

# Modules over Monads and Linearity

André Hirschowitz<sup>1</sup> and Marco Maggesi<sup>2</sup>

<sup>1</sup> LJAD, Université de Nice Sophia-Antipolis, CNRS

<http://math.unice.fr/~ah>

<sup>2</sup> Università degli Studi di Firenze

<http://www.math.unifi.it/~maggesi>

**Abstract.** Inspired by the classical theory of modules over a monoid, we give a first account of the natural notion of module over a monad. The associated notion of morphism of left modules ("linear" natural transformations) captures an important property of compatibility with substitution, in the heterogeneous case where "terms" and variables therein could be of different types as well as in the homogeneous case. In this paper, we present basic constructions of modules and we show examples concerning in particular abstract syntax and lambda-calculus.

## 1 Introduction

Substitution is a major operation. Its relevance to computer sciences has been stressed constantly (see e.g. [7]). Mathematicians of the last century have coined two strongly related notions which capture the formal properties of this operation. The first one is the notion of monad, while the second one is the notion of operad. We focus on the notion of monad. A monad in the category  $\mathbf{C}$  is a monoid in the category of endofunctors of  $\mathbf{C}$  (see 2 below) and as such, has right and left modules. Apriori these are endofunctors (in the same category) equipped with an action of the monad. In fact, we introduce a slightly more general notion of modules over a monad, based on the elementary observation that we can readily extend the notion of a right action of a monad in  $\mathbf{C}$  to the case of a functor from any category  $\mathbf{B}$  to  $\mathbf{C}$ , and symmetrically the notion of a left action of a monad in  $\mathbf{C}$  to the case of a functor from  $\mathbf{C}$  to any category  $\mathbf{D}$ . We are mostly interested in left modules. As usual, the interest of the notion of left module is that it generates a companion notion of morphism. We call morphisms those natural transformations among (left) modules which are compatible with the structure, namely which commute to substitution (we also call these morphisms *linear* natural transformations).

Despite the natural ideas involved, the only mention of modules over monads we have been able to find is on a blog by Urs Schreiber.<sup>3</sup> On the other hand, modules over operads have been introduced by M. Markl ([16, 17]) and are commonly used by topologists (see e.g. [9, 14, 4]). In [8], such modules over operads have been considered, under the name of actions, in the context of semantics.

We think that the notions of module over a monad and linear transformations deserve more attention and propose here a first reference for basic properties of categories of left modules, together with basic examples of morphisms of left modules, hopefully showing the adequacy of the language of left modules for questions concerning in particular (possibly higher-order) syntax and lambda-calculus.

In section 2, we briefly review the theory of monads and their algebras. In section 3, we develop the basic theory of modules. In section 4, we sketch a treatment of syntax (with variable-binding) based on modules. In the remaining sections, we show various linear transformations concerning lists and the lambda-calculus (typed or untyped). The appendix discusses the formal proof in the Coq proof assistant of one of our examples.

## 2 Monads and Algebras

We briefly recall some standard material about monads and algebras. Experienced readers may want to skip this section or just use it as reference for our notations. Mac Lane's book [15] can be used as reference on this material.

<sup>3</sup> <http://golem.ph.utexas.edu/string/archives/000715.html>

Let  $\mathbf{C}$  be a category. A monad over  $\mathbf{C}$  is a monoid in the category  $\mathbf{C} \rightarrow \mathbf{C}$  of endofunctors of  $\mathbf{C}$ . In more concrete terms:

**Definition 1 (Monad).** A monad  $R = \langle R, \mu, \eta \rangle$  is given by a functor  $R: \mathbf{C} \rightarrow \mathbf{C}$ , and two natural transformations  $\mu: R^2 \rightarrow R$  such that the following diagrams commute:

$$\begin{array}{ccc} R^3 & \xrightarrow{R\mu} & R^2 \\ \mu R \downarrow & & \downarrow \mu \\ R^2 & \xrightarrow{\mu} & R \end{array} \quad \begin{array}{ccccc} I \cdot R & \xrightarrow{\eta R} & R^2 & \xleftarrow{R\eta} & R \cdot I \\ & \searrow 1_R & \downarrow \mu & \swarrow 1_R & \\ & & R & & \end{array}$$

The  $\mu$  and  $\eta$  natural transformations are often referred as product (or composition) and unit of the monad  $M$ . In the programming language Haskell, they are noted `join` and `return` respectively.

Given a monad  $R$  and an arrow  $f: X \rightarrow RY$ , we define the function  $\text{bind } f: RX \rightarrow RY$  given by  $\text{bind } f := \mu \cdot Rf$ . The functoriality and the composition of the monad can be defined alternatively in terms of the unit and the bind operator. More precisely, we have the equations

$$\mu_X = \text{bind } 1_X, \quad Rf = \text{bind}(\eta \cdot f).$$

Moreover, we have the following associativity and unity equations for `bind`

$$\text{bind } g \cdot \text{bind } f = \text{bind}(\text{bind } g \cdot f), \quad \text{bind } \eta_X = 1_{RX}, \quad \text{bind } f \cdot \eta = f \quad (1)$$

for any pair of arrows  $f: X \rightarrow RY$  and  $g: Y \rightarrow RZ$ .

In fact, to give a monad is equivalent to give two operators unit and bind as above which satisfy equations 1.

*Example 1 (Lists).* To construct the monad of lists  $L$  (over  $\mathbf{Set}$ ), first take the functor  $L: \mathbf{Set} \rightarrow \mathbf{Set}$

$$L: X \mapsto \Sigma_{n \in \mathbb{N}} X^n = * + X + X \times X + X \times X \times X + \dots$$

So  $L(X)$  is the set of all finite lists with elements in  $X$ . Then consider as composition the natural transformation  $\mu: L \cdot L \rightarrow L$  given by the *join* (or *flattening*) of lists of lists:

$$\mu[[a_1, \dots], [b_1, \dots], \dots, [z_1, \dots]] = [a_1, \dots, b_1, \dots, \dots, z_1, \dots].$$

The unit  $\eta: I \rightarrow L$  is constituted by the singleton map  $\eta_X: x \in X \mapsto [x] \in L(X)$ .

*Example 2 (Lambda Calculus).* This example will be worked out with the necessary details in section 5.1, but let us give early some basic ideas (see also [1]). We denote by  $FV(M)$  the set of free variables of a  $\lambda$ -term  $M$ . For a fixed set  $X$ , consider the collection of  $\lambda$ -terms (modulo  $\alpha$ -conversion) with free variables in  $X$ :

$$\mathbf{LC}(X) := \{M \mid FV(M) \subset X\}.$$

Given a set  $X$  we take as unit morphism  $\eta_X: X \rightarrow \mathbf{LC}(X)$  the application assigning to an element  $x \in X$  the corresponding variable in  $\mathbf{LC}(X)$ . Every map  $f: X \rightarrow Y$  induces a morphism  $\mathbf{LC}(f): \mathbf{LC}(X) \rightarrow \mathbf{LC}(Y)$  (“renaming”) which makes  $\mathbf{LC}$  a functor. The instantiation (or substitution) of free variables gives us a natural transformation

$$\mu_X: \mathbf{LC}(\mathbf{LC}(X)) \rightarrow \mathbf{LC}(X).$$

With this structure  $\mathbf{LC}$  is a monad.

