Theoretical Computer Science 410 (2009) 1319–1327

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Upper bounds and algorithms for parallel knock-out numbers

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ARTICLE INFO

Keywords:
Parallel knock-out schemes
Claw-free graphs
Computational complexity

ABSTRACT

We study parallel knock-out schemes for graphs. These schemes proceed in rounds in each of which each surviving vertex simultaneously eliminates one of its surviving neighbours; a graph is reducible if such a scheme can eliminate every vertex in the graph. We resolve the square-root conjecture, first posed at MFCS 2004, by showing that for a reducible graph G, the minimum number of required rounds is $O(\sqrt{n})$; in fact, our result is stronger than the conjecture as we show that the minimum number of required rounds is $O(\sqrt{\alpha})$, where α is the independence number of G. This upper bound is tight. We also show that for reducible $K_{1,p}$-free graphs at most $p-1$ rounds are required. It is already known that the problem of whether a given graph is reducible is NP-complete. For claw-free graphs, however, we show that this problem can be solved in polynomial time. We also pinpoint a relationship with (locally bijective) graph homomorphisms.

1. Introduction

In this paper, we continue the study on *parallel knock-out schemes* for finite undirected simple graphs introduced in [9] and studied further in [3–5]. Such a scheme proceeds in rounds: in the first round each vertex in the graph selects exactly one of its neighbours, and then all the selected vertices are eliminated simultaneously. In subsequent rounds this procedure is repeated in the subgraph induced by those vertices not yet eliminated. The scheme continues until there are no vertices left, or until an isolated vertex is obtained (since an isolated vertex can never be eliminated).

A graph is *KO-reducible* if there exists a parallel knock-out scheme that eliminates the whole graph. The *parallel knock-out number* of a graph G, denoted by $\mathrm{pko}(G)$, is the minimum number of rounds in a parallel knock-out scheme that eliminates every vertex of G. If G is not KO-reducible, then $\mathrm{pko}(G) = \infty$.

Our main motivation for studying knock-out schemes is the intimate relationship between this concept and well-studied structural graph theoretical concepts such as perfect matchings, hamiltonian cycles and 2-factors (they all yield knock-out schemes of one round). Apart from these structural properties, we are also interested in complexity aspects. Whereas the classical complexity problems related to matchings and hamiltonian cycles have been settled many years ago, the analogous problems related to knock-out schemes have only been resolved recently, and only for general graphs and graphs of bounded tree-width. For many interesting classes, however, these problems on knock-out schemes remain open [4].

Knock-out schemes also have a clear relationship with games on graphs, a topic which has received considerable attention in recent decades [7]. But unlike many games on graphs, knock-out schemes may be motivated by practical settings, e.g., in which objects exchange entities that deactivate the receiving objects, like viruses that paralyse or block computers, or computational tasks that prevent processors or sensors from working on other tasks.

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0304-3975/$ – see front matter
doi:10.1016/j.tcs.2008.03.024

1.1. Our results

In [4], a number of results, conjectures and questions on upper bounds for knock-out numbers were presented. For trees, it was shown that the knock-out number of a tree on n vertices was $O(\log n)$ and a family of trees that met this bound was exhibited. Also presented was a family of bipartite graphs whose knock-out numbers grow proportionally to the square root of the number of vertices, and it was conjectured that for any KO-reducible graph on n vertices the parallel knock-out number is at most $2\sqrt{n}$. In this paper, in Section 3, we prove this conjecture by showing that a KO-reducible n-vertex graph G has

$$\mathrm{pko}(G) \leq \min\left\{-\frac{1}{2}+\sqrt{2n-\frac{7}{4}}, \frac{1}{2}+\sqrt{2\alpha-\frac{7}{4}}\right\},$$

where α denotes the independence number of G.

In [4], a polynomial algorithm was also given that would determine the parallel knock-out number of any tree. In [5] it was shown that the problem of finding parallel knock-out numbers is, for general graphs, NP-complete. In this paper, in Section 4, we present a polynomial algorithm that finds the knock-out number of claw-free graphs, that is, graphs that do not contain an induced $K_{1,3}$; these form a well-studied class of graphs, see [6] for a survey. We also give a tight bound on the knock-out number of KO-reducible $K_{1,p}$-free graphs, generalising a result of [4] on claw-free graphs.

In Section 5, we give an upper bound on the parallel knock-out number of one graph in terms of the parallel knock-out number of another graph: we show that if a graph G allows a so-called locally bijective homomorphism to a smaller graph H then $\mathrm{pko}(G) \leq \mathrm{pko}(H)$. Locally bijective homomorphisms are also called graph coverings. They are well studied and have many applications [1,8].

2. Preliminaries

Graphs in this paper are denoted by $G = (V, E)$. An edge joining vertices u and v is denoted by uv. If not stated otherwise a graph is assumed to be undirected and simple. If a graph G is directed then an arc from a vertex u to a vertex v is denoted by (u, v). For graph terminology not defined below, we refer to [2].

For a vertex $u \in V$ we denote its *neighbourhood*, that is, the set of adjacent vertices, by $N(u) = \{v \mid uv \in E\}$. The *degree* of a vertex is the number of edges incident with it, or, equivalently, the cardinality of its neighbourhood. A subset $U \subseteq V$ is called an *independent set* of G if no two vertices in U are adjacent to each other. The *independence number* α of a graph G is the number of vertices in a maximum independent set of G.

