

Xuzhu Wang, Da Ruan and Etienne E. Kerre

Mathematics of Fuzziness – Basic Issues

Studies in Fuzziness and Soft Computing, Volume 245

Editor-in-Chief

Prof. Janusz Kacprzyk
Systems Research Institute
Polish Academy of Sciences
ul. Newelska 6
01-447 Warsaw
Poland
E-mail: kacprzyk@ibspan.waw.pl

Further volumes of this series can be found on our homepage: springer.com

Vol. 228. Adam Kasperski
Discrete Optimization with Interval Data, 2008
ISBN 978-3-540-78483-8

Vol. 230. Bhanu Prasad (Ed.)
Soft Computing Applications in Business, 2008
ISBN 978-3-540-79004-4

Vol. 231. Michal Baczynski,
Balasubramaniam Jayaram
Fuzzy Implications, 2008
ISBN 978-3-540-69080-1

Vol. 232. Eduardo Massad,
Neli Regina Siqueira Ortega,
Laécio Carvalho de Barros,
Claudio José Struchiner
*Fuzzy Logic in Action: Applications
in Epidemiology and Beyond*, 2008
ISBN 978-3-540-69092-4

Vol. 233. Cengiz Kahraman (Ed.)
*Fuzzy Engineering Economics with
Applications*, 2008
ISBN 978-3-540-70809-4

Vol. 234. Eyal Kolman, Michael Margaliot
*Knowledge-Based Neurocomputing:
A Fuzzy Logic Approach*, 2009
ISBN 978-3-540-88076-9

Vol. 235. Kofi Kissi Dompere
Fuzzy Rationality, 2009
ISBN 978-3-540-88082-0

Vol. 236. Kofi Kissi Dompere
Epistemic Foundations of Fuzziness, 2009
ISBN 978-3-540-88084-4

Vol. 237. Kofi Kissi Dompere
Fuzziness and Approximate Reasoning, 2009
ISBN 978-3-540-88086-8

Vol. 238. Atanu Sengupta, Tapan Kumar Pal
*Fuzzy Preference Ordering of Interval
Numbers in Decision Problems*, 2009
ISBN 978-3-540-89914-3

Vol. 239. Baoding Liu
*Theory and Practice of Uncertain
Programming*, 2009
ISBN 978-3-540-89483-4

Vol. 240. Asli Celikyilmaz, I. Burhan Türksen
Modeling Uncertainty with Fuzzy Logic, 2009
ISBN 978-3-540-89923-5

Vol. 241. Jacek Kluska
*Analytical Methods in Fuzzy
Modeling and Control*, 2009
ISBN 978-3-540-89926-6

Vol. 242. Yaochu Jin, Lipo Wang
*Fuzzy Systems in Bioinformatics
and Computational Biology*, 2009
ISBN 978-3-540-89967-9

Vol. 243. Rudolf Seising (Ed.)
*Views on Fuzzy Sets and Systems from Different
Perspectives*, 2009
ISBN 978-3-540-93801-9

Vol. 244. Xiaodong Liu and Witold Pedrycz
*Axiomatic Fuzzy Set Theory and Its
Applications*, 2009
ISBN 978-3-642-00401-8

Vol. 245. Xuzhu Wang, Da Ruan,
Etienne E. Kerre
Mathematics of Fuzziness – Basic Issues, 2009
ISBN 978-3-540-78310-7

Xuzhu Wang, Da Ruan and Etienne E. Kerre

Mathematics of Fuzziness – Basic Issues

Authors

Prof. Dr. Xuzhu Wang
Department of Mathematics
Taiyuan University of Technology
Taiyuan, Shanxi
China
E-mail: wangxuzhu2006@126.com

Prof. Dr. Da Ruan
Belgian Nuclear Research Centre
(SCK • CEN)
Boeretang 200
2400 Mol
Belgium
E-mail: druan@sckcen.be

Prof. Dr. Etienne E. Kerre
Department of Applied Mathematics &
Computer Science
Ghent University
Krijgslaan 281 (S9)
9000 Gent
Belgium
E-mail: etienne.kerre@ugent.be

&
Department of Applied Mathematics &
Computer Science
Ghent University
Krijgslaan 281 (S9)
9000 Gent
Belgium
E-mail: da.ruan@ugent.be

ISBN 978-3-540-78310-7

e-ISBN 978-3-540-78311-4

DOI 10.1007/978-3-540-78311-4

Studies in Fuzziness and Soft Computing ISSN 1434-9922

Library of Congress Control Number: 2009922105

© 2009 Springer-Verlag Berlin Heidelberg

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable to prosecution under the German Copyright Law.

