Geometry and Computing

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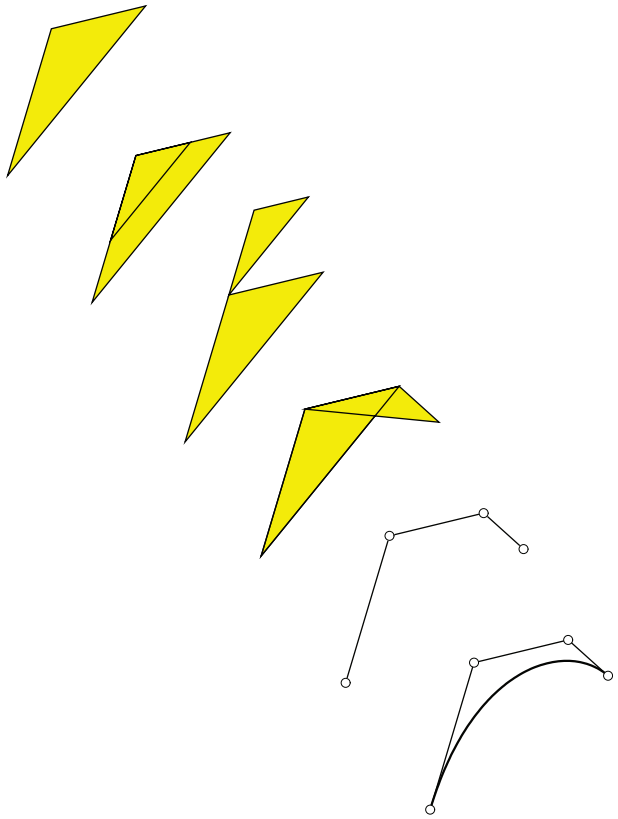
There is no excellent beauty that hath not
some strangeness in the proportion.

Francis Bacon (1561–1626)



Rida T. Farouki

Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable

With 204 Figures and 15 Tables

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Preface

Pythagorean–hodograph curves are characterized by the special property that their "parametric speed" — i.e., the derivative of the arc length with respect to the curve parameter — is a polynomial (or rational) function of the parameter. This distinctive attribute, achieved by *a priori* construction of the hodograph (derivative) components of polynomial or rational curves in $\mathbb{R}^n$ as elements of *Pythagorean* $(n+1)$*–tuples*, endows the Pythagorean–hodograph (PH) curves with many computationally attractive features. For example, it is possible to compute their arc lengths, bending energies, and offset (parallel) curves in an essentially exact manner, without recourse to approximations; and they are exceptionally well–suited to problems of real–time motion control and spatial path planning based on the use of rotation–minimizing frames.

This study surveys and assesses the considerable body of research on PH curves that has accumulated since their inception in 1990. As indicated by the Contents, this research spans a spectrum of topics ranging from elucidation of the basic mathematical theory of PH curves, through development of practical algorithms for their construction and analysis, to the demonstration of their use in computer–aided design and manufacturing applications.

In contrast to the traditional (Bézier/B–spline) schemes of computer–aided geometric design, the PH curves require models that are inherently *non–linear* in nature. However, by use of appropriate algebraic tools — *complex numbers* and *quaternions* for planar and spatial PH curves, and *Clifford algebra* for the most general setting — their construction and analysis is greatly facilitated. The investigation of PH curves thus offers an excellent context and motivation for exploring the pervasive ties between *algebra* and *geometry*.

For ease of access, the material has been organized into seven parts, each comprising a number of chapters. Parts I through III are expository in nature, and serve to establish the required mathematical background. The core theory of planar and spatial PH curves is then developed in Parts IV and V, while Parts VI and VII present practical details on their construction, analysis, and applications. A more detailed synopsis of contents may be found in Chapter 1.

It is inevitable that a study of this nature will lean toward greater emphasis on the author's own contributions, if only because they shape his perspective on the subject matter. Nevertheless, an effort has been made to summarize the key ideas (if not the technical details) of all the most significant developments in the field, and give pointers to many others. The subject matter originated in papers co–authored with Takis Sakkalis in 1990. Subsequently, the author has been fortunate to have the opportunity to pursue related research with many other distinguished colleagues — including Gudrun Albrecht, Hyeong In Choi, Paolo Costantini, Carlotta Giannelli, Chang Yong Han, Sung Chul Jee, Song Hwa Kwon, Jairam Manjunathaiah, Carla Manni, Hwan Pyo Moon, Andy Neff, Lyle Noakes, Francesca Pelosi, Christian Perwass, Jörg Peters, Helmut Pottmann, Kazuhiro Saitou, Lucia Sampoli, Thomas Sederberg, Alessandra Sestini, and Tait Smith, and also a number of graduate students (Mohammad al–Kandari, Bryan Feldman, Bethany Kuspa, David Nicholas, Sagar Shah, Sebastian Timar, Yi–Feng Tsai, and Guo–Feng Yuan). Much of the material presented in Parts IV through VII is a direct outcome of these enlightening, fruitful, and always enjoyable collaborations.

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Since a perusal of the Table of Contents may leave the reader wondering as to whether this volume was intended as a textbook, research monograph, or historical treatise, some explanatory remarks are perhaps in order. There was, in fact, no conscious intent to aim for any of these — but what has transpired seems, in part, each of them. In the pursuit of research and scholarly endeavor, there is nothing remiss in simply pursuing one's intuition — on the contrary, this often proves the most enjoyable and rewarding *modus operandi*.

Davis, California, June 2007 Rida T. Farouki

Some books are to be tasted, others to be swallowed, and some few are to be chewed and digested; that is some books are to be read only in parts; others to be read but not curiously; and some few to be read wholly, and with diligence and attention. Some books also may be read by deputy, and extracts of them made by others.

Francis Bacon (1561–1626)

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