A *complete bipartite* graph $K_{|X|,|Y|}$ is a bipartite graph with the maximum number of edges between its bipartite classes X and Y. If $|X| = 1$, then it is a *star* and the vertex in X is the *centre vertex* and the vertices in Y are *leaves*. If $|X| = |Y| = 1$ we arbitrarily choose one of the star's two vertices to be the centre vertex. A graph G that does not contain a $K_{1,p}$ as an induced subgraph for some $p \geq 1$ is said to be *$K_{1,p}$-free*. A $K_{1,3}$-free graph is also called *claw-free*.

Now we give a more formal definition of knock-out schemes. First, for a graph $G = (V, E)$ and set of vertices $W \subseteq V$, a *KO-selection* is a function $f : W \to W$ with $f(v) \in N(v)$ for all $v \in W$. If $f(v) = u$, we say that vertex v *fires at* vertex u, or that vertex u *is knocked out* by vertex v. We also say that u is a *victim* of v, and that v is an *assassin* of u. For each $u \in W$, we denote the set of assassins of u by $A(u)$; that is, $v \in A(u)$ if and only if $f(v) = u$. If $A(u)$ contains a single vertex v (that is, v is the only vertex that fires at u), then we call u the *unique* victim of v. If $A(u) = \emptyset$, we say that u is a *survivor* of f and the set of all survivors of f is denoted $B(f)$. For a subset $U \subseteq W$ we use the shorthand notation $A(U) = \bigcup_{u\in U} A(u)$, and we say that U is *knocked out* by a subset $Z \subseteq W$ if $A(U) \subseteq Z$, that is, if every vertex in U is knocked out by a vertex in Z.

For $G = (V, E)$, a *KO-reduction scheme* S is a finite sequence of r_S KO-selections $f_1, \ldots f_{r_S}$ where the domain of f_1 is V and the domain of f_i, $2 \leq i \leq r_S$ is $B(f_{i-1})$ and $B(f_{r_S}) = \emptyset$. Each selection in the sequence is called a *round*, or a *firing*, of the KO-reduction scheme and so r_S denotes the number of rounds in the scheme (we omit the subscript when there is no ambiguity). Thus a KO-reduction scheme for a graph is a sequence of firings such that every vertex fires in the first round and in each subsequent round every surviving vertex fires. At the end of the scheme no vertex survives; they have all been knocked out.

If a KO-reduction scheme exists for G, then G is called *KO-reducible*. The parallel knock-out number of G, $\mathrm{pko}(G)$, is either the smallest number r for which such a sequence with r rounds exists or, if no such sequence exists, $\mathrm{pko}(G) = \infty$. For a KO-reduction scheme S we denote the set of vertices that are victims of a vertex v (over all rounds) by $L(v)$. For a subset $Z \subseteq V$, we use the shorthand notation $L(Z) = \bigcup_{v\in Z} L(v)$.

Note that if S is a KO-reduction scheme for G, then it may be possible to obtain further schemes by making small changes to some of the KO-selections. For example, if in some round i, the victim u of a vertex v is not unique, and v has another neighbour w that does not survive round i, then it makes no difference if v fires at w instead of u. So we can obtain another valid KO-reduction scheme by letting $f_i(v) = w$ (instead of having $f_i(v) = u$). In such a case, we might say informally that we are adjusting the firing.

An *in-tree* is a directed tree that contains a *root* u that can be reached from any other vertex by a directed path. Note that a graph containing only one vertex is an in-tree.

Given a KO-reduction scheme, we denote the subset of vertices knocked out in round i, $i = 1, \ldots, r$, by R_i. Let G_i be the directed graph with vertex set R_i and an arc from a vertex u to a vertex v if and only if $f_i(u) = v$. We may also use G_i to denote the underlying undirected graph; it will always be clear which from the context.

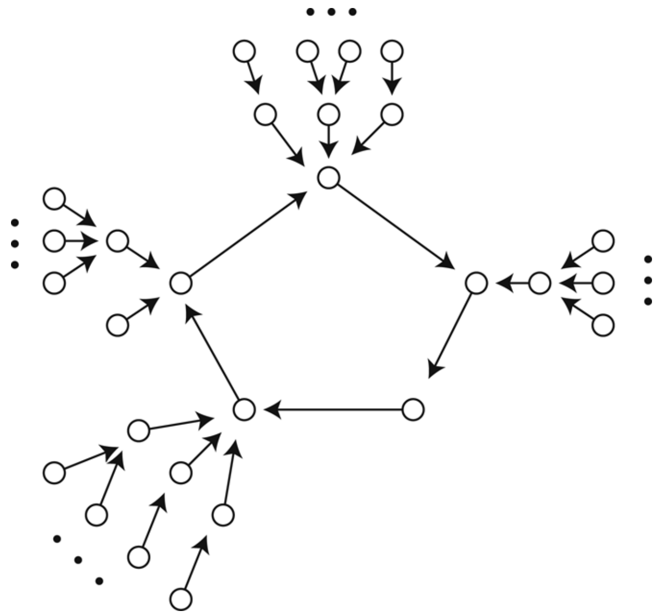

Fig. 1. A component of a graph G_i.

