Computational Interpretations of Classical Linear Logic

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Abstract. We survey several computational interpretations of classical linear logic based on two-player one-move games. The moves of the games are higher-order functionals in the language of finite types. All interpretations discussed treat the exponential-free fragment of linear logic in a common way. They only differ in how much advantage one of the players has in the exponentials games. We discuss how the several choices for the interpretation of the modalities correspond to various well-known functional interpretations of intuitionistic logic, including Gödel's Dialectica interpretation and Kreisel's modified realizability.

1 Introduction

This article surveys several interpretations [3,16,17,18] of classical linear logic based on one-move two-player (Eloise and Abelard) games. As we will see, these are related to functional interpretations of intuitionistic logic such as Gödel's Dialectica interpretations [11] and Kreisel's modified realizability [14].

The intuition behind the interpretation is that each formula A defines an adjudication relation between arguments pro (Eloise's move) and against (Abelard's move) the truth of A. If the formula is in fact true, then Eloise should have no problem in winning the game. The interpretation of each of the logical connectives, quantifiers and exponentials corresponds to constructions that build new games out of given games. Given the symmetry of the interpretation, the game corresponding to the linear negation of A is simply the game A with the roles of the two players swapped. The simplest interpretation of the exponential games view these as games where only one player needs to make a move. For instance, in the game $?A$, only Abelard makes a move, and Eloise will win in case she has a winning move for the game A with the given Abelard's move. A symmetric situation occurs in the case of the game $!A$, only that Abelard now has the advantage. The idea is that the exponentials $?$ and $!$ serve as trump cards for Eloise and Abelard, respectively.

The paper is organised as follows. The basic interpretation of the exponential-free fragment of classical linear logic is presented in Section 2, and soundness of the interpretation is proved. Completeness of the interpretation is presented in Section 3. A simple form of branching quantifier is used for the proof of completeness. In Section 4, we discuss the various possibilities for the interpretation of the exponentials.

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For an introduction to modified realizability see chapter III of [20] or the book chapter [21]. Background on Gödel's Dialectica interpretation can be obtained in [1]. For an introduction to linear logic see Girard's original papers [9,10].

1.1 Classical Linear Logic LL^ω

We work with an extension of classical linear logic to the language of all finite types. The set of *finite types* $\mathcal{T}$ is inductively defined as follows:

- $o \in \mathcal{T}$;
- if $\rho, \sigma \in \mathcal{T}$ then $\rho \to \sigma \in \mathcal{T}$.

For simplicity, we deal with only one basic finite type o.

We assume that the terms of LL^ω contain all typed λ-terms, i.e. variables x^ρ for each finite type ρ; λ-abstractions $(\lambda x^\rho.t^\sigma)^{\rho\to\sigma}$; and term applications $(t^{\rho\to\sigma}s^\rho)^\sigma$. Note that we work with the standard typed λ-calculus, and not with a linear variant thereof. The atomic formulas of LL^ω are $A_{\mathsf{at}}, B_{\mathsf{at}}, \ldots$ and $A_{\mathsf{at}}^\perp, B_{\mathsf{at}}^\perp, \ldots$. For simplicity, the standard propositional constants $0, 1, \perp, \top$ of linear logic have been omitted, since the realizability interpretation of atomic formulas is trivial (see Definition 1).

Table 1. Structural rules

$$A_{\mathsf{at}}, A_{\mathsf{at}}^\perp \quad (\text{id}) \qquad \frac{\Gamma, A \qquad \Delta, A^\perp}{\Gamma, \Delta}\,(\text{cut}) \qquad \frac{\Gamma}{\pi\{\Gamma\}}\,(\text{per})$$

Formulas are built out of atomic formulas $A_{\mathsf{at}}, B_{\mathsf{at}}, \ldots$ and $A_{\mathsf{at}}^\perp, B_{\mathsf{at}}^\perp, \ldots$ via the connectives $A \mathbin{⅋} B$ (par), $A \otimes B$ (tensor), $A \diamond_z B$ (if-then-else), and quantifiers $\forall x A$ and $\exists x A$ (exponentials are treated in Section 4). The *linear negation* $A^\perp$ of an arbitrary formula A is an abbreviation as follows:

$$\begin{array}{llll} (A_{\mathsf{at}})^\perp & \equiv A_{\mathsf{at}}^\perp & (A_{\mathsf{at}}^\perp)^\perp & \equiv A_{\mathsf{at}} \\ (\exists z A)^\perp & \equiv \forall z A^\perp & (\forall z A)^\perp & \equiv \exists z A^\perp \\ (A \mathbin{⅋} B)^\perp & \equiv A^\perp \otimes B^\perp & (A \otimes B)^\perp & \equiv A^\perp \mathbin{⅋} B^\perp \\ (A \diamond_z B)^\perp & \equiv A^\perp \diamond_z B^\perp. & & \end{array}$$

So, $(A^\perp)^\perp$ is syntactically equal to A.

The structural rules of linear logic (shown in Table 1) do not contain the usual rules of weakening and contraction. These are added separately, in a controlled manner via the use of modalities (cf. Section 4). The rules for the multiplicative connectives and quantifiers are shown in Table 2, with the usual side condition in the rule $(\forall)$ that the variable z must not appear free in Γ.

We will deviate from the standard formulation of linear logic, in the sense that we will use the if-then-else logical constructor $A \diamond_z B$ instead of standard additive

Table 2. Rules for multiplicative connectives and quantifiers

$$\frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \otimes B}\,(\otimes) \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}\,(\parr) \qquad \frac{\Gamma, A}{\Gamma, \forall z A}\,(\forall) \qquad \frac{\Gamma, A[t/z]}{\Gamma, \exists z A}\,(\exists)$$

conjunction and disjunction[1]. The logical rules for $\Diamond_z$ are shown in Table 3, where $(z)(\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1)$ denotes a conditional λ-term which reduces to either $\boldsymbol{\gamma}_0$ or $\boldsymbol{\gamma}_1$ depending on whether the boolean variable z reduces to true or false, respectively. The standard additives can be defined as

$$A \wedge B :\equiv \forall z(A \Diamond_z B)$$

$$A \vee B :\equiv \exists z(A \Diamond_z B)$$

with the help of quantification over booleans.