Moreover, by taking the quotient  $\Lambda(X)$  of  $\mathbf{LC}(X)$  modulo  $\beta\eta$ -conversion we still obtain a monad (i.e., the composition and the unit of the monad are compatible with  $\beta\eta$ -conversions).

**Definition 2 (Maybe monad).** In a category  $\mathbf{C}$  with finite sums and a final object (like  $\mathbf{Set}$ ), the functor  $X \mapsto X + *$  which takes an object and “adds one point” has a natural structure of monad on  $\mathbf{C}$ . Borrowing from the terminology of the library of the programming language Haskell, we call it the **Maybe monad**.

**Definition 3 (Derivative).** We define the derivative  $F'$  of a functor  $F: \mathcal{C} \rightarrow \mathcal{C}$  to be the functor  $F' = F \cdot \text{Maybe}$ . We can iterate the construction and denote by  $F^{(n)}$  the  $n$ -th derivative.<sup>4</sup>

**Definition 4 (Morphisms of monads).** A morphism of monads is a natural transformation between two monads  $\phi: P \rightarrow R$  which respects composition and unit, i.e., such that the following diagrams commute:

$$\begin{array}{ccc} P^2 & \xrightarrow{\phi\phi} & R^2 \\ \downarrow \mu_R & & \downarrow \mu_R \\ P & \xrightarrow{\phi} & R \end{array} \quad \begin{array}{ccc} P & \xrightarrow{\phi} & R \\ \eta_P \swarrow & & \searrow \eta_R \\ & I & \end{array}$$

It can be easily seen that morphisms of monads form a category.

For our purpose it is relevant to observe that there are a number of natural transformations which arise in the above examples which fail to be morphisms of monads. We take the following as paradigmatic example.

*Example 3 (Abstraction is not a morphism of monads).* Abstraction on  $\lambda$ -terms gives a natural transformation  $\text{abs}: \text{LC}' \rightarrow \text{LC}$  which takes a  $\lambda$ -term  $M \in \text{LC}(X + *)$  and binds the “variable”  $*$ . This fails to be a morphism of monads because it does not respect substitution in the sense of monads: a careful inspection reveals that the transformation

$$\text{LC}(\text{LC}(X + *) + *) \xrightarrow{\mu} \text{LC}(X + *) \xrightarrow{\text{abs}} \text{LC}(X)$$

binds all stars under a single abstraction while

$$\text{LC}(\text{LC}(X + *) + *) \xrightarrow{\text{abs abs}} \text{LC}(\text{LC}(X)) \xrightarrow{\mu} \text{LC}(X)$$

not. In fact, we will see later that  $\text{LC}'$  is a left module over  $\text{LC}$  and  $\text{abs}$  is a  $\text{LC}$ -linear morphism.

Now let  $R$  be a monad over  $\mathcal{C}$ .

**Definition 5 (Algebra).** An algebra over  $R$  is given by an object  $A$  and a morphism  $\rho: R(A) \rightarrow A$  in  $\mathcal{C}$  such that the following diagrams commute:

$$\begin{array}{ccc} R^2(A) & \xrightarrow{R\rho} & R(A) \\ \mu_A \downarrow & & \downarrow \rho \\ R(A) & \xrightarrow{\rho} & A \end{array} \quad \begin{array}{ccc} A & \xrightarrow{\eta_A} & R(A) \\ & \searrow 1_A & \downarrow \rho \\ & & A \end{array}$$

**Definition 6.** Let  $A, B$  be two algebras over a monad  $R$ . An arrow  $f: A \rightarrow B$  in  $\mathcal{C}$  is said to be a morphism of algebras if it is compatible with the associated actions, i.e., the two induced morphisms from  $R(A)$  to  $B$  are equal:

$$\rho_B \cdot R(f) = f \cdot \rho_A$$

As we will see later, algebras can be regarded as special kind of right modules.

*Example 4 (Monoids).* In the category of sets, algebras over the monad  $L$  of lists are sets equipped with a structure of monoid; given a monoid  $A$ , the corresponding action  $L(A) \rightarrow A$  is the product (sending a list to the corresponding product).

### 3 Modules over monads

Being a monoid in a suitable monoidal category, a monad has associated left and right modules which, a-priori, are objects in the same category, acted upon by the monoid.

Although we are mostly interested in left modules, let us remark that from this classical point of view, algebras over a monad are not (right-)modules. We give a slightly more general definition of modules which is still completely natural. According to this extended definition, algebras turn out to be right-modules.

<sup>4</sup> This corresponds to the `MaybeT` monad transformer in Haskell.

### 3.1 Left modules

We start first by concentrating ourselves on left modules over a given monad  $R$  over a category  $\mathbf{C}$ .

**Definition 7 (Left modules).** A left  $R$ -module in  $\mathbf{D}$  is given by a functor  $M: \mathbf{C} \rightarrow \mathbf{D}$  equipped with a natural transformation  $\rho: M \cdot R \rightarrow M$ , called action, which is compatible with the monad composition, more precisely, we require that the following diagrams commute

$$\begin{array}{ccc} M \cdot R^2 & \xrightarrow{M\mu} & M \cdot R \\ \rho R \downarrow & & \downarrow \rho \\ M \cdot R & \xrightarrow{\rho} & M \end{array} \quad \begin{array}{ccc} M \cdot R & \xleftarrow{M\eta} & M \cdot I \\ \rho \downarrow & \nearrow 1_M & \\ M & & \end{array}$$

We will refer to the category  $\mathbf{D}$  as the range of  $M$ .

*Remark 1.* The companion definition of modules over an operad (c.f. e.g. [17,9]) follows easily from the observation [19] that operads are monoids in a suitable monoidal category. This monoidal structure is central in [6].

Given a left  $R$ -module  $M$ , we can introduce the `mbind` operator which, to each arrow  $f: X \rightarrow RY$ , associates an arrow `mbind`  $f := MX \rightarrow MY$  defined by `mbind`  $f := \rho \cdot Mf$ . The axioms of left module are equivalent to the following equations over `mbind`:

$$\text{mbind } g \cdot \text{mbind } f = \text{mbind}(\text{bind } g \cdot f), \quad \text{mbind } \eta_X = 1_X$$

*Example 5.* We can see our monad  $R$  as a left module over itself (with range  $\mathbf{C}$ ), which we call the *tautological* module.

*Example 6.* Any constant functor  $\underline{W}: \mathbf{C} \rightarrow \mathbf{D}$ ,  $W \in \mathbf{D}$  is a trivial left  $R$ -module.

*Example 7.* For any functor  $F: \mathbf{D} \rightarrow \mathbf{E}$  and any left  $R$ -module  $M: \mathbf{C} \rightarrow \mathbf{D}$ , the composition  $F \cdot M$  is a left  $R$ -module (in the evident way).

**Definition 8 (Derived module).** As for functors and monads, derivation is well-behaved also on left modules: for any left  $R$ -module  $M$ , the derivative  $M' = M \cdot \text{Maybe}$  has a natural structure of left  $R$ -module where the action  $M' \cdot R \rightarrow M'$  is the composition

$$M \cdot \text{Maybe} \cdot R \xrightarrow{M\gamma} M \cdot R \cdot \text{Maybe} \xrightarrow{\rho \text{Maybe}} M \cdot \text{Maybe}$$

and  $\gamma$  is the natural arrow  $\text{Maybe} \cdot R \rightarrow R \cdot \text{Maybe}$ .