Let us make some simple observations about G_i. Let $i \geq 1$ and let u be a vertex in G_i. By definition of G_i, u is knocked out in round i. It may happen that u is knocked out by vertices that survive round i; that is, $A(u) \cap R_i = \emptyset$. Then u has *in-degree* zero in G_i. On the other hand, $A(u) \cap R_i$ may contain one or more vertices in G_i that fire at u. The vertex u itself fires at exactly one vertex in round i. By definition of G_i, the victim of u is in G_i. Hence, u has *out-degree* exactly one in G_i. We conclude that every component of G_i is a directed graph with out-degree equal to one. It is easy to see that this implies the following; see Fig. 1 for an illustration.

Observation 1. *Let S be a KO-reduction scheme for a graph G. For $i = 1, \ldots, r$, each component of G_i is formed by a directed cycle D on at least two vertices, such that each vertex on D is the root of some pendant in-tree.*

Another observation we will use is the following.

Observation 2. *If a graph G contains two distinct vertices of degree 1 that share the same neighbour, then G is not KO-reducible.*

Note that when referring to, for example, G_i, it is implicit that we know with respect to which KO-reduction scheme this graph is defined. We wish to avoid the cumbersome notation necessary to make it explicit. Sometimes we will be considering pairs of schemes S and S' and will write, for instance, that G_2 has fewer vertices under S' than under S. By this we mean that the number of vertices of G that are knocked out in the second round when we apply scheme S' is less than the number of vertices of G that are knocked out in the second round when we apply scheme S.

3. Resolving the square-root conjecture

Let S be a KO-reduction scheme for a KO-reducible graph G. In this section we prove the square-root conjecture by constructing schemes that knock out vertices "as early as possible". Let us make this notion precise. Let

$$w(S) = \sum_{i=1}^{r_S} i|R_i|,$$

and we say that S is a *minimal* KO-reduction scheme for G if $w(S)$ is minimum over all KO-reduction schemes for G.

For a minimal KO-reduction scheme S of a graph G, we can make a number of further assumptions. We use the following terminology. If G_i has a component C that consists of two vertices u and v we call C a *two-component* of G_i. Note that there must be arcs (u, v) and (v, u) between the vertices u and v of a two-component C. If G_i has a component C that consists of vertices $u, v_1, \ldots, v_p$ for some $p \geq 2$ and arcs $(u, v_1), (v_1, u), (v_2, u), \ldots, (v_p, u)$ then we call C a *star-component* of G_i with *centre vertex* u. The vertices $v_1, \ldots, v_p$ are called the *leaves* of C, and v_1 is called the *centre-victim*, and the other leaves are called *centre-free*. Finally, if G_i has a component that is a directed cycle with an odd number of vertices then we call such a component an *odd cycle-component* of G_i.

Lemma 3. *If G is KO-reducible, then G admits a minimal KO-reduction scheme S with the following properties:*

(i) *Each component C of G_1 is either a two-component, a star-component or an odd cycle-component.*
(ii) *For $2 \leq i \leq r - 1$, every component of G_i is either a two-component or a star-component.*

(iii) *Every component of G_r is a two-component.*
(iv) *If C is an odd cycle-component (in G_1) then no vertices of $R_2, \dots, R_r$ fire at vertices of C in round 1.*
(v) *For $1 \le i \le r-1$, there is no edge in G between any two leaves of the same star-component or of two different star-components in G_i.*

Proof. Let G be a KO-reducible graph. Then G admits a KO-reduction scheme S. Let C be a component in G_i for some $1 \le i \le r$. We start the proof by showing that if S is minimal, then we can assume that C is either a two-component, a star-component or an odd cycle-component. By Observation 1, C is formed by a directed cycle D on vertices $u_1, \dots, u_p$ for some $p \ge 2$, such that each u_i is the root of some pendant in-tree T_i.

Suppose that p is even and $p \ge 4$. We adjust the firing by letting the vertices of V_D fire at each other according to a perfect matching of D. Hence, we may assume that this case does not occur.

Suppose that $p \ge 3$ is odd. If D contained a vertex that is knocked out by some vertex v in its corresponding pendant in-tree, then we can adjust the firing by letting the vertices of $V_D \cup \{v\}$ fire at each other according to a perfect matching of this subgraph. Hence, we may assume that $C = D$ is an odd cycle-component.

Suppose that $p = 2$. Then the underlying undirected graph of C is a tree, and it is obvious that it can be decomposed into two-components and star-components (and that we can let these components define the firing).

By Observation 2, we have that G_r cannot contain any star-components. To complete the proof of (i)–(iii), we must show that odd cycle-components only occur in G_1. To do this we shall first prove a claim which also immediately implies (iv): for any odd cycle-component D we may assume that $A(D) = D$; that is, vertices in D are only knocked out by each other. Suppose D is an odd cycle-component on vertices $u_1, \dots, u_p$ in some G_i for $i \ge 1$, such that there exists a vertex $v \in A(D) \backslash D$ and v fires at u_1. We adjust the firing by replacing the arc (u_p, u_1) by (u_p, u_{p-1}) and return to a previous case. Hence, we may assume that this case does not occur.

