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Cardinality in Allegories

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Abstract. In this paper we want to investigate two notions of the cardinality of relations in the context of allegories. The different axiom systems are motivated on the existence of injective and surjective functions, respectively. In both cases we provide a canonical cardinality function and show that it is initial in the category of all cardinality functions over the given allegory.

1 Introduction

The calculus relations, and its categorical versions in particular, are often used to model programming languages, classical and non-classical logics and different methods of data mining (see for example [1–3, 7, 8]). In certain applications the cardinality of those relations is of interest. For example, finite trees can be characterized as those connected graphs satisfying the numerical equation $e = n - 1$ relating the number of edges e and vertices n. Since graphs can be considered as binary relation an abstract formulation of the property above in the theory of allegories needs a notion of cardinality.

In this paper we want to investigate two notions of the cardinality of relations in the context of allegories. The first notion is motivated by the standard cardinal (pre)ordering of sets, i.e. a set A is smaller than a set B if there is an injective function from A to B. The second notion will be based on surjective function, i.e. we consider a set A smaller than a set B if there is a surjective function from B to A. Ignoring the empty set, the two notions are equivalent in regular set theory with the axiom of choice. Since the theory of allegories is much weaker we cannot expect such a result in general.

In both cases we provide a canonical cardinality function and show that it is initial in the category of all cardinality functions over the given allegory. Last but not least, we give an additional axiom characterizing the canonical cardinality function (up to isomorphism).

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2 Categories of Relations

Given a category $\mathcal{C}$ we denote its collection of objects by $\mathrm{Obj}_\mathcal{C}$ and its collection of morphisms by $\mathrm{Mor}_\mathcal{C}$. To indicate that a morphism f has source A and target B we usually write $f : A \to B$. The collection of all morphisms between A and B is denoted by $\mathcal{C}[A, B]$. We use ; for composition of morphisms, which has to be read from left to right, i.e. $f; g$ means first f then g. The identity morphism on the object A is written as $\mathbb{I}_A$.

Definition 1. *An allegory $\mathcal{R}$ is a category satisfying the following:*

1. *For all objects A and B the class $\mathcal{R}[A, B]$ is a lower semi-lattice. Meet and the induced ordering are denoted by $\sqcap, \sqsubseteq$,respectively. The elements in $\mathcal{R}[A, B]$ are called relations.*
2. *There is a monotone operation $^\smile$ (called the converse operation) such that for all relations $Q, R : A \to B$ and $S : B \to C$ the following holds*
$$(Q; S)^\smile = S^\smile; Q^\smile \quad \text{and} \quad (Q^\smile)^\smile = Q.$$
3. *For all relations $Q : A \to B$, $R, S : B \to C$ we have $Q; (R \sqcap S) \sqsubseteq Q; R \sqcap Q; S$.*
4. *For all relations $Q : A \to B, R : B \to C$ and $S : A \to C$ the following modular law holds $Q; R \sqcap S \sqsubseteq Q; (R \sqcap Q^\smile; S)$.*

A relation $R : A \to B$ is called univalent (or a partial function) iff $R^\smile; R \sqsubseteq \mathbb{I}_B$ and total iff $\mathbb{I}_A \sqsubseteq R; R^\smile$. Functions are total and univalent relations and are usually denoted by lower letters. Furthermore, R is called injective iff $R^\smile$ is univalent and surjective iff $R^\smile$ is total. In the following lemma we have summarized several basic properties of relations used in this paper. A proof can be found in [4, 7, 8].

Lemma 1. *Let $\mathcal{R}$ be an allegory. Then we have:*

1. *$Q; R \sqcap S \sqsubseteq (Q \sqcap S; R^\smile); (R \sqcap Q^\smile; S)$ for all relations $Q : A \to B, R : B \to C$ and $S : A \to C$ (Dedekind formula);*
2. *If $Q : A \to B$ is univalent, then $Q; (R \sqcap S) = Q; R \sqcap Q; S$ for all relations $R, S : B \to C$;*
3. *If $R : B \to C$ is univalent, then $Q; R \sqcap S = (Q \sqcap S; R^\smile); R$ for all relations $Q : A \to B$ and $S : A \to C$.*

Another important property of commuting squares of function is as follows:

Lemma 2. *Let $\mathcal{R}$ be an allegory, and $f : A \to B, g : A \to C, h : B \to D$ and $k : C \to D$ be function with $f^\smile; g = h; k^\smile$. Then we have $f; h = g; k$.*