**Definition 9 (Morphisms of left modules).** We say that a natural transformation of left  $R$ -modules  $\tau: M \rightarrow N$  is linear if it is compatible with substitution:

$$\begin{array}{ccc} M \cdot R & \xrightarrow{\tau R} & N \cdot R \\ \rho_M \downarrow & & \downarrow \rho_N \\ M & \xrightarrow{\tau} & N \end{array}$$

We take linear natural transformations as left module morphisms.

*Remark 2.* Here the term *linear* refers to linear algebra: linear applications between modules over a ring are group morphisms compatible with the action of the ring. It is compatible with the usual flavor of the word linear (no duplication, no junk) as the following example shows.

*Example 8.* We consider the monad  $M$  on **Set** generated by two binary constructions  $+$  and  $*$  and we build (by recursion) a natural transformation  $n: M \rightarrow M$  as follows: for a variable  $x$ ,  $n(x)$  is  $x+x$ , while for the other two cases we have  $n(a+b) = n(a)*n(b)$  and  $n(a*b) = n(a)+n(b)$ . It is easily verified that  $n$  is a non-linear natural transformation (check the diagram against  $n(\text{var}(x * x))$ ).

*Example 9.* We easily check that the natural transformation of a left module into its derivative is linear. Note that there are two natural inclusions of the derivative  $M'$  into the second derivative  $M''$ . Both are linear.

*Example 10.* Consider again the monad of lists  $L$ . The concatenation of two lists is a  $L$ -linear morphism  $L \times L \rightarrow L$ .

**Definition 10 (Category of left modules).** We check easily that linear morphisms between left  $R$ -modules with the same range yield a subcategory of the functor category. We denote by  $\text{Mod}^D(R)$  the category of left  $R$ -modules with range  $D$ .

**Definition 11 (Product of left modules).** We check easily that the cartesian product of two left  $R$ -modules as functors (having as range a cartesian category  $D$ ) is naturally a left  $R$ -module again and is the cartesian product also in the category  $\text{Mod}^D(R)$ . We also have finite products as usual. The final left module  $*$  is the product of the empty family.

*Example 11.* Given a monad  $R$  on  $\text{Set}$  and a left  $R$ -module  $M$  with range in a fixed cartesian category  $D$ , we have a natural “evaluation” morphism  $\text{eval}: M' \times R \rightarrow M$ , where  $M'$  is the derivative of  $M$ .

**Proposition 1.** Derivation yields a cartesian endofunctor on the category of left  $R$ -modules with range in a fixed cartesian category  $D$ .

### 3.2 Right modules

Let  $R$  be a monad over a category  $C$ . The definition of right module is similar to that of left module.

**Definition 12 (Right modules).** A right  $R$ -module (from  $D$ ) is given by a functor  $M: D \rightarrow C$  equipped with a natural transformation  $\rho: R \cdot M \rightarrow M$  which makes the following diagrams commutative

$$\begin{array}{ccc} R^2 \cdot M & \xrightarrow{\mu_M} & R \cdot M \\ R\rho \downarrow & & \downarrow \rho \\ R \cdot M & \xrightarrow{\rho} & M \end{array} \quad \begin{array}{ccc} I \cdot M & \xrightarrow{\eta_M} & R \cdot M \\ & \searrow 1_M & \downarrow \rho \\ & & M \end{array}$$

As for left modules, we will call corange of  $M$  the category  $D$ .

We remark that for any right  $R$ -module  $M$  and any object  $X \in D$  the image  $M(X)$  is an  $R$ -algebra. Then a right  $R$ -module is simply a functor from the corange category  $D$  to the category of  $R$ -algebras.

*Example 12.* Our monad  $R$  is a right module over itself.

*Example 13.* If  $A$  is an  $R$ -algebra, then for any category  $D$  the constant functor  $\underline{A}: X \mapsto A$  has a natural structure of right  $R$ -module. In particular, we can identify our algebra with the corresponding functor  $\underline{A}: * \rightarrow C$ , where  $*$  is the category with one object and one arrow.

*Example 14.* Let  $\phi: R \rightarrow P$  be a morphism of monads. Then  $P$  is a right and left  $R$ -module with actions given respectively by  $P \cdot R \xrightarrow{\phi_R} R \cdot R \xrightarrow{\mu_R} R$  and  $R \cdot P \xrightarrow{P\phi} R \cdot R \xrightarrow{\mu_R} R$ .

**Definition 13 (Morphisms of right modules).** A morphism of right  $R$ -modules is a natural transformation  $\tau: M \rightarrow N$  which is compatible with substitution, i.e., such that the following diagram commutes:

$$\begin{array}{ccc} R \cdot M & \xrightarrow{R\tau} & R \cdot N \\ \rho_M \downarrow & & \downarrow \rho_N \\ M & \xrightarrow{\tau} & N \end{array}$$

**Definition 14 (Category of right  $R$ -modules).** We check easily that module morphisms among right  $R$ -modules with the same corange yield a subcategory of the functor category.

### 3.3 Limits and colimits of left modules

Limits and colimits in the category of left modules can be constructed pointwise. For instance:

**Lemma 1 (Limits and colimits of left modules).** *If  $\mathcal{D}$  is complete (resp. cocomplete), then  $\text{Mod}^{\mathcal{D}}(R)$  is complete (resp. cocomplete).*

*Proof.* Suppose first that  $\mathcal{D}$  be a complete category and  $G: \mathcal{I} \rightarrow \text{Mod}^{\mathcal{D}}(R)$  be a diagram of left modules over the index category  $\mathcal{I}$ . For any object  $X \in \mathcal{C}$  we have a diagram  $G(X): \mathcal{I} \rightarrow \mathcal{D}$  and for any arrow  $f: X \rightarrow Y$  in  $\mathcal{C}$  we have a morphism of diagrams  $G(X) \rightarrow G(Y)$ . So define

$$U(X) := \lim G(X)$$

Next, given an arrow  $f: X \rightarrow R(Y)$ , we have an induced morphism of diagrams  $G(X) \rightarrow G(Y)$  by the module structure on each object of the diagram. This induces a morphism  $\text{mbind } f: U(X) \rightarrow U(Y)$ . It is not hard to prove that  $\text{mbind}$  satisfies the module axioms and that  $U$  is the limit of  $G$ . The colimit construction is carried analogously.

### 3.4 Base change

**Definition 15 (Base change).** *Given a morphism  $f: A \rightarrow B$  of monads and a left  $B$ -module  $M$ , we have an  $A$ -action on  $M$  given by*

$$M \cdot A \xrightarrow{Mf} M \cdot B \xrightarrow{\rho_M} M.$$

We denote by  $f^*M$  the resulting  $A$ -module and we refer to  $f^*$  as the base change operator.

**Lemma 2.** *The base change of a left module is a left module.*

*Proof.* Our thesis is the commutativity of the diagram

$$\begin{array}{ccccc} M \cdot B \cdot A & \xleftarrow{MfA} & M \cdot A \cdot A & \xrightarrow{M\mu} & M \cdot A \\ \downarrow \rho A & & \downarrow Mff & & \downarrow Mf \\ & & M \cdot B \cdot B & \xrightarrow{M\mu} & M \cdot B \\ & & \downarrow \rho B & & \downarrow \rho \\ M \cdot A & \xrightarrow{Mf} & M \cdot B & \xrightarrow{\rho} & M \end{array}$$

which follows from the commutativity of the three pieces:  $M$  is a left  $B$ -module, the map from  $M(B(-)) \rightarrow M(-)$  is functorial, and  $f$  is a morphism.