Now suppose that a graph G_i, $i \ge 2$, contains an odd cycle-component D. First suppose that in round $i-1$ all vertices in D fire at vertices in R_{i-1} that either are centre vertices of star-components, or else belong to two-components or odd cycle-components. Since we just saw that no vertices in $R_{i+1} \cup \dots \cup R_r$ fire at D, we can move D to G_{i-1} (since all victims of D in R_{i-1} are not unique, it does not matter if the vertices of D fire at each other instead). This way we obtain a KO-reduction scheme S' with $w(S') < w(S)$. This contradicts the minimality of S. In the remaining case, there exists a vertex u in D that fires at a leaf w in a star-component in R_{i-1}. We let u and w fire at each other in round $i-1$, so we are able to move u to R_{i-1} as $A(D) = D$. We let the other vertices in D fire at each other in round i according to a perfect matching of $D - u$. This way we again obtain a KO-reduction scheme S' with $w(S') < w(S)$, contradicting the minimality of S.

To finish the claim we prove (v). Suppose that u and v are leaves in G_i for some $1 \le i \le r-1$, such that u and v are adjacent in G. In case u and v are leaves of different star-components, we adjust the firing by letting u and v fire at each other, and, if necessary, changing the centre-victims to be vertices other than u and v. Suppose that u and v are leaves of the same star-component C. Let z be the centre vertex of C. If C has a third leaf, then we again let u and v fire at each other and let another leaf be the centre-victim. Otherwise we can form an odd cycle-component and return to a previous case. □

We call a minimal KO-reduction scheme S of a graph G that satisfies the properties (i)–(v) of Lemma 3 a *simple* KO-reduction scheme of G. We will continue to find further properties of simple KO-reduction schemes.

Observation 4. *Let S be a simple KO-reduction scheme for a graph G. Let u, v be, respectively, vertices of R_i and R_j, $i < j$, such that u is the unique victim of v. Then u is a centre-free leaf of a star-component in G_i.*

Proof. By Lemma 3, u cannot be a vertex of an odd cycle-component. If u is in a two-component, or u is the centre vertex or centre-victim of a star-component, then there are at least two vertices firing at u. Hence u must be a centre-free leaf of a star-component. □

Lemma 5. *Let S be a simple KO-reduction scheme for a graph G with $r \ge 2$. Let C be a two-component in G_r. Then in rounds $1, \dots, r-1$ all victims of one of the two vertices of G_r are not unique, and all victims of the other one are unique.*

Proof. For $i = 1, \dots, r-1$, let x_i be the victim of u in round i, and let y_i be the victim of v in round i.

Suppose that both x_{r-1} and y_{r-1} are not unique victims. We show that this means that it is possible to move u and v to R_{r-1}. If $x_{r-1} \neq y_{r-1}$ or $x_{r-1} = y_{r-1}$ is the victim of vertices other than u and v, then let u and v fire at each other in round $r-1$. If $x_{r-1} = y_{r-1}$ is fired at by only u and v, then it is a centre-free vertex of a star-component and we can adjust the firing to let u, v and x_{r-1} form an odd cycle-component in G_{i-1}. Either way we obtain a new KO-reduction scheme S' with $w(S') < w(S)$, contradicting the minimality of S. Hence we can assume that y_{r-1} is a unique victim.

We show that all victims of u are not unique by contradiction. Let h be the largest index such that x_h is unique. By Observation 4, vertices x_h and y_{r-1} are centre-free leaf vertices of star-components. Since centre vertices are not unique victims, we can let u and x_h fire at each other in round h, and we can let v and y_{r-1} fire at each other in round $r-1$. This way we obtain a new KO-reduction scheme S' with $w(S') < w(S)$. This contradicts the minimality of S.

Now we again find a contradiction to show that all victims of v are unique. Let h be the largest index such that y_h is not a unique victim. Then we let v fire at y_j in round $j-1$ for $j = h+1, \dots, r-1$ (so we move those vertices from R_j to R_{j-1}), and v does not fire at y_h anymore. Since x_{r-1} is not a unique victim, we can then let u and v fire at each other in round $r-1$. This way we obtain a new KO-reduction scheme S' with $w(S') < w(S)$. This contradicts the minimality of S and completes the proof of the lemma. □

Lemma 6. *Let S be a simple KO-reduction scheme for a graph G with $r \geq 2$. For each $i \geq 2$, R_i contains a vertex v_i whose victims in round $1, \ldots, i-1$ are all unique. Let u_r be the (unique) neighbour of v_r in G_r. Then $\bigcup_{i=2}^{r} L(v_i) \cup \{u_r\}$ is an independent set of cardinality $\frac{r^2-r+2}{2}$ in G.*

