in [1], in analogy with the concept of fuzzy numbers [2], as fuzzy sets over the domain of truth-values {true, false}. A fuzzy Boolean is denoted as (a,b), where a and b are numbers from the interval [0,1], a shorthand for the conventional notation "a/true + b/false". The truth-values 'true' and 'false' are represented by (1,0) and (0,1) respectively. The advantage of fuzzy booleans is that they allow fuzzy reasoning with concepts 'contradiction' and 'undefined'. For instance, when interpreted as possibilities, (1,1) is 'undefined', and (0,0) is 'contradiction'; when interpreted as necessities, this is just the other way around.

Let AND and OR be the fuzzy equivalents of the crisp Boolean operators and and or, respectively. They may be defined by means of approximate reasoning. For instance, AND is defined by the following four fuzzy rules:

IF X_1=(1,0) AND X_2=(1,0) THEN Y=(1,0) (1a)
IF X_1=(1,0) AND X_2=(0,1) THEN Y=(0,1) (1b)
IF X_1=(0,1) AND X_2=(1,0) THEN Y=(0,1) (1c)
IF X_1=(0,1) AND X_2=(0,1) THEN Y=(0,1) . (1d)

With approximate reasoning, using the CRI, we obtain

$$\text{AND}\ ((a,b),(c,d)) = (\min(a,c), \max\ \{\min\ (a,d), \min(b,c), \min(b,d)\}) . \qquad (2)$$

Using the abbreviations T = (1,0) (True), F = (0,1) (False), C = (1,1) (Contradiction) and U = (0,0) (Unknown), we obtain for AND:

AND	T	F	C	U
T	T	F	C	U
F	F	F	F	U
C	C	F	C	U
U	U	U	U	U .

This is not in accordance with our intuitive understanding of the AND-operation. For instance, AND (F,U) = U, where we would expect AND (F,U) = F. Indeed, since U is the empty set, the CRI will give U whenever one of the arguments of AND is U. The same thing happens when we define AND with Zadeh's extension principle: whenever one of the arguments of AND is U, the result is U.

The aim of this paper is to introduce and motivate a slight adaptation of the CRI, which only effects the results for non-normal fuzzy sets, and show that the adapted CRI gives the expected results for the standard Boolean operations on fuzzy Booleans. As a second application, we will show that the adapted CRI enables a don't-care value in approximate reasoning. From the close connection between the CRI and Zadeh's extension principle, we will derive an adaptation of the extension principle, which, like the modified CRI, also gives the expected Boolean operations on fuzzy Booleans.

This paper is organised as follows. In the next section, we will introduce the adaptation of the CRI. In section 3 we describe approximate reasoning with the adapted CRI. In section 4 we show that the adapted CRI gives results for the standard Boolean operations on fuzzy Booleans which are in accordance with our intuitive understanding of the AND-operation. In section 5 we show that the adapted CRI enables a don't-care value in approximate reasoning. In section 6 we derive a adaptation of the extension principle, and show that with the adapted extension principle we also obtain the expected Boolean operations on fuzzy Booleans. Section 7 concludes the paper.

2 Adaptation of the compositional rule of inference

Given a fuzzy set A on a domain U and a fuzzy relation R on the domain U⊗V, the CRI gives the fuzzy set B on V which is given by

$$B(v) = \sup_u \min\ (A(u), R(u,v)) . \qquad (3)$$

If A is non-normal, B is non-normal as well. If A is empty, B is empty as well. If A is the crisp singleton set containing only u_0, then $B(v) = R(u_0,v)$. So, for each crisp singleton set A

$$B(v) >= \inf_u R(u,v) . \quad (4)$$

Our adaptation of the CRI is such that eq. (4) holds for every fuzzy set A. Instead of eq. (3), we thus propose

$$B(v) = \sup_u \max(\inf_u R(u,v), \min (A(u), R(u,v)) . \quad (5)$$

This adaptation can only give a different result when A is non-normal. Indeed, if $A(u_0) = 1$ for some u_0 in U then $\sup_u \min (A(u), R(u,v)) >= \min (A(u_0), R(u_0,v)) = R(u_0,v) >= \inf_u R(u,v)$, and so eq. (5) reduces to eq. (3). Also, when $R(u,v) = 0$ for some u in U there is no difference between eq. (5) and eq. (3).