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Typeset & Cover Design: Scientific Publishing Services Pvt. Ltd., Chennai, India.

Printed in acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Foreword

The word *fuzziness*, as introduced by our friend and mentor Professor Lotfi A. Zadeh in 1965, has evolved to characterize the information associated with the human language. The human language which carries a lot of information and is being used in our everyday decision making processes, in a mathematical sense, is not very precise. For example, in Saskatoon in the month of January day time temperature is usually around -15°C to -20°C . A resident of Saskatoon will classify this temperature as '*not very cold*'. By a friend just arriving from India, his classification for this temperature would be '*very- very cold*'.

Professor Zadeh has given us a mathematical tool called the *fuzzy mathematics*, which gives precision to uncertainty inherent in our human language. The meaning of the wordings that we use in our everyday life are context dependent, subjective, and have a cognitive flavor. Thus, we can say that *the fuzzy set theory is a mathematical tool that provides a figure of certainty to a cognitive type of uncertain phenomenon inherent in our human language*.

Since the inception of notion of fuzzy sets in 1965, many research papers and books have appeared in the field of fuzzy uncertainty. In the present book entitled '*Mathematics of Fuzziness, Basic Issues*' the authors, Drs. Xuzhu Wang, Da Ruan and Etienne Kerre who are among the *pioneers* in the field of fuzzy mathematics, have provided a lucid mathematical characterization of 'fuzzy uncertainty'. In this book the authors introduce a basic notion of '*fuzziness*' and provide a conceptual mathematical framework to characterize such fuzzy phenomena.

In general, mathematical methods have evolved in order to characterize the phenomena we are surrounded with. This book which contains six chapters in 281 pages lays the basic mathematical basis of fuzzy phenomena, the phenomena which are inherent in our human perception and cognitive processes. The authors start with the preliminaries on sets and relations, in Chapter 1. Then in Chapter 2 they provide some conceptual basics of fuzzy sets. In Chapter 3 they provide a detailed description of the methodology of fuzzy relations with applications in several fields such as fuzzy clustering, information retrieval and multiple attribute decision making analysis. Following this basic introduction, the authors move to Chapter 4 to describe the extension principle and fuzzy numbers, in Chapter 5 some mathematical topics such as fuzzy measures and fuzzy integrals, and finally in Chapter 6 some applications oriented topics such as fuzzy inference and fuzzy control. Each chapter is very well appended with exercises and reference material.

In the years ahead, the notion of *fuzziness* is likely to be an integral part for dealing with cognitive uncertainty in a computationally effective way. Thus, this

book on mathematics of fuzziness which has evolved from the authors' lecture notes to both undergraduate and graduate students provides a basic mathematical exposition to this growing field of soft-computing. The authors and the publisher, Springer Verlag, have produced a treatise that addresses, with high authority and high level of expertise, the mathematics devoted in the studies of fuzziness and soft-computing. For this very informative and timely book the authors deserve our heartiest congratulations!

University of Saskatchewan
January 28, 2009

Madan M. Gupta

Preface

This book is intended to serve as a basic textbook for a fuzzy set and system course at the undergraduate level and as a preliminary material for further research and applications on some special topics. Accordingly, the readers may contain undergraduate as well as graduate students in mathematics, computer science and engineering, engineers and researchers in the related fields. The seven aspects feature our book.