Proof. Consider the following computation

$$\begin{aligned}
f; h &\sqsubseteq g; g^\smile; f; h && g \text{ total} \\
&= g; k; h^\smile; h && \text{assumption} \\
&\sqsubseteq g; k && h \text{ univalent} \\
&\sqsubseteq f; f^\smile; g; k && f \text{ total} \\
&= f; h; k^\smile; k && \text{assumption} \\
&\sqsubseteq f; h. && k \text{ univalent}
\end{aligned}$$

This completes the proof. $\square$

Two functions $f : C \to A$ and $g : C \to B$ with common source are said to tabulate a relation $R : A \to B$ iff $R = f^{\smile}; g$ and $f; f^{\smile} \sqcap g; g^{\smile} = \mathbb{I}_C$. If for all relations of an allegory $\mathcal{R}$ there is tabulation, then $\mathcal{R}$ is called tabular. Notice that a function $f : A \to B$ and its converse $f^{\smile} : B \to A$ always have a tabulation. The tabulation is given by $(\mathbb{I}_A, f)$ and $(f, \mathbb{I}_B)$, respectively.

Lemma 3. *Let $\mathcal{R}$ be an allegory, and $R : A \to B$ a relation that is tabulated by $f : C \to A$ and $g : C \to B$. Furthermore, let $h : D \to A$ and $k : D \to B$ be functions with $h^{\smile}; k \sqsubseteq R$, and define $l := h; f^{\smile} \sqcap k; g^{\smile} : D \to C$. Then we have the following:*

1. *l is the unique function with $h = l; f$ and $k = l; g$.*
2. *If $h^{\smile}; k = R$, then l is surjective.*
3. *If $h : D \to A$ and $k : D \to B$ is a tabulation, i.e. $h; h^{\smile} \sqcap k; k^{\smile} = \mathbb{I}_D$, then l is injective.*
4. *If R is a partial identity, i.e. $A = B$ and $R \sqsubseteq \mathbb{I}_A$, then f (or g) is a tabulation of R, i.e. $R = f^{\smile}; f$ and $f; f^{\smile} = \mathbb{I}_C$.*

Proof. 1. This was already shown in 2.143 of [4].

2. Assume $h^{\smile}; k = R$. Then we have

$$\begin{aligned}
\mathbb{I}_C &= \mathbb{I}_C \sqcap f; f^{\smile}; g; g^{\smile} && f, g \text{ total} \\
&= \mathbb{I}_C \sqcap f; h^{\smile}; k; g^{\smile} && \text{assumption} \\
&\sqsubseteq (f; h^{\smile} \sqcap g; k^{\smile}); (h; f^{\smile} \sqcap k; g^{\smile}) && \text{Lemma 1(1)} \\
&= l^{\smile}; l.
\end{aligned}$$

3. Assume $h; h^{\smile} \sqcap k; k^{\smile} = \mathbb{I}_D$. Then we have

$$\begin{aligned}
l; l^{\smile} &= (h; f^{\smile} \sqcap k; g^{\smile}); (f; h^{\smile} \sqcap g; k^{\smile}) \\
&\sqsubseteq h; f^{\smile}; f; h^{\smile} \sqcap k; g^{\smile}; g; k^{\smile} \\
&\sqsubseteq h; h^{\smile} \sqcap k; k^{\smile} && f, g \text{ univalent} \\
&= \mathbb{I}_D. && \text{assumption}
\end{aligned}$$

4. This was already shown in 2.145 of [4]. $\square$

The previous lemma also implies that tabulations are unique up to isomorphism.

The next lemma is concerned with a tabulation of the meet of two relations.

Lemma 4. *Let $\mathcal{R}$ be an allegory, and $Q_i : A \to B$ be relations tabulated by $f_i : C_i \to A$ and $g_i : C \to B$ for $i = 1, 2$. If $f : D \to A$ and $g : D \to B$ is a tabulation of $Q_1 \sqcap Q_2$, then there are unique injections $h_i : D \to C_i$ $(i = 1, 2)$ satisfying the following:*

1. *$h_i; f_i = f$ and $h_i; g_i = g$;*

2. *If there are functions $k_i : E \to C$ with $k_1; f_1 = k_2; f_2$ and $k_1; g_1 = k_2; g_2$, then there is a unique function $m : E \to D$ with $k_i = m; h_i$ $(i = 1, 2)$.*

$$\begin{tikzcd}
A \arrow[rr, "Q_i"] & & B \\
& C_i \arrow[ul, "f_i"'] \arrow[ur, "g_i"] & \\
& D \arrow[u, "h_i"] \arrow[uul, bend left, "f"] \arrow[uur, bend right, "g"'] & \\
& E \arrow[u, "m"] \arrow[uu, bend right, "k_i"'] &
\end{tikzcd}$$