Proof. Since R_r is non-empty, there exists a two-component C in G_r. Let u_r and v_r be the two vertices of C. By Lemma 5, we may assume that all victims of u_r in rounds $i = 1, \ldots, r-1$ are not unique, and all victims of v_r are unique. Denote the victims of v_r in rounds $i = 1, \ldots, r-1$ by $y_1^r, \ldots, y_{r-1}^r$, respectively. By Observation 4, every y_i^r is a centre-free leaf vertex of a star-component C_i^r. For $i = 2, \ldots, r-1$, let v_i be the centre vertex of C_i^r and for $h = 1, \ldots i-1$, let y_h^i be the victim of v_i in round h. We claim that these victims y_h^i are all unique. For $i = r$, this is already shown. We prove the rest of the statement by contradiction. Let $2 \leq i \leq r-1$. Let h be the largest index such that y_h^i is not a unique victim of v_i. We adjust the firing as follows. Since y_h^i is not a unique victim of v_i, we do not have to let v_i fire at it. Then we let v_i fire at y_j^i in round $j-1$ for $j = h+1, \ldots, i-1$, so we move y_j^i to R_{j-1} for $j = h+1, \ldots, i-1$. In round $i-1$ we let v_i fire at y_i^r, so we move y_i^r to R_{i-1}. Then we do not have to let v_r fire at y_i^r. Hence, we can let v_r fire at y_j^r in round $j-1$ for $j = i+1, \ldots, r-1$, so we move y_j^r to round $j-1$ for $j = i+1, \ldots, r-1$. Finally, we let u_r and v_r fire at each other in round $r-1$. This is possible, because the victim of u_r in round $r-1$ is not unique, due to Lemma 5. This way we have obtained a new KO-reduction scheme S' with $w(S') < w(S)$, contradicting the minimality of S.

We will now prove that

$$L = \bigcup_{i=2}^{r} L(v_i) = \bigcup_{i=2}^{r} \bigcup_{h=1}^{i-1} y_h^i$$

is an independent set. We first note that

$$|L| = \left| \bigcup_{i=2}^{r} \bigcup_{h=1}^{i-1} y_h^i \right| = \sum_{i=2}^{r} \sum_{h=1}^{i-1} 1 = \frac{r^2 - r}{2},$$

since all vertices in L are unique victims.

As S is simple, by Lemma 3, there is no edge between any two vertices y_h^i and y_h^j. Suppose that there is an edge $y_h^i y_j^r$, where $h \neq j$. If $h < j$, then we move y_j^r to R_h, each y_k^r for $k = j+1, \ldots, r-1$ to R_{k-1}, and finally u_r and v_r to R_{r-1}. We can adjust the firing and obtain a new KO-reduction scheme S' with $w(S') < w(S)$. This contradicts the minimality of S. If $h > j$, then we move y_h^i to R_j, each y_k^r for $k = i, \ldots, r-1$ to R_{k-1}, and finally u_r and v_r to R_{r-1}. We adjust the firing and obtain the same contradiction as before. Suppose that there exists an edge between two vertices y_h^i and y_j^k with $h < j$ and $r \notin \{i, j\}$. We move y_j^k to R_h, each y_ℓ^r for $\ell = j, \ldots, r-1$ to $R_{\ell-1}$, and finally u_r and v_r to R_{r-1}. We adjust the firing and obtain the same contradiction as before.

Now suppose that u_r is adjacent to a vertex y_h^i of L. By Lemma 5, all victims of u_r are not unique. Then we can let u_r fire at y_h^i in round i. Then y_h^i is no longer a unique victim and we find a KO-reduction scheme S' with $w(S') < w(S)$ as before. This final contradiction completes the proof. □

We are now ready to state our main theorem, which proves (and strengthens) the square-root conjecture posed in [4].

Theorem 7. *Let G be a KO-reducible graph. Then*

$$pko(G) \leq \min\left\{ -\frac{1}{2} + \sqrt{2n - \frac{7}{4}},\ \frac{1}{2} + \sqrt{2\alpha - \frac{7}{4}} \right\}.$$

Proof. It is straightforward to check that the statement holds for a graph G with $\text{pko}(G) = 1$. Let S be a simple KO-reduction scheme for a graph G with $r \geq \text{pko}(G) \geq 2$. By Lemma 6, we find an independent set L' of G that has cardinality $|L'| = \frac{1}{2}(r^2 - r + 2) \leq \alpha$. Note that R_1 contains a centre vertex of a star-component. This, together with Lemmas 5 and 6, implies that $n \geq |L'| + r - 1 + 1 = \frac{1}{2}(r^2 - r + 2) + r$. Solving both inequalities gives us the required upper bound. □

We note that the bound mentioned in Theorem 7 is asymptotically tight. In [4], it has been proven that for all $p \geq 1$, $\text{pko}(K_{p,q}) = p = \Theta(\sqrt{n}) = \Theta(\sqrt{\alpha})$ for all complete bipartite graphs on $n = p + q$ vertices with $q = \frac{1}{2}p(p+1)$.

4. Claw-free graphs

It is known that claw-free graphs can be knocked out in at most two rounds [4] if they are KO-reducible (not all claw-free graphs are, take for example an isolated vertex or a path on three vertices). We generalise this result for $K_{1,p}$-free graphs for any $p \geq 2$. This solves a question in [4].

Theorem 8. *Let $p \geq 1$. If a $K_{1,p}$-free graph G is KO-reducible then $pko(G) \leq p - 1$.*

Proof. The case $p = 1$ is trivial. For $p \geq 2$, the statement follows directly from Lemma 6. □

This result is the best possible. In [4, Section 4], a tree Y_ℓ is defined for each integer $\ell \geq 1$, and it is shown that $\text{pko}(Y_\ell) = \ell$. It is also easy to check that Y_ℓ is $K_{1,\ell+1}$-free. We omitted the details.