Table 3. Rules for if-then-else connective

$$\frac{\Gamma[\boldsymbol{\gamma}_0], A \quad \Gamma[\boldsymbol{\gamma}_1], B}{\Gamma[(z)(\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1)], A \Diamond_z B}\,(\Diamond_z) \qquad \frac{\Gamma, A}{\Gamma, A \Diamond_{\mathrm{t}} B}\,(\Diamond_{\mathrm{t}}) \qquad \frac{\Gamma, B}{\Gamma, A \Diamond_{\mathrm{f}} B}\,(\Diamond_{\mathrm{f}})$$

Notation. We use bold face variables $\boldsymbol{f}, \boldsymbol{g}, \ldots, \boldsymbol{x}, \boldsymbol{y}, \ldots$ for tuples of variables, and bold face terms $\boldsymbol{a}, \boldsymbol{b}, \ldots, \boldsymbol{\gamma}, \boldsymbol{\delta}, \ldots$ for tuples of terms. Given sequence of terms $\boldsymbol{a}$ and $\boldsymbol{b}$, by $\boldsymbol{a}(\boldsymbol{b})$, we mean the sequence of terms $a_0(\boldsymbol{b}), \ldots, a_n(\boldsymbol{b})$. Similarly for $\boldsymbol{a}[\boldsymbol{b}/\boldsymbol{x}]$.

2 Basic Interpretation

To each formula A of the exponential-free fragment of linear logic we associate a quantifier-free formula $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$, where $\boldsymbol{x}, \boldsymbol{y}$ are fresh-variables not appearing in A. The variables $\boldsymbol{x}$ in the superscript are called the *witnessing variables*, while the subscript variables $\boldsymbol{y}$ are called the *challenge variables*. Intuitively, the interpretation of a formula A is a two-player (Eloise and Abelard) one-move game, where $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ is the adjudication relation. We want that Eloise has a winning move whenever A is provable in LL^ω. Moreover, the linear logic proof of A will provide Eloise's winning move $\boldsymbol{a}$, i.e. $\forall \boldsymbol{y} |A|^{\boldsymbol{a}}_{\boldsymbol{y}}$.

Definition 1 (Basic interpretation). *Assume we have already defined* $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ *and* $|B|^{\boldsymbol{v}}_{\boldsymbol{w}}$, *we define*

[1] See Girard's comments in [9] (p13) and [10] (p73) on the relation between the additive connectives and the if-then-else construct.

$$
\begin{aligned}
|A \parr B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{v},\boldsymbol{w}} &:\equiv |A|^{\boldsymbol{f}\boldsymbol{w}}_{\boldsymbol{v}} \parr |B|^{\boldsymbol{g}\boldsymbol{v}}_{\boldsymbol{w}} \\
|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{f},\boldsymbol{g}} &:\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{v}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{g}\boldsymbol{x}} \\
|A \diamond_z B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} &:\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|\exists z A(z)|^{\boldsymbol{x},z}_{\boldsymbol{f}} &:\equiv |A(z)|^{\boldsymbol{x}}_{\boldsymbol{f}z} \\
|\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{y},z} &:\equiv |A(z)|^{\boldsymbol{f}z}_{\boldsymbol{y}}.
\end{aligned}
$$

The interpretation of atomic formulas are the atomic formulas themselves, i.e.

$$
\begin{aligned}
|A_{\mathsf{at}}| &:\equiv A_{\mathsf{at}} \\
|A_{\mathsf{at}}^{\perp}| &:\equiv A_{\mathsf{at}}^{\perp}.
\end{aligned}
$$

Notice that for atomic formulas the tuples of witnesses and challenges are both empty.

It is easy to see that $|A^{\perp}|^{\boldsymbol{y}}_{\boldsymbol{x}} \equiv (|A|^{\boldsymbol{x}}_{\boldsymbol{y}})^{\perp}$. We now prove the soundness of the basic interpretation, i.e. we show how Eloise's winning move in the game $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ can be extracted from a proof of A in classical linear logic (exponentials treated in Section 4).

Theorem 1 (Soundness). *Let $A_0, \ldots, A_n$ be formulas of LL^{ω}, with $\boldsymbol{z}$ as the only free-variables. If*

$$A_0(\boldsymbol{z}), \ldots, A_n(\boldsymbol{z})$$

is provable in LL^{ω} then from this proof terms $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n$ can be extracted such that

$$|A_0(\boldsymbol{z})|^{\boldsymbol{a}_0}_{\boldsymbol{x}_0}, \ldots, |A_n(\boldsymbol{z})|^{\boldsymbol{a}_n}_{\boldsymbol{x}_n}$$

is also provable in LL^{ω}, where $\mathsf{FV}(\boldsymbol{a}_i) \in \{\boldsymbol{z}, \boldsymbol{x}_0, \ldots, \boldsymbol{x}_n\} \backslash \{\boldsymbol{x}_i\}$.

Proof. The proof is by induction on the derivation of $A_0, \ldots, A_n$. The only relevant rule where free-variables matter is the universal quantifier rule. Therefore, for all the other rules we will assume the tuple of parameters $\boldsymbol{z}$ is empty. The cases of the axiom, permutation rule and if-then-else are trivial. The other rules are treated as follows:

Multiplicativies.

$$
\dfrac{\dfrac{\dfrac{|\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{x}]}_{\boldsymbol{v}}, |A|^{\boldsymbol{a}}_{\boldsymbol{x}}}{|\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{f}\boldsymbol{b}]}_{\boldsymbol{v}}, |A|^{\boldsymbol{a}}_{\boldsymbol{f}\boldsymbol{b}}}\,[\tfrac{\boldsymbol{f}\boldsymbol{b}}{\boldsymbol{x}}] \quad \dfrac{|\Delta|^{\boldsymbol{\delta}[\boldsymbol{y}]}_{\boldsymbol{w}}, |B|^{\boldsymbol{b}}_{\boldsymbol{y}}}{|\Delta|^{\boldsymbol{\delta}[\boldsymbol{g}\boldsymbol{a}]}_{\boldsymbol{w}}, |B|^{\boldsymbol{b}}_{\boldsymbol{g}\boldsymbol{a}}}\,[\tfrac{\boldsymbol{g}\boldsymbol{a}}{\boldsymbol{y}}]}{|\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{f}\boldsymbol{b}]}_{\boldsymbol{v}}, |\Delta|^{\boldsymbol{\delta}[\boldsymbol{g}\boldsymbol{a}]}_{\boldsymbol{w}}, |A|^{\boldsymbol{a}}_{\boldsymbol{f}\boldsymbol{b}} \otimes |B|^{\boldsymbol{b}}_{\boldsymbol{g}\boldsymbol{a}}}\,(\otimes)}{|\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{f}\boldsymbol{b}]}_{\boldsymbol{v}}, |\Delta|^{\boldsymbol{\delta}[\boldsymbol{g}\boldsymbol{a}]}_{\boldsymbol{w}}, |A \otimes B|^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{f},\boldsymbol{g}}}\,(\mathrm{D1})
\qquad
\dfrac{\dfrac{|\Gamma|^{\boldsymbol{\gamma}}_{\boldsymbol{v}}, |A|^{\boldsymbol{a}[\boldsymbol{y}]}_{\boldsymbol{x}}, |B|^{\boldsymbol{b}[\boldsymbol{x}]}_{\boldsymbol{y}}}{|\Gamma|^{\boldsymbol{\gamma}}_{\boldsymbol{v}}, |A|^{\boldsymbol{a}[\boldsymbol{y}]}_{\boldsymbol{x}} \parr |B|^{\boldsymbol{b}[\boldsymbol{x}]}_{\boldsymbol{y}}}\,(\parr)}{|\Gamma|^{\boldsymbol{\gamma}}_{\boldsymbol{v}}, |A \parr B|^{\lambda \boldsymbol{y}.\boldsymbol{a}[\boldsymbol{y}], \lambda \boldsymbol{x}.\boldsymbol{b}[\boldsymbol{x}]}_{\boldsymbol{x},\boldsymbol{y}}}\,(\mathrm{D1})
$$