Proof. From Lemma 3 (1) and (3) we get $h_i = f; f_i^{\smile} \sqcap g_i; g_i^{\smile}$. It just remains to verify the second property. Assume $k_i : E \to C$ are as required, and let $p := k_1; f_1 = k_2; f_2$ and $q := k_1; g_1 = k_2; g_2$. Then we have

$$\begin{aligned}
p^{\smile}; q &= p^{\smile}; q \sqcap p^{\smile}; q \\
&= (k_1; f_1)^{\smile}; k_1; g_1 \sqcap (k_2; f_2)^{\smile}; k_2; g_2 && \text{by definition} \\
&= f_1^{\smile}; k_1^{\smile}; k_1; g_1 \sqcap f_2^{\smile}; k_2^{\smile}; k_2; g_2 \\
&\sqsubseteq f_1^{\smile}; g_1 \sqcap f_1^{\smile}; g_1 && k_i \text{ univalent} \\
&= Q_1 \sqcap Q_2.
\end{aligned}$$

Since f, g is a tabulation of $Q_1 \sqcap Q_2$ there is a unique function $m : E \to D$ with $m; f = p$ and $m; g = q$. We conclude $m; h_i; f_i = m; f = p = k_i; f_i$ and $m; h_i; g_i = m; g = q = k_i; g_i$ for $= 1, 2$. This implies

$$\begin{aligned}
m; h_i &= m; h_i; (f_i; f_i^{\smile} \sqcap g_i; g_i^{\smile}) && f_i, g_i \text{ is a tabulation} \\
&= m; h_i; f_i; f_i^{\smile} \sqcap m; h_i; g_i; g_i^{\smile} && \text{Lemma 1(2)} \\
&= k_i; f_i; f_i^{\smile} \sqcap k_i; g_i; g_i^{\smile} && \text{see above} \\
&= k_i; (f_i; f_i^{\smile} \sqcap g_i; g_i^{\smile}) && \text{Lemma 1(2)} \\
&= k_i. && f_i, g_i \text{ is a tabulation}
\end{aligned}$$

Suppose $n : E \to D$ is another function with $n; h_i = k_i$. Then $n; f = n; h_i; f_i = k_i; f_i = p$ and $n; g = n; h_i; g_i = k_i; g_i = q$ so that we conclude $n = m$. □

The last lemma of this section is a technical lemma that will be used in Section 5.

Lemma 5. *Let $\mathcal{R}$ be an allegory, and $Q : A \to B$ and $R : A \to C$ be relations tabulated by $f : D \to A, g : D \to B$ and $h : E \to A, k : E \to C$, respectively. Furthermore, let $h_0 : F \to D, f_0 : F \to E$ be a tabulation of $f; h^{\smile}$. Then $Q; Q^{\smile} \sqcap R; R^{\smile} \sqsubseteq \mathbb{I}_A$ iff $h_0; g; g^{\smile}; h_0^{\smile} \sqcap f_0; k; k^{\smile}; f_0^{\smile} = \mathbb{I}_F$.*

$$\begin{tikzcd}
B & & \\
D \arrow[u, "g"] \arrow[r, "f"'] & A \arrow[ul, "Q"'] \arrow[dr, "R"] & \\
F \arrow[u, "h_0"] \arrow[r, "f_0"'] & E \arrow[u, "h"] \arrow[r, "k"'] & C
\end{tikzcd}$$

Proof. '$\Rightarrow$': Assume $Q;Q^\smile \sqcap R;R^\smile \sqsubseteq \mathbb{I}_A$. Then we have

$$\begin{aligned}
& h_0;g;g^\smile;h_0^\smile \sqcap f_0;k;k^\smile;f_0^\smile \\
& \sqsubseteq h_0;f;f^\smile;g;g^\smile;f;f^\smile;h_0^\smile \sqcap f_0;h;h^\smile;k;k^\smile;h;h^\smile;f_0^\smile && f,h \text{ total} \\
& = f_0;h;f^\smile;g;g^\smile;f;h^\smile;f_0^\smile \sqcap f_0;h;h^\smile;k;k^\smile;h;h^\smile;f_0^\smile && \text{Lemma 2} \\
& = f_0;h;(f^\smile;g;g^\smile;f \sqcap h^\smile;k;k^\smile;h);h^\smile;f_0^\smile && \text{Lemma 1(2)} \\
& = f_0;h;(Q;Q^\smile \sqcap R;R^\smile);h^\smile;f_0^\smile && \text{tabulations} \\
& \sqsubseteq f_0;h;h^\smile;f_0^\smile. && \text{assumption}
\end{aligned}$$