In the rest of this section, we suppose that $G = (V, E)$ is a claw-free graph and show that $\text{pko}(G)$ can be determined in polynomial time. We need the following lemma.

Lemma 9. *Let G be a connected claw-free graph with $pko(G) = 2$. Then there is a simple KO-reduction scheme in which only two vertices u and v survive to the second round.*

Proof. By Lemma 3 and claw-freeness, we know there is a simple two-round KO-reduction scheme S for G such that

(i) each component of G_1 is a two-component, star-component or odd cycle,
(ii) each component of G_2 is a two-component,
(iii) in the first round the vertices of G_2 do not fire at vertices that belong to odd cycles in G_1, and
(iv) the leaves of the star-components in G_1 are not adjacent.

As the leaves of the star-components are not adjacent, we can, by claw-freeness and Lemma 3, further suppose that each star-component is a path on three vertices which we shall call a *three-component*.

Note that among all schemes that satisfy these properties, S is the one with the fewest number of components in G_2 (as it is minimal). To prove the lemma, we show that if, for S, G_2 contains more than one component, then we can find a scheme S' that admits fewer components to G_2.

For S, let the vertex sets of the two-components of G_2 be $\{\{u_i, v_i\} \mid i = 1, \ldots, q\}$. By Lemma 5, we can assume that the victim of u_i in G_1 is not unique, but that of v_i is unique. By Observation 4, v_i fires at the centre-free leaf of a three-component, say y_i. Let x_i be the victim of u_i. Suppose that x_i is the centre vertex of a three-component. Then there is also an edge from u_i to one of the leaves, say w, of the three-component (else, by (iv), x_i, u_i and the leaves of the three-component induce a claw). Let z be the other leaf of the three-component.

Suppose that $y_i = w$. Then let S' be a scheme identical to S except that in the first round

- v_i fires at y_i,
- y_i fires at u_i,
- u_i fires at v_i,
- x_i and z fire at each other.

Thus S' has one fewer two-component in G_2 than S.

Suppose that $y_i = z$. Then let S' be a scheme identical to S except that in the first round

- v_i and y_i fire at each other,
- u_i fires at x_i,
- x_i fires at w,
- w fires at u_i.

Thus S' has one fewer two-component in G_2 than S.

Suppose that $y_i \notin \{w, z\}$. Then let S' be a scheme identical to S except that in the first round

- v_i and y_i fire at each other,
- u_i and w fire at each other, and
- x_i and z fire at each other.

Thus S' has one fewer two-component in G_2 than S. Hence, we have proven that x_i is not the centre-vertex of a three-component.

Suppose that x_i is the leaf of a three-component. If y_i also belongs to this three-component, then, since $x_i \neq y_i$, we have that u_i, v_i and the three-component of their victims lie on a 5-cycle in G. Then let S' be a scheme identical to S except that in the first round these five vertices fire according to an orientation of this 5-cycle. Thus S' has one fewer two-component in G_2 than S.

If x_i is the leaf of a three-component that does not contain y_i, then u_i, v_i and the components containing their first round victims lie on a path of length 8 in G so can be matched. So let S' be a scheme identical to S except that in the first round these eight vertices fire according to this matching. Thus S' has one fewer two-component in G_2 than S.

Thus x_i is not the leaf of a three-component, and, by (iii), x_i belongs to a two-component.

Thus u_i and v_i combined with the components of G_1 containing their victims lie on a path of length 7 in G. We call such a path a *seven-component*. Let us motivate this choice of name by showing that the seven-components are vertex-disjoint.

The vertices v_i, $1 \leq i \leq r$, fire at distinct three-components in the first round (as their victims are unique and one of the leaves of each three-component is the centre-victim). We must also show that the victims x_i of the vertices u_i, $1 \leq i \leq r$, belong to distinct two-components. Suppose that x_i and x_j, $i \neq j$, are distinct but belong to the same two-component in G_1. Then let S' be a scheme identical to S except that in the first round

- v_i and y_i fire at each other,
- v_j and y_j fire at each other,

- u_i and x_i fire at each other, and
- u_j and x_j fire at each other.

Again S' has fewer two-components in G_2 than S. Now suppose that $x_i = x_j$. If either u_i or u_j is adjacent to the other vertex in x_i's two-component, then we have the previous case. Otherwise, there is an edge u_iu_j (else there is a claw). So let S' be a scheme identical to S except that in the first round

- v_i and y_i fire at each other,
- v_j and y_j fire at each other, and
- u_i and u_j fire at each other.

Again S' has fewer two-components in G_2 than S.

We have shown that the seven-components are vertex-disjoint. Note that all the three-components in G_1 contain a victim of a vertex in G_2 and so must be a subgraph of a seven-component. Thus we can represent S as a collection of vertex-disjoint seven-components, two-components and odd cycles that span G. We denote such a representation G^*. Note that the number of two-components in G_2 is equal to the number of seven-components in G^*. Thus to prove the lemma we show that if for S there is more than one seven-component in G^*, then we can find another scheme with fewer seven-components.