Cut.

$$
\dfrac{\dfrac{|\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{x}]}_{\boldsymbol{v}}, |A|^{\boldsymbol{a}}_{\boldsymbol{x}}}{|\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{a}^-]}_{\boldsymbol{v}}, |A|^{\boldsymbol{a}}_{\boldsymbol{a}^-}}\,[\tfrac{\boldsymbol{a}^-}{\boldsymbol{x}}] \quad \dfrac{\dfrac{|\Delta|^{\boldsymbol{\delta}[\boldsymbol{x}^-]}_{\boldsymbol{w}}, |A^{\perp}|^{\boldsymbol{a}^-}_{\boldsymbol{x}^-}}{|\Delta|^{\boldsymbol{\delta}[\boldsymbol{a}]}_{\boldsymbol{w}}, |A^{\perp}|^{\boldsymbol{a}^-}_{\boldsymbol{a}}}\,[\tfrac{\boldsymbol{a}}{\boldsymbol{x}^-}]}{|\Delta|^{\boldsymbol{\delta}[\boldsymbol{a}]}_{\boldsymbol{w}}, (|A|^{\boldsymbol{a}}_{\boldsymbol{a}^-})^{\perp}}}{|\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{a}^-]}_{\boldsymbol{v}}, |\Delta|^{\boldsymbol{\delta}[\boldsymbol{a}]}_{\boldsymbol{w}}}\,(\mathrm{cut})
$$

Note that the assumption that the tuple of variables $\boldsymbol{x}$ (respectively $\boldsymbol{x}^-$) does not appear free in $\boldsymbol{a}$ (respectively $\boldsymbol{a}^-$) is used crucially in the soundness of the cut rule in order remove any circularity in the two simultaneous substitutions.

Quantifiers.

$$\frac{|\Gamma|_{\boldsymbol{v}}^{\boldsymbol{\gamma}[z]}, |A|_{\boldsymbol{x}}^{\boldsymbol{a}[z]}}{|\Gamma|_{\boldsymbol{v}}^{\boldsymbol{\gamma}[z]}, |\forall z A|_{\boldsymbol{x},z}^{\lambda z.\boldsymbol{a}[z]}}\ (\mathrm{D1}) \qquad \frac{\dfrac{|\Gamma|_{\boldsymbol{v}}^{\boldsymbol{\gamma}[\boldsymbol{x}]}, |A(t)|_{\boldsymbol{x}}^{\boldsymbol{a}}}{|\Gamma|_{\boldsymbol{v}}^{\boldsymbol{\gamma}[\boldsymbol{g}t]}, |A(t)|_{\boldsymbol{g}t}^{\boldsymbol{a}}}\ [\tfrac{\boldsymbol{g}t}{\boldsymbol{x}}]}{|\Gamma|_{\boldsymbol{v}}^{\boldsymbol{\gamma}[\boldsymbol{g}t]}, |\exists z A(z)|_{\boldsymbol{g}}^{\boldsymbol{a},t}}\ (\mathrm{D1})$$

This concludes the proof. □

3 Completeness

In this section we investigate the completeness of the interpretation given above. More precisely, we ask the question: for which extension of classical linear logic $\mathsf{LL}^\star$ it is the case that if there are terms $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n$ such that $|A_0(\boldsymbol{z})|_{\boldsymbol{x}_0}^{\boldsymbol{a}_0}, \ldots, |A_n(\boldsymbol{z})|_{\boldsymbol{x}_n}^{\boldsymbol{a}_n}$ is provable in $\mathsf{LL}^\star$ then the sequence $A_0, \ldots, A_n$ is also provable in $\mathsf{LL}^\star$? The idea that a formula A is interpreted as a *symmetric* game $|A|_{\boldsymbol{y}}^{\boldsymbol{x}}$ between two players suggests that A is equivalent to $\exists\!\!\!/_{\boldsymbol{y}}^{\boldsymbol{x}} |A|_{\boldsymbol{y}}^{\boldsymbol{x}}$, using a simple form of branching quantifier to ensure that no player has an advantage over the other. This simple branching quantifier (we will refer to these as *simultaneous quantifiers*[2]) can be axiomatised with the rule:

$$\frac{A_0(\boldsymbol{a}_0, \boldsymbol{y}_0), \ldots, A_n(\boldsymbol{a}_n, \boldsymbol{y}_n)}{\exists\!\!\!/_{\boldsymbol{y}_0}^{\boldsymbol{x}_0} A_0(\boldsymbol{x}_0, \boldsymbol{y}_0), \ldots, \exists\!\!\!/_{\boldsymbol{y}_n}^{\boldsymbol{x}_n} A_n(\boldsymbol{x}_n, \boldsymbol{y}_n)}\ (\exists\!\!\!/)$$

with the side-condition: $\boldsymbol{y}_i$ may only appear free in the terms $\boldsymbol{a}_j$, for $j \neq i$. In particular, we will have that each $\boldsymbol{y}_i$ will not be free in the conclusion of the rule.

The standard quantifier rules can be obtained from this single rule. The rule ($\forall$) can be obtained in the case when only the tuple $\boldsymbol{y}_n$ is non-empty. The rule ($\exists$) can be obtained in the case when only the tuple $\boldsymbol{x}_n$ is non-empty. Hence, for the rest of this section we will consider that standard quantifiers $\forall x A$ and $\exists x A$ are in fact abbreviations for $\exists\!\!\!/_x A$ and $\exists\!\!\!/^x A$, respectively.

In terms of games, the new quantifier embodies the idea of the two players performing their moves simultaneously. The most interesting characteristic of this simultaneous quantifier is with respect to linear negation, which is defined as

$$(\exists\!\!\!/_{\boldsymbol{y}}^{\boldsymbol{x}} A)^\perp \equiv \exists\!\!\!/_{\boldsymbol{x}}^{\boldsymbol{y}} A^\perp$$

and corresponds precisely to the switch of roles between the players. Let us refer to the extension of LL^ω with the new simultaneous quantifier by $\mathsf{LL}_{\mathsf{q}}^\omega$.