**Definition 16 (Base change (functoriality)).** *We upgrade the base change operator into a functor  $f^*: \text{Mod}^{\mathcal{D}}(B) \rightarrow \text{Mod}^{\mathcal{D}}(A)$  by checking that if  $g: M \rightarrow N$  is a morphism of left  $B$ -modules, then so is the natural transformation  $f^*g: f^*M \rightarrow f^*N$ .*

**Proposition 2.** *The base change functor commutes with products and with derivation.*

**Proposition 3.** *Any morphism of monads  $f: A \rightarrow B$  yields a morphism of left  $A$ -modules, still denoted  $f$ , from  $A$  to  $f^*B$ .*

## 4 Initial Algebra Semantics

To ease the reading of the forthcoming sections, we collect in this section some classical ideas about Initial Algebra Semantics.

Given a category  $\mathbf{C}$  and an endofunctor  $T: \mathbf{C} \rightarrow \mathbf{C}$ , a  $T$ -algebra<sup>5</sup> is given by an object  $A \in \mathbf{C}$  and an arrow

$$\sigma_A: TA \rightarrow A.$$

A morphism of  $T$ -algebras is an arrow  $f: A \rightarrow B$  which commutes with the *structural morphism*  $\sigma$

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \sigma_A \downarrow & & \downarrow \sigma_B \\ A & \xrightarrow{f} & B \end{array}$$

This defines the category of  $T$ -algebras. Notice that, for any  $T$ -algebra  $A$ , there is an induced  $T$ -algebra structure on  $TA$  given by  $T\sigma_A: T(TA) \rightarrow TA$ , turning  $\sigma_A$  into a morphism of algebras. An initial  $T$ -algebra is called a *(least) fixpoint* of  $T$ . Given one such fixpoint  $U$  and any other  $T$ -algebra  $A$  we denote by  $\text{fold}_A: U \rightarrow A$  the induced initial morphism. We observe that  $\sigma_U$  is an isomorphism whose inverse is  $\text{fold}_{TU}$  since  $\sigma_U \cdot \text{fold}_{TU} = 1_U$  by the universal property of  $U$  and from the naturality of  $\text{fold}$  follows that the diagram

$$\begin{array}{ccc} TU & \xrightarrow{T \text{fold}_{TU}} & T(TU) \\ \sigma_U \downarrow & \searrow 1_{TU} & \downarrow T\sigma_U \\ U & \xrightarrow{\text{fold}_{TU}} & TU \end{array}$$

is commutative.

Let us show how this general framework can work in the case of (polymorphic) lists.

*Example 15.* Take  $\mathbf{C} = \mathbf{Set} \rightarrow \mathbf{Set}$  the category of endofunctors of  $\mathbf{Set}$  and consider the functor  $T: (\mathbf{Set} \rightarrow \mathbf{Set}) \rightarrow (\mathbf{Set} \rightarrow \mathbf{Set})$  defined by

$$T(F) := X \mapsto * + X \times FX.$$

The least fix point of  $T$  is (the underlying functor of) the monad of lists  $L$  introduced in section 2. The  $T$ -algebra structure  $* + X \times LX = TL \simeq L$  gives the constructors (`nil`, `cons`) and the corresponding destructors. We would like to recognise this structural isomorphism as an  $L$ -linear morphism. Unfortunately, we do not have on  $TL$  a structure of left  $L$ -module corresponding to our expectation (notice that the identity functor is not an  $L$ -module in a natural way). We will explain in section 6 how this phenomenon can be considered a consequence of the lack of typing.

## 5 Monads over sets

In this section we consider more examples of linear morphisms over monads on the category of sets.

### 5.1 Untyped Syntactic Lambda Calculus

Consider the functor  $T := (\mathbf{Set} \rightarrow \mathbf{Set}) \rightarrow (\mathbf{Set} \rightarrow \mathbf{Set})$  given by

$$TF: X \mapsto X + FX \times FX + F'X$$

where  $F'$  denotes the derived functor  $X \mapsto F(X + *)$ . It can be shown that  $T$  possesses a least fixpoint that we denote by  $\mathbf{LC}$  ( $\mathbf{LC}$  standing for  $\lambda$ -calculus, cfr. the example in section 2). We consider  $\mathbf{LC}(X)$  as the set of  $\lambda$ -terms with free variables taken from  $X$  (see also [3]). In fact,

<sup>5</sup> There is a lexical conflict here with algebra of monads introduced in section 2, which is deeply rooted in the literature anyway. We hope that this will not lead to any confusion.

the structural morphism  $TLC \rightarrow LC$  gives the familiar constructors for  $\lambda$ -calculus in the *locally nameless* encoding, namely, the natural transformations

$$\text{var}: I \rightarrow LC, \quad \text{app}: LC \times LC \rightarrow LC, \quad \text{abs}: LC' \rightarrow LC.$$

As already observed, the substitution (instantiation) of free variables gives a monad structure on  $LC$  where  $\text{var}$  is the unit.

We would like to express that these constructors are well behaved with respect to substitution. Again, as in the case of lists,  $TLC$  has no natural structure of left  $LC$ -module. However, we can consider the functor  $T$  as built of two parts  $TF = I + T_0F$  where  $T_0F := F \times F + F'$  (in other words we are tackling apart  $\text{var}$ , the unit of the monad, from the other two constructors  $\text{app}$  and  $\text{abs}$ ). Now  $T_0LC$  is a left  $LC$ -module and we can observe that the algorithm of substitution is carried precisely in such a way that the induced morphism

$$\text{app}, \text{abs}: T_0LC \rightarrow LC$$

is  $LC$ -linear or, equivalently, the natural transformations  $\text{app}: LC \times LC \rightarrow LC$  and  $\text{abs}: LC' \rightarrow LC$  are  $LC$ -linear. To make the idea clearer, we reproduce a short piece of code in the *Haskell* programming language which implements the algorithm of substitution.

```
module LC where
import Monad (liftM)

data LC a = Var a | App (LC a) (LC a) | Abs (LC (Maybe a))

instance Monad LC where
  return = Var
  Var x >>= f = f x
  App x y >>= f = App (x >>= f) (y >>= f)
  Abs x >>= f = Abs (x 'mbind' f)

mbind :: LC (Maybe a) -> (a -> LC b) -> LC (Maybe b)
mbind x f = x >>= maybe (Var Nothing) (liftM Just . f)
```

In the above code, `mbind` constitutes the left  $LC$ -module structure on  $LC'$ . It is now evident that the recursive calls in the definition of `(>>=)` are exactly those given by the linearity of `app` and `abs`.

We can go further and try to make the linearity more explicit in the syntactic definition of  $\lambda$ -calculus. This can be done as follows.

**Theorem 1.** *Consider the category  $\text{Mon}^{T_0}$  where objects are monads  $R$  over sets endowed with a  $R$ -linear morphism  $T_0R \rightarrow R$  while arrows are given by commutative diagrams*

$$\begin{array}{ccc} T_0R & \xrightarrow{T_0f} & f^*T_0P \\ \downarrow & & \downarrow \\ R & \xrightarrow{f} & f^*P \end{array}$$

where all morphisms are  $R$ -linear (we are using implicitly the fact that the base change functor commutes with derivation and products). The monad  $LC$  is initial in  $\text{Mon}^{T_0}$ .