Next we will define how to adapt a composition of two applications of the CRI as in

$$B(v) = \sup_{(u1,u2)} \min(A_1(u_1), \min (A_2(u_2), R((u_1,u_2),v))) . \quad (6)$$

Here it is not appropriate to adapt this in a single step by first writing this as

$$B(v) = \sup_u \min(A_1 \otimes A_2 (u), R(u,v))) \quad (7)$$

where $A_1 \otimes A_2$ is the Cartesean product of A_1 and A_2 and $u = (u_1,u_2)$. Instead, the adaptation should be applied twice, which leads to

$$\begin{aligned} B(v) = \sup_{(u1,u2)} \max(&\inf_{u1} \max(\inf_{u2} R((u_1,u_2),v), \min(A_2(u_2), R((u_1,u_2),v))), \\ &\min(A_1(u_1), \max(\inf_{u2} R((u_1,u_2),v), \min(A_2 (u_2), R((u_1,u_2),v))))) . \quad (8) \end{aligned}$$

3 Approximate reasoning with the adapted CRI

Consider the fuzzy rule

$$\text{IF } X = A' \text{ THEN } Y = B' . \quad (9)$$

Given the fact X = A, approximate reasoning with the standard CRI gives Y = B, where B is given by eq. (3), and R(u,v) is given by

$$R(u,v) = Q (A'(u), B'(v)) . \quad (10)$$

In case of approximate reasoning with the interpolation method [4], the operator Q is a t-norm; the most commonly used t-norm is the minimum operator. In case of approximate reasoning with the implication method [3], the operator Q is an implication operator.

In case of multiple fuzzy rules, one calculates a relation as in eq. (10) for each fuzzy rule, and then aggregates the results. Aggregation is done with the maximum operator in the interpolation method, and with the minimum operator in the implication method. The resulting aggregated relation is then used to calculate the inference results with eq. (3). Using the adapted CRI means that eq. (5) should be used instead of eq. (3).

Consider next the fuzzy rule with two antecedents

$$\text{IF } X_1 = A'_1 \text{ AND } X_2 = A'_2 \text{ THEN } Y = B' . \tag{11}$$

With the standard CRI we have, given the input $X_1 = A_1$ AND $X_2 = A_2$, the output Y = B, where B is given by eq. (6), and

$$R((u_1,u_2),v) = Q\ (\min\ (A'_1(u_1),\ A'_2(u_2)),\ B'(v)) \tag{12}$$

In case of multiple fuzzy rules, one calculates a relation as in eq. (12) for each fuzzy rule, and then aggregates the results. The resulting aggregated relation is then used to calculate the inference results with eq. (6). Using the adapted CRI means that eq. (8) should be used instead of eq. (6). Generalisation to three or more antecedents is straightforward.

Note that we described here the FATI (first aggregate, then inference) approach. In Mamdani's original approach [4], the interpolation method with the minimum operator as t-norm, the FITA (first inference, then aggregate) approach is used. Indeed, it happens that the inference results of FATI and FITA are the same in this case. This is no longer true when the adapted CRI is used. So, with the adapted CRI, one should always adopt the FATI approach.

4 Application to fuzzy Booleans

In this section we will use the results of the previous section to compute the inference results for the four fuzzy rules of eq. (1) with the adapted CRI.

First we compute the four relations R_1, R_2, R_3 and R_4 for the fuzzy rules of eqs. (1a,1b,1c,1d) respectively, and their aggregation R in the interpolation method, using eq. (12):

	((t,t),t)	((t,f),t)	((f,t),t)	((f,f),t)	(t,t),f)	((t,f),f)	((f,t),f)	((f,f),f)
R_1	1	0	0	0	0	0	0	0
R_2	0	0	0	0	0	1	0	0
R_3	0	0	0	0	0	0	1	0
R_4	0	0	0	0	0	0	0	1
R	1	0	0	0	0	1	1	1

Here we used the abbreviations t and f for true and false, respectively.

Next we compute the four relations R_1, R_2, R_3 and R_4 and their aggregation R in the implication method:

	((t,t),t)	((t,f),t)	((f,t),t)	((f,f),t)	(t,t),f)	((t,f),f)	((f,t),f)	((f,f),f)
R_1	1	1	1	1	0	1	1	1
R_2	1	0	1	1	1	1	1	1
R_3	1	1	0	1	1	1	1	1
R_4	1	1	1	0	1	1	1	1
R	1	0	0	0	0	1	1	1

Note that the aggregated relation R is the same for both methods, and is independent of Q. This is in accordance with in general result in [1], where it is proved that this is always the case when the set of fuzzy rules is a complete set of fuzzy rules with crisp antecedents.