We conclude

$$\begin{aligned}
& h_0;g;g^\smile;h_0^\smile \sqcap f_0;k;k^\smile;f_0^\smile \\
& = h_0;g;g^\smile;h_0^\smile \sqcap f_0;h;h^\smile;f_0^\smile \sqcap f_0;k;k^\smile;f_0^\smile \sqcap f_0;h;h^\smile;f_0^\smile && \text{see above} \\
& = h_0;g;g^\smile;h_0^\smile \sqcap h_0;f;f^\smile;h_0^\smile \sqcap f_0;k;k^\smile;f_0^\smile \sqcap f_0;h;h^\smile;f_0^\smile && \text{Lemma 2} \\
& = h_0;(g;g^\smile \sqcap f;f^\smile);h_0^\smile \sqcap f_0;(k;k^\smile \sqcap h;h^\smile);f_0^\smile && \text{Lemma 1(2)} \\
& = h_0;h_0^\smile \sqcap f_0;f_0^\smile && \text{tabulations} \\
& = \mathbb{I}_F. && \text{tabulation}
\end{aligned}$$

'$\Leftarrow$': Now, assume $h_0;g;g^\smile;h_0^\smile \sqcap f_0;k;k^\smile;f_0^\smile = \mathbb{I}_F$. Then we have

$$\begin{aligned}
& Q;Q^\smile \sqcap R;R^\smile \\
& = f^\smile;g;g^\smile;f \sqcap h^\smile;k;k^\smile;h && \text{tabulation} \\
& = f^\smile;(g;g^\smile;f;h^\smile \sqcap f;h^\smile;k;k^\smile);h && \text{Lemma 1(3)} \\
& = f^\smile;(g;g^\smile;h_0^\smile;f_0 \sqcap h_0^\smile;f_0;k;k^\smile);h && \text{tabulation} \\
& = f^\smile;h_0^\smile;(h_0;g;g^\smile;h_0^\smile \sqcap f_0;k;k^\smile;f_0^\smile);f_0;h && \text{Lemma 1(3)}
\end{aligned}$$

$$\begin{aligned}
& = f^\smile;h_0^\smile;f_0;h && \text{assumption} \\
& = f^\smile;f;h^\smile;h && \text{tabulation} \\
& \sqsubseteq \mathbb{I}_A. && f,h \text{ univalent}
\end{aligned}$$

This completes the proof. $\square$

Notice that in the situation of the previous lemma we always have

$$g^{\smile}; h_0^{\smile}; f_0; k = g^{\smile}; f; h^{\smile}; k = Q^{\smile}; R$$

so that the assertion could be formulated alternatively as follows:

$$Q; Q^{\smile} \sqcap R; R^{\smile} \sqsubseteq \mathbb{I}_A \text{ iff } h_0; g \text{ and } f_0; k \text{ is a tabulation of } Q^{\smile}; R.$$

3 Cardinal Preorderings on Objects

In this section we want to study two notions of preordering on the class of objects of an allegory.

Definition 2. *Let $\mathcal{R}$ be an allegory. Then the relations $\lesssim_i$ and $\lesssim_s$ on the class of objects of $\mathcal{R}$ are defined by*

1. *$A \lesssim_i B$ iff there is an injective function $f : A \to B$;*
2. *$A \lesssim_s B$ iff there is a surjective function $f : B \to A$.*

By $\sim_i$ and $\sim_s$ we denote the equivalence relations on the class of objects induced by $\lesssim_i$ and $\lesssim_s$, respectively.

In set theory (with the axiom of choice) both notions are equivalent for non-empty sets. Since the theory of allegories is much weaker we cannot expect the same for arbitrary allegories. We want to give several examples showing that $\lesssim_i$ and $\lesssim_s$ are different in general - even in the case of tabular allegories.

Example 1. Consider the structure consisting of two sets $A := \{1\}$ and $B := \{1, 2\}$ as objects and the following morphisms:

- The identity relations on A and B.
- The inclusion function $f := \{(1, 1)\}$ from A to B and its converse.
- The partial identity $f^{\smile}; f = \{(1, 1)\}$ on B.

The structure can be visualized by the following graph:

$$\begin{tikzcd} A \arrow[loop left, "\mathbb{I}_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "f^{\smile}"] \arrow[loop right, "\mathbb{I}_B, f^{\smile}; f"] \end{tikzcd}$$

It is easy to verify that this structure is closed under composition, converse and intersection, and is, therefore, an allegory. Furthermore, this allegory is tabular. The only relation that is not a function or a converse of a function is $f^{\smile}; f$, which is tabulated by the pair (f, f).

f is an injective function so that we get $A \lesssim_i B$. On the other hand, there is no surjective function from B to A so that $A \lesssim_s B$ does not hold. The order structure induced by $\lesssim_s$ is discrete whereas the order structure induced by $\lesssim_i$ is a linear.