- (1) The most fundamental subjects in fuzzy set theory such as fuzzy sets, fuzzy relations and fuzzy numbers are introduced in detail.
- (2) The most important applications of fuzzy sets such as pattern recognition, clustering analysis and fuzzy control are explicitly highlighted.
- (3) The well-developed pure mathematical branches such as fuzzy measures, fuzzy integrals, fuzzy algebra, and fuzzy topology are briefly discussed.
- (4) While most parts of the book are devoted to the introduction to the basics of fuzzy set theory and its applications, some parts contain advanced materials based our research.
- (5) Remarks are made to indicate further reading in the end of some sections particularly with important and theoretical sections.
- (6) The whole book is basically self-contained. There is no difficulty for the reader with basic knowledge of mathematical analysis and linear algebra to get through all chapters except Chapter 5.
- (7) Many exercises arranged as the last section of every chapter help the reader to understand the concepts, approaches and theories in the chapter and serve as a supplement to main results in the chapter.

The book is structured as follows. In Chapter 1, we recall basic concepts of set theory and abstract algebra including set, relation, isomorphism, lattice, Boolean algebra, and soft algebra. These concepts play an important role in the investigation of fuzzy sets and fuzzy relations. In Chapter 2, we introduce the basics of fuzzy set theory. Considering general fuzzy logic connective operations are visible everywhere in the current fuzzy literature, a lot of space is

assigned to deal with negations, t -norms, t -conorms, fuzzy implications and equivalencies besides the Zadeh's set-theoretic operations. Then we expose the links between fuzzy sets and crisp sets which include the decomposition theorems and mathematical representations of fuzzy sets in terms of a nest of sets. Afterwards, fuzzy sets are extended to L -fuzzy sets and a similar investigation is carried out. Finally, an important application of fuzzy set theory – fuzzy pattern recognition – is introduced. In Chapter 3, we deal with fuzzy relations. We begin with the investigation of various compositions of fuzzy relations and followed by the introduction of fuzzy equivalence relations and fuzzy tolerance relations. Then comes an exhaustive investigation into the main properties of fuzzy relations: negativity, semi-transitivity, Ferrers property, consistency, weak transitivity and acyclicity. After that, we discuss a type of composite fuzzy relation equations, and finally mention several applications of fuzzy relations. In Chapter 4, three tasks are fulfilled. The first one is the introduction of the Zadeh's extension principle. The second one is the discussion of fuzzy numbers and their algebraic operations. The last one is the detailed investigation of ranking methods for fuzzy numbers, which can find wide applications when fuzzy data are involved and processed. In Chapter 5, we introduce three well-developed fuzzified mathematical branches: (i) fuzzy measures and fuzzy integrals; (ii) fuzzy algebraic structures including fuzzy groups, fuzzy rings and fuzzy ideals; (iii) fuzzy topology. The chapter is especially ready for the reader with some crisp mathematical prerequisites and supplies them with primary fuzzification approaches of a mathematical theory, which will open a door to further theoretical research. The major subject of Chapter 6 is fuzzy control. We outline the principle of fuzzy control by a typical application case. Around fuzzy control, we briefly introduce some concepts related to fuzzy inference such as linguistic variables, hedges, fuzzy propositions, IF-THEN rules and special fuzzy inference approaches. Considering that fuzzification and defuzzification often play an important role in the design of fuzzy controllers, we also investigate them in this chapter.

This book has evolved from our lectures for both undergraduate and graduate students. We are thankful to our students and colleagues for their constructive suggestions. We are deeply indebted to the authors whose works are employed and not explicitly cited, but certainly listed in our bibliography. We gratefully acknowledge the research support by the China-Flanders Foundation and the Natural Science Foundation of Shanxi Province.

Xuzhu Wang
Da Ruan
Etienne E. Kerre

Contents

1	Preliminaries	1
1.1	Sets	1
1.2	Relations	3
1.3	Mappings and Algebraic Systems	7
1.4	Lattices	10
1.5	Special Lattices	13
1.6	Exercises	17
2	Basics of Fuzzy Sets	21
2.1	Fuzzy Sets and Their Set-Theoretic Operations	21
2.2	General Fuzzy Logic Connectives	26
2.2.1	Fuzzy Negations	26
2.2.2	Triangular Norms and Conorms	28
2.2.3	Fuzzy Implications	32
2.2.4	Fuzzy Equivalencies	38
2.3	Decomposition of a Fuzzy Set	39
2.4	Mathematical Representation of Fuzzy Sets	43
2.5	L -Fuzzy Sets	46
2.5.1	Pseudo-complements	46
2.5.2	L -Fuzzy Sets and Their Set-Theoretic Operations	47
2.5.3	Decomposition of an L -Fuzzy Set	49
2.5.4	Mathematical Representation of L -Fuzzy Sets	51
2.6	Fuzzy Pattern Recognition	54
2.6.1	Type I Fuzzy Pattern Recognition	55
2.6.2	Type II Fuzzy Pattern Recognition	56
2.7	Exercises	60