Substituting this relation in eq. (8) now gives

$$\text{AND}\ ((a,b),(c,d)) = (\min(a,c),\ \max(b,d)) \tag{13}$$

whereas with the standard CRI (eq. (6)) we would have obtained eq.(2). We can verify that eq. (13) is in accordance with our intuitive understanding of the AND-operation. Indeed, eq. (13) just says that the "trueness" of AND P Q is the trueness of both P and Q, and the "falseness" of AND P Q is the falseness of either P or Q. This should be compared with eq. (2), where the falseness of P does not imply the falseness of AND P Q; the falseness of AND P Q follows only if in addition to the falseness of P we also have either the trueness or the falseness of Q. The table in the introduction is replaced by

AND	T	F	C	U
T	T	F	C	U
F	F	F	F	F
C	C	F	C	F
U	U	F	F	U .

In the same way, we obtain the expression

$$\text{OR}\ ((a,b),(c,d)) = (\max(a,c),\min(b,d)) \tag{14}$$

and the table

OR	T	F	C	U
T	T	T	T	T
F	T	F	C	U
C	T	C	C	T
U	T	U	T	U

which are in accordance with our intuitive understanding of the OR-operation. Finally, the NOT-operation, given by NOT (a,b) = (b,a), is not affected by our adaptation of the CRI.

5 Don't-care value in approximate reasoning

As a second application of the adapted CRI, we will show in this section that with the adapted CRI there exists a don't-care value in approximate reasoning. Consider first the fuzzy rule of eq. (9). Given the fact X = A, approximate reasoning with the CRI gives Y = B, where B is given by

$$B(v) = \sup_u \min (A(u), Q (A'(u),B'(v))) \ . \tag{15}$$

We will consider first the interpolation method, i.e. Q is a t-norm. The fuzzy set A' is a don't-care value if B(v) = B'(v) for all v in V and all fuzzy sets A. Since B(v) <= B'(v) for all v in V, and B(v) increases if A' increases, the best value for A' is the universe U itself, i.e A'(u) = 1 for all u in U. Then eq. (15) becomes

$$B(v) = \sup_u \min (A(u), B'(v)) \tag{16}$$

which means that we have the desired property only if we restrict the input A to be normal. So, a don't-care value does not exist.

The inference result of the fuzzy rule in eq. (9) with the adapted CRI is

$$B(v) = \sup_u \max (\inf_u (Q (A'(u),B'(v))), \min (A(u), Q (A'(u),B'(v)))) \ . \tag{17}$$

Substituting A'(u) = 1 for all u in U gives B(v) = B'(v), which shows that with the adapted CRI A' has the required property, even for non-normal input A. Therefore, the universe U can be taken as don't-care value.

Consider next the following two fuzzy rules:

$$\text{IF } X_1 = A_1 \text{ THEN } Y = B_1 \tag{18a}$$
$$\text{IF } X_2 = A_2 \text{ THEN } Y = B_2 \tag{18b}$$

where the domains of X_1,X_2 and Y are U_1,U_2 and V respectively. For instance when one wants to compile a single relation for both fuzzy rules, one would like to write this as

$$\text{IF } X_1 = A_1 \text{ AND } X_2 = DC_2 \text{ THEN } Y = B_1 \qquad (19a)$$
$$\text{IF } X_1 = DC_1 \text{ AND } X_2 = A_2 \text{ THEN } Y = B_2 \qquad (19a)$$

where DC denotes a don't-care fuzzy set, i.e. the results for the fuzzy rules in eq. (19) should be the same as the results for the fuzzy rules in eq. (18). As above, such a don't-care fuzzy set does not exist. It is however a straightforward exercise to verify, by using the inference results with the adapted CRI (eqs. (5,8)), that the fuzzy rule

$$\text{IF } X_1 = A_1 \text{ AND } X_2 = U_2 \text{ THEN } Y = B \tag{20}$$

gives the same result as the rule

$$\text{IF } X_1 = A_1 \text{ THEN } Y = B \tag{21}$$

and that the rule

$$\text{IF } X_1 = U_1 \text{ AND } X_2 = A_2 \text{ THEN } Y = B \tag{22}$$

gives the same result as

$$\text{IF } X_2 = A_2 \text{ THEN } Y = B \tag{23}$$

showing that in case of fuzzy rules with two antecedents the universe can be taken as a don't-care fuzzy set. Generalisation to three or more antecedents is straightforward.

Next we consider the implication method. Then Q, in eq. (15), is an implication operator. Analogously to the reasoning above, we find that the fuzzy set A' with A'(u) = 0 for all u in U, i.e. the empty set, is a don't-care value.