In fact, the previous theorem can be generalized as follows (interested readers may also want to look at other works on higher order abstract syntax, e.g., [6, 13, 10] see also our [12]). Let  $R$  be a monad over  $\text{Set}$ . We define an *arity* to be a list of nonnegative integers. We denote by  $\mathbb{N}^*$  the set of arities. For each arity  $(a_1, \dots, a_r)$ , and for any  $R$ -module  $M$ , we define the  $R$ -module  $T^aM$  by

$$T^aM = M^{(a_1)} \times \dots \times M^{(a_r)},$$

where  $M^{(n)}$  denotes the  $n$ -th derivative of  $M$ , and we say that a linear morphism  $T^aR \rightarrow R$  is a  *$R$ -representation* of  $a$  (or a representation of  $a$  in  $R$ ). For instance, the `app` and `abs` constructors are  $LC$ -representations of the arities  $(0, 0)$  and  $(1)$  respectively.



Next, we consider *signatures* which are family of arities. For each signature  $\Sigma: I \rightarrow \mathbb{N}^*$ , and for any  $R$ -module  $M$ , we define the  $R$ -module  $T^\Sigma M$  by

$$T^\Sigma M = \sum_{i \in I} T^{\Sigma_i} M$$

and we say that a linear morphism  $T^\Sigma R \rightarrow R$  is a  $R$ -representation of  $\Sigma$  (or a representation of  $\Sigma$  in  $R$ ). Altogether **app** and **abs** give a LC-representation of the signature  $((0, 0), (1))$ .

As in the special case of the  $\lambda$ -calculus, representations of a given signature  $\Sigma$  form a category.

**Theorem 2.** *For any signature  $\Sigma$ , the category of  $\Sigma$ -representations has an initial object.*

## 5.2 Untyped Semantic Lambda Calculus

For any set  $X$ , consider the equivalence relation  $\equiv_{\beta\eta}$  on  $\mathbf{LC}(X)$  given by the reflexive symmetric transitive closure of  $\beta$  and  $\eta$  conversions and define  $\Lambda(X) := \mathbf{LC}(X) / \equiv_{\beta\eta}$ . It can be shown that  $\equiv_{\beta\eta}$  is compatible with the structure of  $\mathbf{LC}$  so  $\Lambda$  has a structure of monad, the projection  $\mathbf{LC} \rightarrow \Lambda$  is a morphism of monads, and we have an induced morphism  $T_0 \Lambda \rightarrow \Lambda$  which is  $\Lambda$ -linear.

Now the key fact is that the abstraction **abs**:  $\Lambda' \rightarrow \Lambda$  is a linear isomorphism! In fact, it is easy to construct its inverse **app**<sub>1</sub>:  $\Lambda \rightarrow \Lambda'$ :

$$\mathbf{app}_1 x = \mathbf{app}(\hat{x}, *)$$

where  $x \mapsto \hat{x}$  denotes the natural inclusion  $\Lambda \rightarrow \Lambda'$ . The equation

$$\mathbf{abs} \cdot \mathbf{app}_1 = 1_\Lambda$$

clearly corresponds to the  $\eta$ -rule while the other equation

$$\mathbf{app}_1 \cdot \mathbf{abs} = 1_{\Lambda'}$$

can be considered the ultimate formulation of the  $\beta$ -rule. In fact, there is a more classical formulation of the  $\beta$ -rule which can be stated as the commutativity of the diagram

$$\begin{array}{ccc} \Lambda' \times \Lambda & \xrightarrow{\mathbf{abs} \times \Lambda} & \Lambda \times \Lambda \\ & \searrow \text{subst} & \downarrow \mathbf{app} \\ & & \Lambda \end{array}$$

Again, we can present this situation from a more syntactical point of view. For this, consider the category of *exponential monads*: an exponential monad is a monad  $R$  endowed with a  $R$ -linear isomorphism with its derivative  $\exp_R: R' \rightarrow R$ . An arrow is a monad morphism  $f$  such that

$$\begin{array}{ccc} R' & \xrightarrow{f'} & f^* P' \\ \exp_R \downarrow & & \downarrow \exp_{P'} \\ R & \xrightarrow{f} & f^* P \end{array}$$

is a commutative diagram of  $R$ -modules (we are implicitly using the commutativity of base change with derivation).

**Theorem 3.** *The monad  $\Lambda$  is initial in the category of exponential monads.*

We have developed a formal proof of the above theorem in the Coq proof assistant [5] which is discussed in the appendix.

## 6 Monads over types

So far we mostly considered examples of monads and modules on the category  $\mathbf{C} = \mathbf{Set}$  of small sets. Other interesting phenomena can be captured by taking into account monads and modules on other categories. In this section we consider the case  $\mathbf{C} = \mathbf{Set}_\tau$  the category of sets fibered over a fixed set  $\tau$ . This is the category given by maps  $\phi_X: X \rightarrow \tau$ , called  $\tau$ -sets, where arrows  $\langle X, \phi_X \rangle \rightarrow \langle Y, \phi_Y \rangle$  are given by maps  $f: X \rightarrow Y$  which commute with the structural morphisms, i.e.,  $\phi_Y \cdot f = \phi_X$ . For each  $t \in \tau$  and each  $\tau$ -set  $X$ , we denote by  $X_t := \phi_X^{-1}(t)$  the preimage of  $t$  in  $X$ . We regard  $\tau$  as a “set of types” and the fibers  $X_t$  as a set of “terms of type  $t$ ”.

### 6.1 Typed lists

Here we show how, in the context of typed lists, the constructors `nil` and `cons` may appear as linear. To this effect, we introduce a distinction between the base type  $*$ , the type of lists `list`, the type of lists of lists, etc. Thus we take  $\tau = \mathbb{N}$  the inductive set of types generated by the grammar  $\tau = * \mid \text{list } \tau$ , and consider the category  $\mathbf{Set}_\tau$ .

For each  $t \in \tau$  we define  $\mathcal{L}_t: \mathbf{Set}_\tau \rightarrow \mathbf{Set}$  by setting  $\mathcal{L}_t(X)$  to be the set of terms of type  $t$  built out from (typed) variables in  $X$  by adding, as usual, terms obtained through the `nil` and `cons` constructions. By glueing these  $\mathcal{L}_t$  together, we obtain an endofunctor  $\mathcal{L}$  in  $\mathbf{Set}_\tau$ . It is easily seen to be a monad (the present structure of monad has nothing to do with flattening).

For each  $t \in \tau$ ,  $\mathcal{L}_t$  is a left  $\mathcal{L}$ -module (by Example 7). The `nil` and `cons` constructors constitute a family of natural transformations parametrized by  $t \in \tau$

$$\text{nil}_t: * \longrightarrow \mathcal{L}_{\text{list } t}, \quad \text{cons}_t: \mathcal{L}_t \times \mathcal{L}_{\text{list } t} \longrightarrow \mathcal{L}_{\text{list } t}.$$

Hence we have here examples of heterogeneous modules since  $*$ ,  $\mathcal{L}_t$  and  $\mathcal{L}_{\text{list } t}$  are  $\mathcal{L}$ -modules in  $\mathbf{Set}$ . And  $\text{nil}_t$  and  $\text{cons}_t$  are easily seen to be morphisms among these modules.