[2] According to Hyland [13] (footnote 18) "the identification of a sufficiently simple tensor as a Henkin quantifier is a common feature of a number of interpretations of linear logic". The simultaneous quantifier can be viewed as a simplification of Henkin's (branching) quantifier [4,12], in which no alternation of quantifiers is allowed on the two branches. See Bradfield [5] as well, where this simple form of branching quantifier is also used.

Theorem 2. *Extend the interpretation (Definition 1) to the system* $\mathsf{LL}^\omega_\mathsf{q}$ *by defining*

$$|\exists\!\!\!\forall^{\boldsymbol{v}}_{\boldsymbol{w}} A(\boldsymbol{v}, \boldsymbol{w})|^{\boldsymbol{f}, \boldsymbol{v}}_{\boldsymbol{g}, \boldsymbol{w}} :\equiv |A(\boldsymbol{v}, \boldsymbol{w})|^{\boldsymbol{f}\boldsymbol{w}}_{\boldsymbol{g}\boldsymbol{v}}.$$

Theorem 1 holds for the extended system $\mathsf{LL}^\omega_\mathsf{q}$.

Proof. The rule for the simultaneous quantifier is handled as follows:

$$\dfrac{\dfrac{|A_0(\boldsymbol{a}_0, \boldsymbol{w}_0)|^{\boldsymbol{b}_0}_{\boldsymbol{y}_0}, \ldots, |A_n(\boldsymbol{a}_n, \boldsymbol{w}_n)|^{\boldsymbol{b}_n}_{\boldsymbol{y}_n}}{|A_0(\boldsymbol{a}_0, \boldsymbol{w}_0)|^{\boldsymbol{b}'_0}_{\boldsymbol{g}_0 \boldsymbol{a}_0}, \ldots, |A_n(\boldsymbol{a}_n, \boldsymbol{w}_n)|^{\boldsymbol{b}'_n}_{\boldsymbol{g}_n \boldsymbol{a}_n}} \ \frac{\boldsymbol{g}_i \boldsymbol{a}_i}{\boldsymbol{y}_i}}{|\exists\!\!\!\forall^{\boldsymbol{v}_0}_{\boldsymbol{w}_0} A_0(\boldsymbol{v}_0, \boldsymbol{w}_0)|^{\lambda \boldsymbol{w}_0 . \boldsymbol{b}'_0, \boldsymbol{a}_0}_{\boldsymbol{g}_0, \boldsymbol{w}_0}, \ldots, |\exists\!\!\!\forall^{\boldsymbol{v}_n}_{\boldsymbol{w}_n} A_n(\boldsymbol{v}_n, \boldsymbol{w}_n)|^{\lambda \boldsymbol{w}_n . \boldsymbol{b}'_n, \boldsymbol{a}_n}_{\boldsymbol{g}_n, \boldsymbol{w}_n}} \ (\mathrm{D1})$$

where $\boldsymbol{b}'_j$ is the sequence of terms $\boldsymbol{b}_j$ after the substitutions $\boldsymbol{g}_i \boldsymbol{a}_i / \boldsymbol{y}_i$, for $i \neq j$. □

In fact, since the simultaneous quantifiers are eliminated, we obtain an interpretation of $\mathsf{LL}^\omega_\mathsf{q}$ into LL^ω. Let us proceed now to define an extension of $\mathsf{LL}^\omega_\mathsf{q}$ which is complete with respect to the interpretation of Section 2. First we need some simple facts about $\mathsf{LL}^\omega_\mathsf{q}$.

Lemma 1. *The following are derivable in* $\mathsf{LL}^\omega_\mathsf{q}$

$$\exists\!\!\!\forall^{\boldsymbol{f}}_{\boldsymbol{y}, z} A(\boldsymbol{f} z, \boldsymbol{y}, z) \multimap \forall z \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A(\boldsymbol{x}, \boldsymbol{y}, z)$$

$$\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A(\boldsymbol{y}) \otimes \exists\!\!\!\forall^{\boldsymbol{v}}_{\boldsymbol{w}} B(\boldsymbol{w}) \multimap \exists\!\!\!\forall^{\boldsymbol{x}, \boldsymbol{v}}_{\boldsymbol{f}, \boldsymbol{g}} (A(\boldsymbol{f} \boldsymbol{v}) \otimes B(\boldsymbol{g} \boldsymbol{x})).$$

Proof. These can be derived as

$$\dfrac{\dfrac{A^\perp(\boldsymbol{f} z, \boldsymbol{y}, z), A(\boldsymbol{f} z, \boldsymbol{y}, z)}{\exists\!\!\!\forall^{\boldsymbol{y}, z}_{\boldsymbol{f}} A^\perp(\boldsymbol{f} z, \boldsymbol{y}, z), \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A(\boldsymbol{x}, \boldsymbol{y}, z)} \ (\exists\!\!\!\forall)}{\exists\!\!\!\forall^{\boldsymbol{y}, z}_{\boldsymbol{f}} A^\perp(\boldsymbol{f} z, \boldsymbol{y}, z), \forall z \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A(\boldsymbol{x}, \boldsymbol{y}, z)} \ (\forall)$$

and

$$\dfrac{\dfrac{\dfrac{A^\perp(\boldsymbol{f} \boldsymbol{v}), A(\boldsymbol{f} \boldsymbol{v}) \quad B^\perp(\boldsymbol{g} \boldsymbol{x}), B(\boldsymbol{g} \boldsymbol{x})}{A^\perp(\boldsymbol{f} \boldsymbol{v}), B^\perp(\boldsymbol{g} \boldsymbol{x}), A(\boldsymbol{f} \boldsymbol{v}) \otimes B(\boldsymbol{g} \boldsymbol{x})} \ (\otimes)}{\exists\!\!\!\forall^{\boldsymbol{y}}_{\boldsymbol{x}} A^\perp(\boldsymbol{y}), \exists\!\!\!\forall^{\boldsymbol{w}}_{\boldsymbol{v}} B^\perp(\boldsymbol{w}), \exists\!\!\!\forall^{\boldsymbol{x}, \boldsymbol{v}}_{\boldsymbol{f}, \boldsymbol{g}} (A(\boldsymbol{f} \boldsymbol{v}) \otimes B(\boldsymbol{g} \boldsymbol{x}))} \ (\exists\!\!\!\forall)}{\exists\!\!\!\forall^{\boldsymbol{y}}_{\boldsymbol{x}} A^\perp(\boldsymbol{y}) \parr \exists\!\!\!\forall^{\boldsymbol{w}}_{\boldsymbol{v}} B^\perp(\boldsymbol{w}), \exists\!\!\!\forall^{\boldsymbol{x}, \boldsymbol{v}}_{\boldsymbol{f}, \boldsymbol{g}} (A(\boldsymbol{f} \boldsymbol{v}) \otimes B(\boldsymbol{g} \boldsymbol{x}))} \ (\parr)$$

respectively. □

The converse of the implications in Lemma 1, however, require extra logical principles. Consider the following principles for the simultaneous quantifier