So the conclusion of this section is that with the adapted CRI there exists a don't-care value for approximate reasoning; this don't-care value is the universe in case of the interpolation method and it is the empty set in case of the implication method.

6 Adaptation of Zadeh's extension principle

Zadeh’s extension principle, developed by Zadeh [6] and elaborated by Yager [5], extends functions from their domain to fuzzy sets on their domain. Let f be a function from the universe U onto the universe V. By Zadeh's extension principle, f maps each fuzzy set A on U onto a fuzzy set B on V which is given by

$$B(v) = \sup_{u:f(u)=v} A(u)\ . \tag{24}$$

There exists an intimate relation between the extension principle and the CRI: When the relation R is defined by

$$\forall u \in U : R\,(u,f(u)) = 1 \tag{25a}$$
$$\forall u \in U\ \forall v \in V : v \neq f(u) \Rightarrow R\,(u,v) = 0 \tag{25b}$$

then eq. (24) is an immediate consequence of eq. (3).

We derive the adapted extension principle in the same way from the adapted CRI. When the relation of eq. (25) is substituted in eq. (5) we find

$$B(v) = 1, \text{ if } \forall u \in U: f(u) = v \tag{26a}$$
$$B(v) = \sup_{u:f(u)=v} A(u), \text{ otherwise .} \tag{26b}$$

So there is a difference with the standard extension principle only in case f is a constant function. Then the membership value of f(u) is equal to 1, while it is equal to $\sup_u A(u)$ according to the standard extension principle. So, if A is normal there is no difference. We feel that this adaptation makes sense; indeed, where f maps each crisp element u of U to the same element v of V, there is no doubt that a fuzzy set on U should be mapped to the singleton set containing v.

This adaptation holds for functions with a single argument. In the case where f is a function with two arguments, whose domains are U_1 and U_2 respectively, we derive the adapted extension principle from eq.(8). We find that

$$f(A_1,A_2)(v) = \sup_{(u1,u2):f(u1,u2)=v} C(u_1,u_2) \tag{27a}$$
$$C(u_1,u_2) = 1, \text{ if } \forall u_3 \in U_1, \forall u_4 \in U_2: f(u_3, u_4) = v \tag{27b}$$
$$C(u_1,u_2) = \max(A_1(u1), A_2(u_2)),$$
$$\text{if } \forall u_4 \in U_2: f(u_1,u_4) = v \ \& \ \forall u_3 \in U_1: f(u_3,u_2) = v \tag{27c}$$
$$C(u_1,u_2) = A_1(u_1), \text{ if } \forall u_4 \in U_2: f(u_1,u_4) = v \tag{27d}$$
$$C(u_1,u_2) = A_2(u_2), \text{ if } \forall u_3 \in U_1: f(u_3,u_2) = v \tag{27e}$$
$$C(u_1,u_2) = \max(A_1(u_1), A_2(u_2)), \text{ otherwise .} \tag{27f}$$

Here the conditions of the five clauses for $C(u_1,u_2)$ should be checked from above.

As an example, let us again compute AND ((a,b),(c,d)). From eq. (25) we find that

$$\text{AND ((a,b),(c,d)) (true)} = C(\text{true,true}) \tag{28a}$$
$$\text{AND ((a,b),(c,d)) (false)} =$$
$$\max (C(\text{true,false}), C(\text{false,true}), C(\text{false,false})) \tag{28b}$$

where C is given by

$$C(\text{true,true}) = \min ((a,b)(\text{true}), (c,d)(\text{true})) = \min (a,c) \tag{29a}$$
$$C(\text{true,false}) = (c,d) (\text{false}) = d \tag{29b}$$
$$C(\text{false,true}) = (a,b) (\text{false}) = b \tag{29c}$$
$$C(\text{false,false}) = \max((a,b)(\text{false}), (c,d)(\text{false}) = \max (b,d) \tag{29d}$$

which leads to

$$\text{AND ((a,b),(c,d))} = (\min (a,c), \max (b,d)), \tag{30}$$

which is, of course, the same result as the one obtained in section 4.

7 Conclusion

We have defined adaptations for the compositional rule of inference and Zadeh's extension principle, which have effect in case of non-normal fuzzy sets. We have demonstrated the usefulness of the adaptations by means of two applications. Firstly, we have shown that we obtain the expected results for the standard Boolean operations on fuzzy Booleans. Secondly, we have shown that we obtain a don't-care value in approximate reasoning. Of course, from these two applications it cannot (yet) be concluded that our adaptations should replace the standard CRI and extension principle. Therefore, it is interesting to examine our adaptations in other applications where non-normal fuzzy sets are used.

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