Let $A = a_1 \cdots a_7$ and $B = b_1 \cdots b_7$ be a pair of seven-components in G^*. First we consider the case where, in G, A and B are joined by an edge a_ib_j for some i, j. We shall show that this implies that the vertices of A and B admit a perfect matching; thus we can replace two seven-components by seven two-components.

If i and j are both odd, then we match a_i with b_j and the remaining vertices and edges of A and B form paths of even length, so can clearly be matched. If i is even and j is odd, then, if either a_{i-1} or a_{i+1} is adjacent to b_j, we have the previous case. Otherwise, by claw-freeness, there is an edge $a_{i-1}a_{i+1}$ and we include both this and a_ib_j in the matching, and, again, what remains of A and B are paths of even length. Finally suppose that i and j are both even. If there are any other edges from a vertex in $\{a_{i-1}, a_i, a_{i+1}\}$ to a vertex in $\{b_{j-1}, b_j, b_{j+1}\}$, then we have an earlier case. Otherwise, claw-freeness implies edges $a_{i-1}a_{i+1}$ and $b_{j-1}b_{j+1}$, and we include these and a_ib_j in the matching to again leave only even-length paths.

So we can assume that no pair of seven-components in S are joined by an edge in G. Now let us assume that S is such that we can find seven-components A and B such that the length of the shortest path in G between them is minimum (that is, there is no pair of seven-components in any other simple scheme separated by a shorter path).

Suppose that a shortest path from A to B meets A at a_i and the next vertex along is w. In G^*, w must belong to either a two-component or an odd cycle.

First suppose that w is in a two-component C whose other vertex is z. We describe how to use the vertices of A and C to find a seven-component A' and two-component C' such that w is in A'; thus A' is closer to B than A contradicting our choice of A and B. By symmetry, there are four cases according to which vertex of A neighbours w. Suppose that a_1 is adjacent to w. Then replace A and C with $A' = zwa_1 \cdots a_5$ and $C' = a_6a_7$. If a_2 is adjacent to w, then claw-freeness implies that one of the edges a_1a_3, a_1w or a_3w is present. Let C' be, respectively, a_6a_7, a_6a_7 or a_1a_2, and in each case we find a path of length 7 on the remaining vertices to be A'. If a_3 is adjacent to w, then let $A' = zwa_3 \cdots a_7$ and $C' = a_1a_2$. If a_4 is adjacent to w, then one of a_3a_5, a_3w or a_5w is present. Let C' be, respectively, a_1a_2, a_1a_2 or a_6a_7, and in each case we find a path of length 7 on the remaining vertices to be A'.

Finally suppose that w belongs to an odd cycle. If a_i, i odd, is joined to w, then there is a perfect matching on the vertices of A and the cycle and we have a scheme with fewer seven-components. Suppose that a_i, i even, is adjacent to w. If either a_{i-1} or a_{i+1} is joined to w, then we have the previous case. Otherwise, there must be an edge $a_{i-1}a_{i-1}$, and if we match both this pair of vertices and a_i and w, then the remaining vertices of A and the cycle induce even-length paths and a perfect matching can again be found. □

Theorem 10. *Computing the parallel knock-out number of a claw-free graph can be done in polynomial time.*

Proof. By Theorem 8, it is sufficient to present methods for checking whether or not $\mathrm{pko}(G)$ is equal to 1 or 2, since if it is neither it must be ∞. Deciding whether a graph can be knocked-out in a single round can be solved in polynomial time [4]. So we need only show how to check whether G can be knocked out in two rounds.

Suppose that $\mathrm{pko}(G) = 2$. By Lemma 9, we can assume that there is a two-round simple KO-reduction scheme for G in which only two vertices, say u and v, survive to the second round, and, by the proof of the lemma, there is exactly one three-component in G_1.

Let w be the first round victim of v. Then $G-\{u, v, w\}$ has a spanning subgraph comprising two-components and odd cycles (that is, $G_1 - w$) and can thus be knocked out in one round. Therefore the following is a necessary condition for $\mathrm{pko}(G) = 2$: there are three vertices u, v and w in V such that

- there are edges uv and vw,
- u and w have neighbours other than v and each other, and
- $\mathrm{pko}(G - \{u, v, w\}) = 1$.

It is easy to see that this condition is also sufficient. Therefore to decide whether or not $\mathrm{pko}(G) = 2$, we look for a set of three vertices that satisfies this condition. This can be done in polynomial time. □

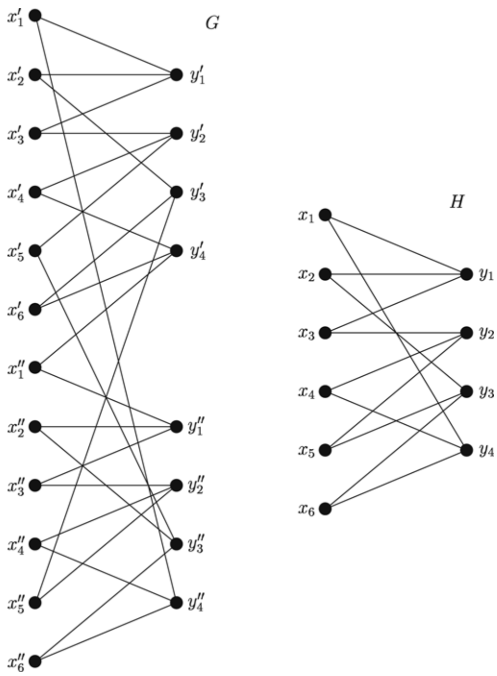

Fig. 2. Two graphs G, H with $G \xrightarrow{B} H$ and $\text{pko}(G) < \text{pko}(H)$.