3	Fuzzy Relations	65
3.1	Basic Concepts of Fuzzy Relations	65
3.2	Compositions of Fuzzy Relations	69
3.2.1	Round Composition of Fuzzy Relations	69
3.2.2	Subcomposition, Supercomposition and Square Composition of Fuzzy Relations	72
3.3	Fuzzy Equivalence Relations	74
3.4	Closures	77
3.4.1	The Concept of a Closure	78
3.4.2	The Transitive Closure of a Fuzzy Relation	79
3.5	Fuzzy Tolerance Relations	80
3.6	Other Special Fuzzy Relations	82
3.6.1	Crisp Negative Transitivity, Semitransitivity, and Ferrers Property	83
3.6.2	Negative S -Transitivity, T - S -Semitransitivity, and T - S -Ferrers Property	85
3.6.3	Consistency, Weak Transitivity and Acyclicity	91
3.7	Fuzzy Relation Equations	94
3.8	Some Applications of Fuzzy Relations	99
3.8.1	Fuzzy Clustering Analysis	99
3.8.2	An Application to Information Retrieval	102
3.8.3	An Application to Multiple Attribute Decision Making Analysis	104
3.9	Exercises	106
4	Extension Principle and Fuzzy Numbers	113
4.1	Unary Extension Principle	114
4.2	n -Ary Extension Principle	119
4.3	Convex Fuzzy Quantities	122
4.4	Fuzzy Numbers	126
4.4.1	The Concept of a Fuzzy Number	126
4.4.2	Properties of Algebraic Operations on Fuzzy Numbers	130
4.5	Ranking of Fuzzy Numbers	132
4.5.1	Ranking Fuzzy Numbers by a Ranking Function	133
4.5.2	Ranking Fuzzy Numbers According to the Closeness to a Reference Set	134
4.5.3	Ranking Fuzzy Numbers Based on Pairwise Comparisons	136
4.5.4	Ranking Axioms	140
4.6	An Application of Fuzzy Numbers	142
4.7	Exercises	146

5	A Brief Introduction to Some Pure Mathematical Topics	153
5.1	Fuzzy Measures and Fuzzy Integrals	153
5.1.1	Fuzzy Measures	153
5.1.2	Fuzzy Integrals	157
5.2	Fuzzy Algebra	163
5.2.1	Fuzzy Subgroups	163
5.2.2	Normal Fuzzy Subgroups	167
5.2.3	Fuzzy Subrings	171
5.2.4	Fuzzy Ideals	173
5.3	Fuzzy Topology	175
5.3.1	Definitions	175
5.3.2	Characterization of a Fuzzy Topology in Terms of Preassigned Operations	177
5.3.3	Characterization of a Fuzzy Topology in Terms of Closed Sets	179
5.3.4	Characterization of a Fuzzy Topology Using the Interior Operator	180
5.3.5	Characterization of a Fuzzy Topology by Means of a Closure Operator	182
5.3.6	Characterization of a Fuzzy Topology by Means of Neighborhood Systems	182
5.3.7	Normality in Fuzzy Topological Spaces	184
5.3.8	Some Examples of Fuzzy Topological Spaces	185
5.4	Exercises	186
6	Fuzzy Inference and Fuzzy Control	189
6.1	Linguistic Variables and Hedges	189
6.2	Fuzzy Propositions and IF-THEN Rules	191
6.3	Fuzzy Inference Rules	193
6.4	The Calculation of Inference Results	194
6.5	Fuzzification and Defuzzification	198
6.6	The Principle of Fuzzy Control	201
6.7	Exercises	204
	References	207
	Index	217