We may also want to glue for instance these  $\text{cons}_t$  into a single `cons`. For this, we need shifts. Given  $X \in \mathbf{Set}_\tau$  we have the associated *shifts*  $X[n]$  which are obtained by adding  $n$  to the structural map  $X \rightarrow \mathbb{N}$ . The shift  $(\cdot)[n]: X \mapsto X[n]$  gives an endofunctor over  $\mathbf{Set}_\tau$ . Given a functor  $F$  from any category to the category  $\mathbf{Set}_\tau$ , we consider the *shifted functors*  $F[n]$  obtained as composition of  $F$  followed by  $(\cdot)[n]$ . From the remarks of section 3, it follows at once that if  $F$  is an  $\mathcal{L}$ -module, then so is  $F[n]$ . With these notations, glueing yields

$$\text{nil}: * [1] \longrightarrow \mathcal{L}, \quad \text{cons}: \mathcal{L}[1] \times \mathcal{L} \longrightarrow \mathcal{L}$$

where  $*$  denotes the final functor in the category of endofunctors of  $\mathbf{Set}_\tau$ . Again `nil` and `cons` are easily checked to be  $\mathcal{L}$ -linear.

### 6.2 Simply Typed Lambda Calculus

Our second example of typed monad is the simply-typed  $\lambda$ -calculus. We denote by  $\tau$  the set of simple types  $\tau := * \mid \tau \rightarrow \tau$ . Following [22], we consider the syntactic typed  $\lambda$ -calculus as an assignment  $V \mapsto \mathbf{LC}_\tau(V)$ , where  $V = \sum_{t \in \tau} (V_t)$  is a (variable) set (of typed variables) while

$$\mathbf{LC}_\tau(V) = \sum_{t \in \tau} (\mathbf{LC}_\tau(V))_t$$

is the set of typed  $\lambda$ -terms (modulo  $\alpha$ -conversion) built on free variables taken in  $V$ .

Given a type  $t$  we set  $\mathbf{LC}_t(X) := (\mathbf{LC}_\tau(X))_t$  which gives a functor over  $\tau$ -sets, which is equipped with substitution, turning it into a (heterogeneous) left module over  $\mathbf{LC}_\tau$ . And given two types  $s, t$ , we have

$$\text{app}_{s,t}: \mathbf{LC}_{s \rightarrow t} \times \mathbf{LC}_s \longrightarrow \mathbf{LC}_t$$

which is linear.

For the `abs` construction, we need a notion of partial derivative for a module. For a left module  $M$  over  $\tau$ -sets, and a type  $t \in \tau$ , we set

$$\delta_t M(V) := M(V + *_t)$$

where  $V + *_t$  is obtained from  $V$  by adding one element with type  $t$ . It is easily checked how  $\delta_t M$  is again a left module. Now, given two types  $s$  and  $t$ , it turns out that

$$\text{abs}_{s,t} : \delta_s \text{LC}_t \longrightarrow \text{LC}_{s \rightarrow t}$$

is linear.

As in the untyped case, we can consider the functor  $\Lambda_\tau$  obtained by quotienting modulo  $\beta\eta$  conversion. This is again a monad over the category of  $\tau$ -sets and the natural quotient transformation  $\text{LC}_\tau \rightarrow \Lambda_\tau$  is a morphism of monads. For this semantic monad, the above left module morphisms induce semantic counterparts:  $\text{app}_{s,t} : \Lambda_{s \rightarrow t} \times \Lambda_s \longrightarrow \Lambda_t$  and  $\text{abs}_{s,t} : \delta_s \Lambda_t \longrightarrow \Lambda_{s \rightarrow t}$ .

Here we need a new notion of arity and signature, which we will introduce in some future work, in order to state and prove a typed counterpart of our theorem 2. For a typed counterpart of our theorem 3, see [22].

### 6.3 Typed Lambda Calculus

Our final example of typed monad is just a glance to more general typed  $\lambda$ -calculi. The point here is that the set of types is no more fixed. Thus our monads take place in the category **Fam** of set families: an object in **Fam** is an application  $p : I \rightarrow \text{Set}$  while a morphism  $m : p \rightarrow p'$  is a pair  $(m_0, m_1)$  with  $m_0 : I \rightarrow I'$ , and  $m_1 : (i : I)p(i) \rightarrow p'(m_0(i))$ . We say that  $I$  is the set of types of  $p : I \rightarrow \text{Set}$ . From **Fam** there are two forgetful functors  $T, \text{Tot} : \text{Fam} \rightarrow \text{Set}$  where  $T(p : I \rightarrow \text{Set}) := I$  and  $\text{Tot}(p) := \coprod_{i \in T(p)} p(i)$ , and a natural transformation  $\text{proj} : \text{Tot} \rightarrow T$ , defined by  $\text{proj}(p) = p$  in the obvious sense. Given a monad  $R$  on **Fam**, we thus have a morphism of  $R$ -modules  $\text{proj}_R : \text{Tot} \circ R \rightarrow T \circ R$ .

We need also two *may-be* monads on **Fam**: the first one  $F \mapsto F^*$  adds one (empty) type (*tnew*) to  $F$ , while the second one,  $F \mapsto F^{*/ *}$  adds one type (*tnew*) with one element (*new*). Given a monad  $R$  on **Fam**, we thus have two “derived”  $R$ -modules:  $R^* := F \mapsto R(F^*)$  and  $R^{*/ *} := F \mapsto R(F^{*/ *})$ .

Now when should we say that  $R$  is a lambda-calculus in this context? At least we should have a module morphism  $\text{arrow} : (T \circ R)^2 \rightarrow T \circ R$  and a module morphism for abstraction,  $\text{abs} : R^{*/ *} \rightarrow R^*$  (the “arity” for application is not so simple). We hope this example shows the need for new notions of arity and signature, as well as the new room opened by modules for such concepts.

## 7 Monads over preordered sets

Our last example is about monads and modules over the category of preordered sets (sets with a reflexive and transitive binary relation). Preordering is used here to model the relation  $\xrightarrow{\beta\eta}_*$  generated by the reflexive and transitive closure of the  $\beta$  and  $\eta$  conversions. In fact, the construction given in this section can be considered a refinement of those of section 5.1 where we used the reflexive, symmetric and transitive closure  $\equiv_{\beta\eta}$ .

Let us consider again the monad **LC** of  $\lambda$ -terms. Given a preordered set  $X$ , we consider the preordering on  $\text{LC}(X)$  given by the rules

$$\begin{aligned} x \leq y &\implies \text{var } x \leq \text{var } y, \\ S \leq S' \wedge T \leq T' &\implies \text{app}(S, T) \leq \text{app}(S', T'), \\ T \leq T' &\implies \text{abs}(T) \leq \text{abs}(T'), \\ T \xrightarrow{\beta\eta} T' &\implies T \leq T'. \end{aligned}$$

It is not hard to verify that with this new structure **LC** is now a monad over preordered sets. It turns out that the **app** and **abs** constructions are still **LC**-linear with respect to this richer structure.

## 8 Conclusions and related works

We have introduced the notion of module over a monad, and more importantly the notion of linearity for transformations among such modules and we have tried to show that this notion is

ubiquitous as soon as syntax and semantics are concerned. Our thesis is that the point of view of modules opens some new room for initial algebra semantics, as we sketched for typed  $\lambda$ -calculus (see also [12]).