As noted before any graph with $\text{pko}(G) = 1$ has a spanning subgraph consisting of a number of mutually disjoint matching edges and disjoint cycles. For *claw-free* graphs we have found the following characterisation, which directly follows from the proof of Lemma 9.

Corollary 11. *Let G be a connected claw-free graph with $\text{pko}(G) = 2$. Then G has a spanning subgraph consisting of a number of vertex-disjoint matching edges, odd cycles and one path on seven vertices.*

5. Locally bijective homomorphisms

A *graph homomorphism* from $G = (V_G, E_G)$ to $H = (V_H, E_H)$ is a vertex mapping $f : V_G \to V_H$ satisfying the property that for any edge uv in E_G, we have $f(u)f(v)$ in E_H as well, i.e., $f(N_G(u)) \subseteq N_H(f(u))$ for all $u \in V_G$. For two graphs G and H we write $G \xrightarrow{B} H$ if there exists a so-called *locally bijective* homomorphism $f : V_G \to V_H$ satisfying:

$$\text{for all } u \in V_G : f(N_G(u)) = N_H(f(u)) \quad \text{and} \quad |f(N_G(u))| = |N_G(u)|.$$

We compare the parallel knock-out numbers of two graphs G and H with $G \xrightarrow{B} H$. Then we find that $\text{pko}(H)$ is an upper bound for $\text{pko}(G)$.

Proposition 12. *If $G \xrightarrow{B} H$ then $\text{pko}(G) \le \text{pko}(H)$.*

Proof. If $\text{pko}(H) = \infty$ the statement holds. Suppose that $\text{pko}(H) = k$ for some integer k and consider a parallel knock-out scheme that eliminates H in exactly k rounds. Let $f : V_G \to V_H$ be a locally bijective homomorphism. For any pair $x, y \in V_H$ with x firing at y in the first round we do as follows. In G we let each vertex u with $f(u) = x$ fire at its (only) neighbour v with $f(v) = y$. Clearly there is a locally bijective homomorphism from the KO-successor of G to the KO-successor of H (the restriction of f to the remaining vertices is one). Thus we can, in the same way, decide how the vertices of G should fire in the second and subsequent rounds, and so a reduction scheme for G that also has k rounds is obtained. $\square$

We note that the reverse implication is not true. Let P_n denote the path on n vertices. Then we can take $G = P_2$ and $H = P_3$. Clearly, there does not exist a locally bijective homomorphism from G to H. However, $\text{pko}(G) = 1 < \text{pko}(H) = \infty$.

In Fig. 2, we illustrate an example that shows that strict inequality may hold in the statement of Proposition 12: it displays two graphs G and H with $G \xrightarrow{B} H$ and $\text{pko}(G) < \text{pko}(H)$. This can be seen as follows. The mapping $f : V_G \to V_H$ defined by $f(x_i') = f(x_i'') = x_i$, for $1 \le i \le 6$, and $f(y_j') = f(y_j'') = y_j$, for $1 \le j \le 4$, is a locally bijective homomorphism from G to H. Below we show that $\text{pko}(G) = 2 < \infty = \text{pko}(H)$.

We first need some terminology. A bipartite graph G is called $(2, 3)$-*regular* if all vertices in one class of the bipartition have degree 2 and all other vertices have degree 3. Let $F = (V, E)$ be a $(2, 3)$-regular bipartite graph. Let X denote the vertices with degree 2, and Y the vertices with degree 3. Then $|E| = 2|X| = 3|Y|$, so $|Y| = 2\ell$ and $|X| = 3\ell$ for some positive integer ℓ. We call a subset Y^* of Y with ℓ vertices that has the whole set X as its neighbourhood a *star cover* of F. Note that both G and

H are (2, 3)-regular bipartite graphs. Furthermore, G has a star cover $\{y_1', y_4', y_3'', y_2''\}$ while H does not have a star cover. Then $\text{pko}(G) = 2$ and $\text{pko}(H) = \infty$ follow immediately from a result from [5] on (2, 3)-regular bipartite graphs that states that a (2, 3)-regular bipartite graph G is KO-reducible if and only if G has a star cover and in this case $\text{pko}(G) = 2$.

6. Conclusions

We solved the square-root conjecture of [4] by giving a tight upper bound on the parallel knock-out number of a KO-reducible graph G. We also showed that the parallel knock-out number of a KO-reducible $K_{1,p}$-free graph is at most $p - 1$, and that this bound is tight. We also gave an upper bound on the parallel knock-out number of a graph in terms of the parallel knock-out number of a smaller graph, to which a locally bijective homomorphism exists. For claw-free graphs we showed that their parallel knock-out number can be computed in polynomial time. The question of whether the parallel knock-out number for $K_{1,p}$-free graphs with $p \geq 4$ can also be computed in polynomial time remains open.

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