$$\mathsf{AC}_s \quad : \quad \forall z \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A(\boldsymbol{x}, \boldsymbol{y}, z) \multimap \exists\!\!\!\forall^{\boldsymbol{f}}_{\boldsymbol{y}, z} A(\boldsymbol{f} z, \boldsymbol{y}, z)$$

$$\mathsf{AC}_p \quad : \quad \exists\!\!\!\forall^{\boldsymbol{x}, \boldsymbol{v}}_{\boldsymbol{f}, \boldsymbol{g}} (A(\boldsymbol{f} \boldsymbol{v}) \otimes B(\boldsymbol{g} \boldsymbol{x})) \multimap \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A(\boldsymbol{y}) \otimes \exists\!\!\!\forall^{\boldsymbol{v}}_{\boldsymbol{w}} B(\boldsymbol{w})$$

for *quantifier-free* formula A and B. We refer to these as the *sequential choice* AC_s and *parallel choice* AC_p. For those familiar with the modified realizability of intuitionistic logic, the principle AC_s corresponds to the standard (intentional) axiom of choice, while AC_p is a generalisation of the independence of premise principle (case when tuples $\boldsymbol{x}, \boldsymbol{v}$ and $\boldsymbol{w}$ are empty).

Lemma 2. *The principles AC_s and AC_p are sound for our interpretation, i.e. for any instance P of these principles, there are terms $\boldsymbol{t}$ such that $|P|^{\boldsymbol{t}}_{\boldsymbol{y}}$ is derivable in LL^ω.*

Proof. Consider an instance of AC_s

$$\forall z \exists\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A(\boldsymbol{x}, \boldsymbol{y}, z) \multimap \exists\!\!\forall^{\boldsymbol{f}}_{\boldsymbol{y},z} A(\boldsymbol{f}z, \boldsymbol{y}, z).$$

Since A is quantifier-free, it is easy to see that premise and conclusion have the same interpretation, namely

$$|\forall z \exists\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A(\boldsymbol{x}, \boldsymbol{y}, z)|^{\boldsymbol{f}}_{\boldsymbol{y},z} \equiv A(\boldsymbol{f}z, \boldsymbol{y}, z)$$
$$|\exists\!\!\forall^{\boldsymbol{f}}_{\boldsymbol{y},z} A(\boldsymbol{f}z, \boldsymbol{y}, z)|^{\boldsymbol{f}}_{\boldsymbol{y},z} \equiv A(\boldsymbol{f}z, \boldsymbol{y}, z).$$

The same is true for the premise and conclusion of AC_p. □

Let us denote by $\mathsf{LL}^\omega_{\mathsf{q}+}$ the extension of $\mathsf{LL}^\omega_{\mathsf{q}}$ with these two extra schemata AC_s and AC_p. The next lemma shows that, in fact, these extra principles are all one needs to show the equivalence between A and its interpretation $\exists\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$.

Lemma 3. *The equivalence between A and $\exists\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ can be derived in the system $\mathsf{LL}^\omega_{\mathsf{q}}$.*

Proof. By induction on the logical structure of A. Consider for instance the case of $A \otimes B$.

$$A \otimes B \overset{\text{(IH)}}{\leftrightarrow} \exists\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes \exists\!\!\forall^{\boldsymbol{v}}_{\boldsymbol{w}} |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \overset{(\mathsf{L1},\mathsf{AC}_p)}{\leftrightarrow} \exists\!\!\forall^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{f},\boldsymbol{g}} (|A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{v}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{g}\boldsymbol{x}}).$$

The other cases are treated similarly. □

Theorem 3. *Let A be a formula in the language of LL^ω. Then A is derivable in $\mathsf{LL}^\omega_{\mathsf{q}+}$ if and only if $|A|^{\boldsymbol{t}}_{\boldsymbol{y}}$ is derivable in LL^ω, for some sequence of terms $\boldsymbol{t}$.*

Proof. The forward direction can be obtained with an extension of Theorem 1 given by Lemma 2. The converse follows from Lemma 3. □

4 Possible Interpretations of Exponentials

The exponential-free fragment of LL^ω, despite its nice properties, bears little relation to the standard logical systems of classical and intuitionistic logic. In order to recover the full strength of classical logic, we need to add back contraction and weakening. These are recovered in linear logic in a controlled manner, with the help of modalities (exponentials) $?A$ and $!A$ (cf. Table 4). The exponentials are dual to each other, i.e.

$$(?A)^\perp \equiv !A^\perp \qquad\qquad (!A)^\perp \equiv ?A^\perp.$$

Girard's points out in several places (cf. [10] (p84)) that these modalities, contrary to the other connectives, are not canonical. More precisely, if we add new modalities $?'A$ and $!'A$ with the same rules as shown in Table 4, we will not be able to derive the equivalences $?A \leftrightarrow ?'A$ and $!A \leftrightarrow !'A$. This is reflected in the flexibility with which we can interpret these modalities discussed below.

Table 4. Rules for the exponentials

$$\frac{?\Gamma, A}{?\Gamma,\ !A}\,(!) \qquad \frac{\Gamma, A}{\Gamma, ?A}\,(?) \qquad \frac{\Gamma, ?A, ?A}{\Gamma, ?A}\,(\mathrm{con}) \qquad \frac{\Gamma}{\Gamma, ?A}\,(\mathrm{wkn})$$

4.1 Interpretation 1: Kreisel's Modified Realizability

The first alternative for the interpretation of the exponentials we consider is one in which the game $?A$ gives maximal advantage to Eloise, and game $!A$ gives maximal advantage to Abelard. The maximal advantage corresponds to the player in question not needing to make any move, with their best possible move being played for them. More precisely, the interpretation is defined as:

$$|!A|^{\boldsymbol{x}} :\equiv\ !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$
$$|?A|_{\boldsymbol{y}} :\equiv\ ?\exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

It is easy to see that Theorem 1 still holds when Definition 1 is extended in this way. For instance, the soundness of the rules (!) and (con) are obtained as:

$$\dfrac{\dfrac{\dfrac{|?\Gamma|_{\boldsymbol{v}}, |A|^{\boldsymbol{a}}_{\boldsymbol{y}}}{|?\Gamma|_{\boldsymbol{v}}, \forall \boldsymbol{y} |A|^{\boldsymbol{a}}_{\boldsymbol{y}}}\,(\forall)}{|?\Gamma|_{\boldsymbol{v}}, !\forall \boldsymbol{y} |A|^{\boldsymbol{a}}_{\boldsymbol{y}}}\,(!)}{|?\Gamma|_{\boldsymbol{v}}, |!A|^{\boldsymbol{a}}} \qquad \dfrac{\dfrac{|\Gamma|^{\gamma[\boldsymbol{y}_0, \boldsymbol{y}_1]}_{\boldsymbol{v}}, |?A|_{\boldsymbol{y}_0}, |?A|_{\boldsymbol{y}_1}}{|\Gamma|^{\gamma[\boldsymbol{y}, \boldsymbol{y}]}_{\boldsymbol{v}}, |?A|_{\boldsymbol{y}}, |?A|_{\boldsymbol{y}}}\,[\tfrac{\boldsymbol{y}}{\boldsymbol{y}_0}, \tfrac{\boldsymbol{y}}{\boldsymbol{y}_1}]}{|\Gamma|^{\gamma[\boldsymbol{y}, \boldsymbol{y}]}_{\boldsymbol{v}}, |?A|_{\boldsymbol{y}}}\,(\mathrm{con})$$

We have shown [16] that when combined with the embedding of intuitionistic logic into linear logic, this choice for the interpretation of the exponentials corresponds to Kreisel's modified realizability interpretation [14] of intuitionistic logic.