The idea that the notion of monad is suited for modelling substitution concerning syntax (and semantics) has been retained by many recent contributions concerned with syntax (see e.g. [2, 11, 18]) although some other settings have been considered. Notably in [6] the authors work within a setting roughly based on operads (although they do not write this word down; the definition of operad is on Wikipedia; operads and monads are not too far from each other). As they mention, their approach is, to some extent, equivalent to an approach through monads. It has been both applied e.g. in [21] and generalized e.g. in [20]. Another approach to syntax with bindings, initiated by Gabbay and Pitts [10], relies on a systematic consideration of freshness, an issue which is definitely ignored in the monadic or operadic approach.

While the notion of module over a monad has been essentially ignored till now, the notion of module over an operad has been introduced more than ten years ago, and has been incidentally considered in the context of semantics, as we already mentioned in our introduction.

## 9 Appendix: Formal proof of theorem 3

In this section we present our formal proof of theorem 3 in the Coq proof assistant [5]. We recall the statement of the theorem

The monad  $\Lambda$  of semantic untyped  $\lambda$ -calculus is an initial object in the category of exponential monads.

We include here only a small fraction of the code without proofs. The full sources can be found at <http://www.math.unifi.it/~maggesi>.

### 9.1 Structure of the formalisation

The structure of our proof can be outlined in the following four major parts: (1) axioms and support library; (2) formalisation of monads, modules and exponential monads; (3) formalisation of syntactic and semantic  $\lambda$ -calculus; (4) the main theorem.

The second and third part are independent of each other. As for what this paper is concerned, the first part (files `Misc.v`, `Congr.v`) can be considered as an extension of the Coq system for practical purposes. This part contains some meta-logical material (tactics and notations) and declares the following axioms: functional choice, proof irrelevance, dependent extensionality. We include here their declarations:

```
Axiom functional_choice : forall (A B : Type) (R : A -> B -> Prop),
  (forall x : A, exists y : B, R x y) -> exists f : A -> B, (forall x : A, R x (f x)).
Axiom proof_irrelevance : forall (A : Prop) (H1 H2 : A), H1 = H2.
Axiom extens_dep : forall (X : Type) (T : X -> Type) (f g : forall x : X, T x),
  (forall x : X, f x = g x) -> f = g.
```

Moreover, we use an axiomatic definition of quotient types (file `Quot.v`) to construct semantic  $\lambda$ -calculus as quotient of syntactic  $\lambda$ -calculus.

### 9.2 Formalisation of monads and modules

After the preliminary material, our formalisation opens the theory of monads and (left) modules (files `Monad.v`, `Mod.v`, `Derived_Mod.v`). This is constructed starting from a rather straightforward translation of the Haskell monad library. As an example, we report here our definitions of monads and modules in the Coq syntax.

```
Record Monad : Type := {
  monad_carrier :> Set -> Set;
  bind : forall X Y : Set, (X -> monad_carrier Y) -> monad_carrier X -> monad_carrier Y;
  unit : forall X : Set, X -> monad_carrier X;
  bind_bind : forall (X Y Z : Set) (f : X -> monad_carrier Y) (g : Y -> monad_carrier Z)
```

```

    (x : monad_carrier X),
    bind Y Z g (bind X Y f x) = bind X Z (fun u => bind Y Z g (f u)) x;
bind_unit : forall (X Y : Set) (f : X -> monad_carrier Y) (x : X),
    bind X Y f (unit X x) = f x;
unit_bind : forall (X : Set) (x : monad_carrier X), bind X X (unit X) x = x
}.
Notation "x >>= f" := (@bind _ _ _ f x).

Record Mod (U : Monad) : Type := {
  mod_carrier :> Set -> Set;
  mbind : forall (X Y : Set) (f : X -> U Y) (x : mod_carrier X), mod_carrier Y;
  mbind_mbind : forall (X Y Z : Set) (f : X -> U Y) (g : Y -> U Z) (x : mod_carrier X),
    mbind Y Z g (mbind X Y f x) = mbind X Z (fun u => f u >>= g) x;
  unit_mbind : forall (X : Set) (x : mod_carrier X), mbind X X (@unit U X) x = x
}.
Notation "x >>>= f" := (@mbind _ _ _ _ f x).

```

The library also includes the definition of morphism of monads and modules and other related categorical material. Other definitions which are specific to our objective are those of derived module and exponential monad. The latter reads as follows:

```

Record ExpMonad : Type := {
  exp_monad :> Monad;
  exp_abs : Mod_Hom (Derived_Mod exp_monad) exp_monad;
  exp_app : Mod_Hom exp_monad (Derived_Mod exp_monad);
  exp_eta : forall X (x : exp_monad X), exp_abs _ (exp_app _ x) = x;
  exp_beta : forall X (x : Derived_Mod exp_monad X), exp_app _ (exp_abs _ x) = x
}.

```

and comes with its associated notion of morphism:

```

Record ExpMonad_Hom (M N : ExpMonad) : Type := {
  expmonad_hom :> Monad_Hom M N;
  expmonad_hom_app : forall X (x : M X),
    expmonad_hom _ (exp_app M _ x) = exp_app N _ (expmonad_hom _ x);
  expmonad_hom_abs : forall X (x : Derived_Mod M X),
    expmonad_hom _ (exp_abs M _ x) = exp_abs N _ (expmonad_hom _ x)
}.

```

### 9.3 Formalisation of the $\lambda$ -calculus

This part contains the definition of syntactic and semantic  $\lambda$ -calculus (files Slc.v and Lc.v respectively). We use nested datatypes to encode  $\lambda$ -terms in the Calculus of (Co)Inductive Constructions as already shown in the Haskell fragment of section 5.1 for which we report below the equivalent Coq code. Notice that this encoding can be considered a typeful variant of the well-known de Bruijn encoding [3]. As the de Bruijn encoding, it represents  $\lambda$ -terms modulo  $\alpha$ -conversion.

```

Inductive term (X : Set) : Set := var : X -> term X
| app : term X -> term X -> term X
| abs : term (option X) -> term X.

Fixpoint fct (X Y : Set) (f : X -> Y) (x : term X) { struct x } : term Y :=
  match x with var a => var (f a)
| app x y => app (x //- f) (y //- f)
| abs x => abs (x //- (optmap f)) end
where "x //- f" := (@fct _ _ f x).

Definition shift X (x : term X) : term (option X) := x //- @Some X.

Definition comm (X Y : Set) (f : X -> term Y) (x : option X) : term (option Y) :=
  match x with Some a => shift (f a) | None => var None end.

```

```

Fixpoint subst (X Y : Set) (f : X -> term Y) (x : term X) { struct x } : term Y :=
  match x with var x => f x
    | app x y => app (x // f) (y // f)
    | abs x => abs (x // comm f) end
where "x // f" := (@subst _ _ f x).

```

Once the basic definitions are settled, we prove a series of basic lemmas which includes the associativity of substitution

```

Lemma subst_subst : forall (X Y Z : Set) (f : X -> term Y) (g : Y -> term Z) (x : term X),
  x // f // g = x // fun u => f u // g.