Note that given this interpretation for the exponentials the relation $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ is no longer quantifier-free. It is the case, however, that formulas in the image of the interpretation (we call these *fixed formulas*) are also in the kernel of the interpretation. More, precisely, if A is in the kernel of the interpretation then $|A| \equiv A$. The completeness result of Section 3 needs to be calibrated, as the schemata AC_s and AC_p need to be taken for all fixed-formulas (and not just quantifier-free formulas). Moreover, we need an extra principle

$$\mathsf{TA} \quad : \quad !\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A \multimap \exists \boldsymbol{x} ! \forall \boldsymbol{y} A$$

called *trump advantage*, for fixed-formulas A, in order to obtain the equivalences involving exponentials, i.e. equivalence between $!\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A$ and $\exists \boldsymbol{x} ! \forall \boldsymbol{y} A$.

4.2 Interpretation 2: Diller-Nahm Interpretation

Another possibility for the interpretation is to give the player in question a restricted advantage by simply allowing the player to see the opponent's move, and then select a finite set of moves. If any of these is a good move the player wins. This leads to the following interpretation of the exponentials:

$$|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} :\equiv \,!\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$
$$|?A|^{\boldsymbol{f}}_{\boldsymbol{y}} :\equiv \,?\exists \boldsymbol{x} \in \boldsymbol{f}\boldsymbol{y}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

Again, an extension of Definition 1 in this direction would also make the Soundness Theorem 1 valid for the full classical linear logic. For instance, the soundness of the contraction rule is obtained as:

$$\dfrac{\dfrac{\dfrac{|\Gamma|^{\gamma[\boldsymbol{y}_0,\boldsymbol{y}_1]}_{\boldsymbol{v}}, |?A|^{\boldsymbol{a}_0}_{\boldsymbol{y}_0}, |?A|^{\boldsymbol{a}_1}_{\boldsymbol{y}_1}}{|\Gamma|^{\gamma[\boldsymbol{y}_0,\boldsymbol{y}_1]}_{\boldsymbol{v}}, ?\exists \boldsymbol{x}_0 \in \boldsymbol{a}_0\boldsymbol{y}_0 |A|^{\boldsymbol{x}_0}_{\boldsymbol{y}_0}, ?\exists \boldsymbol{x}_1 \in \boldsymbol{a}_1\boldsymbol{y}_1 |A|^{\boldsymbol{x}_1}_{\boldsymbol{y}_1}}\,(\mathrm{def})}{|\Gamma|^{\gamma[\boldsymbol{y},\boldsymbol{y}]}_{\boldsymbol{v}}, ?\exists \boldsymbol{x} \in (\boldsymbol{a}_0\boldsymbol{y}) \cup (\boldsymbol{a}_1\boldsymbol{y}) |A|^{\boldsymbol{x}}_{\boldsymbol{y}},}\,[\tfrac{\boldsymbol{y}}{\boldsymbol{y}_0}, \tfrac{\boldsymbol{y}}{\boldsymbol{y}_1}]}{|\Gamma|^{\gamma[\boldsymbol{y},\boldsymbol{y}]}_{\boldsymbol{v}}, |?A|^{\lambda \boldsymbol{y}((\boldsymbol{a}_0\boldsymbol{y}) \cup (\boldsymbol{a}_1\boldsymbol{y}))}_{\boldsymbol{y}}}\,(\mathrm{con})$$

while the rules (!) is dealt with as follows:

$$\dfrac{\dfrac{|?\Gamma|^{\gamma[\boldsymbol{y}]}_{\boldsymbol{v}}, |A|^{\boldsymbol{a}[\boldsymbol{v}]}_{\boldsymbol{y}}}{|?\Gamma|^{\cup_{\boldsymbol{y} \in \boldsymbol{f}(\boldsymbol{a}[\boldsymbol{v}])}(\gamma[\boldsymbol{y}]\boldsymbol{v})}_{\boldsymbol{v}}, \forall \boldsymbol{y} \in \boldsymbol{f}(\boldsymbol{a}[\boldsymbol{v}])\, |A|^{\boldsymbol{a}[\boldsymbol{v}]}_{\boldsymbol{y}}}\,(\forall)}{|?\Gamma|^{\lambda \boldsymbol{v}.\cup_{\boldsymbol{y} \in \boldsymbol{f}(\boldsymbol{a}[\boldsymbol{v}])}(\gamma[\boldsymbol{y}]\boldsymbol{v})}_{\boldsymbol{v}}, |!A|^{\boldsymbol{a}[\boldsymbol{v}]}_{\boldsymbol{f}}}\,(!)$$

It is clear in this case that enough term construction needs to be added to the verifying system in order to deal with finite sets of arbitrary type. This choice for the treatment of the exponentials corresponds to a variant of Gödel's Dialectica interpretation due to Diller and Nahm [6].

4.3 Interpretation 3: Stein's Interpretation

A hybrid interpretation between options 1 and 2 can also be given for each parameter $n \in \mathbb{N}$. The natural number n dictates from which type level we should use option 2 (Diller-Nahm), and up to which level we should choose option 1 (modified realizability). The slogan is "only higher-type objects are witnessed". Given a tuple of variable $\boldsymbol{x}$, we will denote by $\underline{\boldsymbol{x}}$ the sub-tuple containing the variables in $\boldsymbol{x}$ which have type level $\geq n$, whereas $\overline{\boldsymbol{x}}$ denotes the sub-tuple of the variables in $\boldsymbol{x}$ which have type level $< n$. For each fixed type level $n \in \mathbb{N}$ we can define an interpretation of the exponentials as:

$$|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} :\equiv \,!\forall \underline{\boldsymbol{y}} \in \mathsf{rng}(\boldsymbol{f}\boldsymbol{x})\forall \overline{\boldsymbol{y}}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$
$$|?A|^{\boldsymbol{f}}_{\boldsymbol{y}} :\equiv \,?\exists \underline{\boldsymbol{x}} \in \mathsf{rng}(\boldsymbol{f}\boldsymbol{y})\exists \overline{\boldsymbol{x}}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

where $\forall \boldsymbol{y} \in \mathsf{rng}(\boldsymbol{b}^{(n-1)\to\rho})A[\boldsymbol{y}]$ and $\exists \boldsymbol{y} \in \mathsf{rng}(\boldsymbol{b}^{(n-1)\to\rho})A[\boldsymbol{y}]$ are used as an abbreviation for $\forall i^{n-1} A[\boldsymbol{b}i]$ and $\exists i^{n-1} A[\boldsymbol{b}i]$, respectively ($n-1$ is the pure type of type level $n-1$).

This choice corresponds to Stein's interpretation [19], and again leads to a sound interpretation of full classical linear logic.

4.4 Interpretation 4: Bounded Dialectica Interpretation

In [7,8], a "bounded" variant of Gödel's Dialectica interpretation was developed in order to deal with strong analytical principles in classical feasible analysis. The interpretation makes use of Howard-Bezem's strong majorizabilty relation $\leq^*$ between functionals (cf. [2]). Using the majorizability relation, we can define well-behaved bounded sets which can be used as moves in the treatment of the exponential games. More precisely, the interpretation of the exponentials can also be given as:

$$
\begin{aligned}
|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} &:\equiv {!}\forall \boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|?A|^{\boldsymbol{f}}_{\boldsymbol{y}} &:\equiv {?}\exists \boldsymbol{x} \leq^* \boldsymbol{f}\boldsymbol{y}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.
\end{aligned}
$$

As argued in [15], in this case we must first perform a relativisation of the quantifiers to Bezem's model $\mathcal{M}$ of strongly majorizable functionals. After that, all candidate witnesses and challenges are monotone, and the soundness of the contraction rule can be derived as

$$
\dfrac{\dfrac{\dfrac{|\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{y}_0,\boldsymbol{y}_1]}_{\boldsymbol{v}}, |?A|^{\boldsymbol{a}_0}_{\boldsymbol{y}_0}, |?A|^{\boldsymbol{a}_1}_{\boldsymbol{y}_1}}{|\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{y}_0,\boldsymbol{y}_1]}_{\boldsymbol{v}}, ?\exists \boldsymbol{x}_0 \leq^* \boldsymbol{a}_0\boldsymbol{y}_0 |A|^{\boldsymbol{x}_0}_{\boldsymbol{y}_0}, ?\exists \boldsymbol{x}_1 \leq^* \boldsymbol{a}_1\boldsymbol{y}_1 |A|^{\boldsymbol{x}_1}_{\boldsymbol{y}_1}}\,(\text{def})}{|\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{y},\boldsymbol{y}]}_{\boldsymbol{v}}, ?\exists \boldsymbol{x} \leq^* \max(\boldsymbol{a}_0\boldsymbol{y}, \boldsymbol{a}_1\boldsymbol{y}) |A|^{\boldsymbol{x}}_{\boldsymbol{y}},}\,\left[\frac{\boldsymbol{y}}{\boldsymbol{y}_0}, \frac{\boldsymbol{y}}{\boldsymbol{y}_1}\right]}{|\Gamma|^{\boldsymbol{\gamma}[\boldsymbol{y},\boldsymbol{y}]}_{\boldsymbol{v}}, |?A|^{\lambda \boldsymbol{y}.\max(\boldsymbol{a}_0\boldsymbol{y},\boldsymbol{a}_1\boldsymbol{y})}_{\boldsymbol{y}}}\,(\text{con})
$$

where $\max(\boldsymbol{x}^\rho, \boldsymbol{y}^\rho)$ is the pointwise maximum.

Besides being sound for the principles AC_s, AC_p of Section 3, and the principle TA of Section 4.1, the Dialectica interpretation of LL^ω will also interpret the linear counterpart of the bounded Markov principle

$$
\mathsf{MP}_B \quad : \quad \forall \boldsymbol{z} {!}\forall \boldsymbol{x} \leq^* \boldsymbol{z}\, A \multimap {!}\forall \boldsymbol{x} A
$$

for bounded formulas A. The completeness for the bounded interpretation can then be extended to deal with the exponentials as

$$
\begin{tikzcd}
{!}\exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A \arrow[r, shift left, "\mathsf{TA}"] & \exists \boldsymbol{x} {!}\forall \boldsymbol{y} A \arrow[l, shift left, "\mathsf{LL}^\omega_{\mathsf{q}}"] \arrow[r, shift left, "\mathsf{LL}^\omega"] & \exists \boldsymbol{x} \forall \boldsymbol{z} {!}\forall \boldsymbol{y} \leq^* \boldsymbol{z}\, A \arrow[l, shift left, "\mathsf{MP}_B"] \arrow[r, shift left, "\mathsf{LL}^\omega_{\mathsf{q}}"] & \exists\!\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{f}} {!}\forall \boldsymbol{y} \leq^* \boldsymbol{f}\boldsymbol{x}\, A \arrow[l, shift left, "\mathsf{AC}_s"]
\end{tikzcd}
$$

MP_B in particular implies the majorizability axiom by taking $A(x, y) \equiv (x = y)$. In fact, the principles $\mathsf{AC}_s, \mathsf{AC}_p, \mathsf{TA}, \mathsf{MP}_B$ can be viewed as refinements of the (linear logic versions of) the extra principles used in the bounded functional interpretation [7]. For instance, the bounded Markov principle translates into linear logic as

$$
({!}\forall y A \multimap B) \multimap \exists b ({!}\forall y \leq^* b\, A \multimap B)
$$

for bounded formulas A and B. This can be derived as: By TA we get from ${!}\forall y A \multimap B$ to $\forall b {!}\forall y \leq^* b\, A \multimap B$. Then by AC_p we get $\exists b ({!}\forall y \leq^* b\, A \multimap B)$.

4.5 Interpretation 5: Gödel's Dialectica Interpretation

The most restricted interpretation is the one in which the player's advantage in the exponential game is minimal. The only head-start will be to be able to see the opponents move. Based on the opponent's move the player will then have to make a single move. This leads to an extension of the interpretation given in Definition 1 with the interpretation of the exponentials as

$$|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} :\equiv !|A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}}$$
$$|?A|^{\boldsymbol{f}}_{\boldsymbol{y}} :\equiv ?|A|^{\boldsymbol{f}\boldsymbol{y}}_{\boldsymbol{y}}.$$