```

which is the most important ingredient to prove that the  $\lambda$ -calculus is a monad. Finally, we introduce the beta-eta equivalence relation on lambda terms

```

Inductive lcr (X : Set) : term X -> term X -> Prop :=
| lcr_var : forall a : X, var a == var a
| lcr_app : forall x1 x2 y1 y2 : term X, x1 == x2 -> y1 == y2 -> app x1 y1 == app x2 y2
| lcr_abs : forall x y : term (option X), x == y -> abs x == abs y
| lcr_beta : forall x y : term X, Beta x y -> x == y
| lcr_eta : forall x : term X, abs (app1 x) == x
| lcr_sym : forall x y : term X, y == x -> x == y
| lcr_trs : forall x y z : term X, lcr x y -> lcr y z -> lcr x z
where "x == y" := (@lcr _ x y).

```

and prove some compatibility lemmas for constructors and other operations. The compatibility of substitution is stated as follows:

```

Lemma lcr_subst : forall (X Y : Set) (f g : X -> term Y) (x y : term X),
  (forall u, f u == g u) -> x == y -> x // f == y // g.

```

## 9.4 Proof of the main theorem

The fourth and last part summarises the results proved in the other parts and proves the main theorem. It starts by glueing together the previous two sections by proving that our definitions of syntactic and semantic lambda calculus provides indeed two monads, denoted SLC and LC respectively, and by showing that the two morphisms `abs` and `app1` constitute morphisms of modules:

```

Definition SLC : Monad := Build_Monad term subst var subst_subst subst_var var_subst.

```

```

Definition LC : Monad :=
  Build_Monad lc lc_subst lc_var lc_subst_assoc lc_subst_var lc_var_subst.

```

```

Let abs_hom : Mod_Hom (Derived_Mod LC) LC :=
  Build_Mod_Hom (Derived_Mod LC) LC lc_abs lc_abs_hom.

```

```

Let app1_hom : Mod_Hom LC (Derived_Mod LC) :=
  Build_Mod_Hom LC (Derived_Mod LC) lc_app1 lc_app1_hom.

```

One more glueing step is the proof that LC is an exponential monad, which is stated in Coq through the following definition:

```

Definition ELC : ExpMonad := Build_ExpMonad abs_hom app1_hom lc_eta lc_beta.

```

Next comes the construction of the initial morphism which is initially defined as a fixpoint on terms.

```

Variable M : ExpMonad.
Fixpoint iota_fix X (x : term X) { struct x } : M X :=
  match x with var a => unit M a
    | app x y => exp_app M _ (iota_fix x) >>= default (@unit M X) (iota_fix y)
    | abs x => exp_abs M _ (iota_fix x) end.

```

Then we prove that `iota_fix` is compatible with the  $\beta\eta$  equivalence relation and thus it factors through the monad LC.

```
Let iota X : lc X -> M X := lc_factor (@iota_fix X) (@iota_fix_wd X).
```

The construction of the initial morphism ends with the verification that it is actually a morphism of exponential monads.

```
Let iota_monad : Monad_Hom LC M := Build_Monad_Hom LC M iota iota_subst iota_var.
```

```
Let exp_iota : ExpMonad_Hom ELC M :=  
  Build_ExpMonad_Hom ELC M iota_monad iota_app1 iota_abs.
```

Finally, we prove that `iota_monad` is unique.

```
Theorem iota_unique : forall j : ExpMonad_Hom ELC M, j = exp_iota.
```

The Coq terms `ELC`, `iota_monad` and `iota_unique` altogether form the formal proof of the initiality of the monad `LC` in the category of exponential monads.

## References

1. Thorsten Altenkirch and Bernhard Reus. Monadic presentations of lambda terms using generalized inductive types. In *CSL*, pages 453–468, 1999.
2. Richard Bird and Ross Paterson. Generalised folds for nested datatypes. *Formal Aspects of Computing*, 11(2):200–222, 1999.
3. Richard S. Bird and Ross Paterson. De Bruijn notation as a nested datatype. *Journal of Functional Programming*, 9(1):77–91, 1999.
4. Michael Ching. Bar constructions for topological operads and the Goodwillie derivatives of the identity. *Geom. Topol.*, 9:833–933 (electronic), 2005.
5. The Coq Proof Assistant. <http://coq.inria.fr>.
6. Marcelo Fiore, Gordon Plotkin, and Daniele Turi. Abstract syntax and variable binding (extended abstract). In *14th Symposium on Logic in Computer Science (Trento, 1999)*, pages 193–202. IEEE Computer Soc., Los Alamitos, CA, 1999.
7. Marcelo P. Fiore. On the structure of substitution. Invited address for the 22nd Mathematical Foundations of Programming Semantics Conf. (MFPS XXII), 2006. DISI, University of Genova (Italy).
8. Marcelo P. Fiore and Daniele Turi. Semantics of name and value passing. In *Logic in Computer Science*, pages 93–104, 2001.
9. Benoit Fresse. Koszul duality of operads and homology of partition posets. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 115–215. Amer. Math. Soc., Providence, RI, 2004.
10. Murdoch Gabbay and Andrew Pitts. A new approach to abstract syntax involving binders. In *14th Symposium on Logic in Computer Science (Trento, 1999)*, pages 214–224. IEEE Computer Soc., Los Alamitos, CA, 1999.
11. Neil Ghani and Tarmo Uustalu. Explicit substitutions and higher-order syntax. In *MERLIN '03: Proceedings of the 2003 ACM SIGPLAN workshop on Mechanized reasoning about languages with variable binding*, pages 1–7, New York, NY, USA, 2003. ACM Press.
12. André Hirschowitz and Marco Maggesi. The algebraicity of the lambda-calculus. [arXiv:math/0607427v1](https://arxiv.org/abs/math/0607427v1), 2007.
13. Martin Hofmann. Semantical analysis of higher-order abstract syntax. In *14th Symposium on Logic in Computer Science (Trento, 1999)*, pages 204–213. IEEE Computer Soc., Los Alamitos, CA, 1999.
14. Muriel Livernet. From left modules to algebras over an operad: application to combinatorial Hopf algebras. [arXiv:math/0607427v1](https://arxiv.org/abs/math/0607427v1), 2006.
15. Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
16. Martin Markl. Models for operads. [arXiv:hep-th/9411208v1](https://arxiv.org/abs/hep-th/9411208v1), 1994.
17. Martin Markl. A compactification of the real configuration space as an operadic completion. [arXiv:hep-th/9608067v1](https://arxiv.org/abs/hep-th/9608067v1), 1996.
18. Ralph Matthes and Tarmo Uustalu. Substitution in non-wellfounded syntax with variable binding. *Theor. Comput. Sci.*, 327(1-2):155–174, 2004.
19. V. A. Smirnov. Homotopy theory of coalgebras. *Math. USSR Izv.*, 27:575–592, 1986.
20. Miki Tanaka and John Power. A unified category-theoretic formulation of typed binding signatures. In *MERLIN '05: Proceedings of the 3rd ACM SIGPLAN workshop on Mechanized reasoning about languages with variable binding*, pages 13–24, New York, NY, USA, 2005. ACM Press.
21. Miki Tanaka and John Power. Pseudo-distributive laws and axiomatics for variable binding. *Higher Order Symbol. Comput.*, 19(2-3):305–337, 2006.
22. Julianna Zsidó. Le lambda calcul vu comme monade initiale. Master’s thesis, Université de Nice – Laboratoire J. A. Dieudonné, 2005/06. Mémoire de Recherche – master 2.