Note that in this case the target of the interpretation is again a quantifier-free calculus (as in the basic interpretation of Section 2). For the soundness, however, we must assume that quantifier-free formulas are decidable (a usual requirement for Dialectica interpretations) in order to satisfy the contraction rule

$$\dfrac{\dfrac{\dfrac{|\Gamma|^{\gamma[\boldsymbol{y}_0,\boldsymbol{y}_1]}_{\boldsymbol{v}}, |?A|^{\boldsymbol{a}_0}_{\boldsymbol{y}_0}, |?A|^{\boldsymbol{a}_1}_{\boldsymbol{y}_1}}{|\Gamma|^{\gamma[\boldsymbol{y}_0,\boldsymbol{y}_1]}_{\boldsymbol{v}}, ?|A|^{\boldsymbol{a}_0\boldsymbol{y}_0}_{\boldsymbol{y}_0}, ?|A|^{\boldsymbol{a}_1\boldsymbol{y}_1}_{\boldsymbol{y}_1}}\,(\text{def})}{|\Gamma|^{\gamma[\boldsymbol{y},\boldsymbol{y}]}_{\boldsymbol{v}}, ?|A|^{\boldsymbol{a}\boldsymbol{y}}_{\boldsymbol{y}},}\,[\tfrac{\boldsymbol{y}}{\boldsymbol{y}_0}, \tfrac{\boldsymbol{y}}{\boldsymbol{y}_1}]}{|\Gamma|^{\gamma[\boldsymbol{y},\boldsymbol{y}]}_{\boldsymbol{v}}, |?A|^{\boldsymbol{a}}_{\boldsymbol{y}}}\,(\text{con})$$

where

$$\boldsymbol{a}\boldsymbol{y} := \begin{cases} \boldsymbol{a}_0\boldsymbol{y} & \text{if } ?|A|^{\boldsymbol{a}_0\boldsymbol{y}}_{\boldsymbol{y}} \\ \boldsymbol{a}_1\boldsymbol{y} & \text{otherwise.} \end{cases}$$

The soundness of the weakening rule, and the rules (!) and (?) is trivial. This interpretation corresponds to Gödel's Dialectica interpretation [11] intuitionistic logic, used in connection to a partial realisation of Hilbert's consistency program (the consistency of classical first-order arithmetic relative to the consistence of the quantifier-free calculus T).

Besides being sound for the principles $\mathsf{AC}_s, \mathsf{AC}_p$ (Section 3) and the principle TA (Section 4.1) the Dialectica interpretation of LL^ω will also be sound for the following principle

$$\mathsf{MP}_D \quad : \quad \forall \boldsymbol{x}!A \multimap !\forall \boldsymbol{x}A$$

for quantifier-free formulas A. This is the linear logic counterpart of the (semi) intuitionistic Markov principle. In fact, these are all the extra principles needed to show the equivalence between A and its Dialectica interpretation $\exists\forall^{\boldsymbol{x}}_{\boldsymbol{y}}|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$. For instance, in the case of the exponentials we have

$$!\exists\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A \;\underset{\mathsf{LL}^\omega_{\mathsf{q}}}{\overset{\mathsf{TA}}{\rightleftarrows}}\; \exists \boldsymbol{x}!\forall \boldsymbol{y} A \;\underset{\mathsf{MP}_D}{\overset{\mathsf{LL}^\omega}{\rightleftarrows}}\; \exists \boldsymbol{x}\forall \boldsymbol{y}!A \;\underset{\mathsf{AC}_s}{\overset{\mathsf{LL}^\omega_{\mathsf{q}}}{\rightleftarrows}}\; \exists\forall^{\boldsymbol{x}}_{\boldsymbol{f}}!A[\boldsymbol{f}\boldsymbol{x}/\boldsymbol{y}]$$

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References

1. Avigad, J., Feferman, S.: Gödel's functional ("Dialectica") interpretation. In: Avigad, J., Feferman, S. (eds.) Handbook of proof theory. Studies in Logic and the Foundations of Mathematics, vol. 137, pp. 337–405. North Holland, Amsterdam (1998)
2. Bezem, M.: Strongly majorizable functionals of finite type: a model for bar recursion containing discontinuous functionals. The Journal of Symbolic Logic 50, 652–660 (1985)
3. Blass, A.: Questions and answers – a category arising in linear logic, complexity theory, and set theory. In: Girard, J.-Y., Lafont, Y., Regnier, L. (eds.) Advances in Linear Logic. London Math. Soc. Lecture Notes, vol. 222, pp. 61–81 (1995)
4. Blass, A., Gurevich, Y.: Henkin quantifiers and complete problems. Annals of Pure and Applied Logic 32, 1–16 (1986)
5. Bradfield, J.: Independence: Logic and concurrency. In: Proceedings of Computer Science Logic (2000)
6. Diller, J., Nahm, W.: Eine Variant zur Dialectica-interpretation der Heyting Arithmetik endlicher Typen. Arch. Math. Logik Grundlagenforsch 16, 49–66 (1974)
7. Ferreira, F., Oliva, P.: Bounded functional interpretation. Annals of Pure and Applied Logic 135, 73–112 (2005)
8. Ferreira, F., Oliva, P.: Bounded functional interpretation in feasible analysis. Annals of Pure and Applied Logic 145, 115–129 (2007)
9. Girard, J.-Y.: Linear logic. Theoretical Computer Science 50(1), 1–102 (1987)
10. Girard, J.-Y.: Towards a geomery of interaction. Contemporary Mathematics, 92 (1989)
11. Gödel, K.: Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. Dialectica 12, 280–287 (1958)
12. Henkin, L.: Some remarks on infinitely long formulas. In: Infinitistic methods, pp. 167–183. Pergamon Press, New York, and Polish Scientific Publishers, Warsaw (1961)
13. Hyland, J.M.E.: Proof theory in the abstract. Annals of Pure and Applied Logic 114, 43–78 (2002)
14. Kreisel, G.: Interpretation of analysis by means of constructive functionals of finite types. In: Heyting, A. (ed.) Constructivity in Mathematics, pp. 101–128. North Holland, Amsterdam (1959)
15. Oliva, P.: Unifying functional interpretations. Notre Dame Journal of Formal Logic 47(2), 263–290 (2006)
16. Oliva, P.: Modified realizability interpretation of classical linear logic. In: Proc. of the Eighteenth Annual IEEE Symposium on Logic in Computer Science LICS'07, IEEE Computer Society Press, Los Alamitos (2007)
17. de Paiva, V.C.V.: A Dialectica-like model of linear logic. In: Dybjer, P., Pitts, A.M., Pitt, D.H., Poigné, A., Rydeheard, D.E. (eds.) Category Theory and Computer Science. LNCS, vol. 389, pp. 341–356. Springer, Heidelberg (1989)
18. Shirahata, M.: The Dialectica interpretation of first-order classical linear logic. Theory and Applications of Categories 17(4), 49–79 (2006)
19. Stein, M.: Interpretationen der Heyting-Arithmetik endlicher Typen. Arch. Math. Logik Grundlag 19, 175–189 (1979)
20. Troelstra, A.S.: Metamathematical Investigation of Intuitionistic Arithmetic and Analysis. Lecture Notes in Mathematics, vol. 344. Springer, Heidelberg (1973)
21. Troelstra, A.S.: Realizability. Handbook of proof theory 137, 408–473 (1998)