This example can be extended by adding the objects $\{1, 2, 3\}, \{1, 2, 3, 4\}, \ldots$ and the corresponding inclusion functions. $\lesssim_s$ remains to be discrete and $\lesssim_i$ is linear of length ω.

Example 2. Let $R_n^p \subseteq \omega \times \omega$ with $n \geq 0$ and p an arbitrary integer be defined by

$$(x, y) \in R_n^p :\iff x + p = y \text{ and } \min(x, y) \geq n.$$

It is easy to verify that the following properties are satisfied:

1. $R_0^0 = \mathbb{I}_\omega$,
2. $(R_n^p)^\smile = R_n^{-p}$
3. $R_m^p \sqcap R_n^q = \begin{cases} \emptyset & : p \neq q \\ R^p_{\max(m,n)} & : p = q, \end{cases}$
4. $R_m^p ; R_n^q = R^{p+q}_{\max(m,m-p,n,n+q)}$ for an $l \geq 0$.

The properties above show that the set of relations $\{R_n^p \mid n \geq 0, p \in \mathbb{Z}\}$ is closed under all operations of an allegory.

Consider the allegory given by two copies of the natural numbers ω_1, ω_2 and the morphism sets as indicated in the following diagram:

$$\begin{tikzcd} \omega_1 \arrow[loop left] \arrow[rr, bend left] & & \omega_2 \arrow[loop right] \arrow[ll, bend left] \end{tikzcd}$$

$\{R_n^p \mid p \text{ is even}\}$ $\{R_n^p \mid p \text{ is odd}\}$ $\{R_n^p \mid p \text{ is even}\}$

In this allegory there is an injection $R_0^1 : \omega_1 \to \omega_2$ (the successor function). By the symmetric definition of the allegory the same relation is also an injection from ω_2 to ω_1. The only bijection R_0^0 is not a relation between ω_1 and ω_2 since its exponent is even. Notice that R_0^0 is also the only surjective function in the given set of relations. Consequently, $\omega_1 \sim_i \omega_2$ but we have neither $\omega_1 \lesssim_s \omega_2$ nor $\omega_2 \lesssim_s \omega_1$.

This example is pre-tabular, i.e. every relation is in included in a tabular relation. This follows from the fact that every relation is included in an injection or in the converse of such a relation. The embedding of a pre-tabular in a tabular allegory by splitting partial identities is full. Consequently, the resulting allegory omits the same example as above but is tabular.

Example 3. Again, consider the structure consisting of the two sets $A := \{1\}$ and $B := \{1, 2\}$ as objects and the following morphisms:

- The identity relations on A and B.
- The function $g := \{(1, 1), (2, 1)\}$ from B to A and its converse.
- The universal relation $\mathbb{T}_{BB} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ on B.

The structure can be visualized by the following graph:

$$\begin{tikzcd} A \arrow[loop left, "\mathbb{I}_A"] \arrow[r, bend left, "g^\smile"] & B \arrow[loop right, "{\mathbb{I}_B, \mathbb{T}_{BB}}"] \arrow[l, bend left, "g"] \end{tikzcd}$$

It is easy to verify that this structure is closed under composition, converse and intersection, and is, therefore, an allegory. This allegory is not tabular since $\mathbb{T}_{BB}$ has no tabulation.

g is a surjection so that we get $A \lesssim_s B$, but there is no injective function from A to B so that $A \lesssim_i B$ doe not hold.

There is also an example of tabular allegory omitting two objects A and B with $A \lesssim_s B$ and $A \not\lesssim_i B$. This example uses a substructure of a model of ZF not satisfying the axiom of choice and its tabular closure within the given model of set theory. Details can be found in [6].

4 Cardinality Function (injective case)

We now give the definition of cardinality function motivated by the preordering $\lesssim_i$.

Definition 3. *Let $\mathcal{R}$ be an allegory, and $(\mathcal{C}, \leq)$ be an ordered class. A function $|.|_i : \mathrm{Mor}_{\mathcal{R}} \to \mathcal{C}$ mapping the morphisms of $\mathcal{R}$ to elements of $\mathcal{C}$ is called a (injective) cardinality function iff*

C0: $|R^{\smile}|_i = |R|_i$ *for all relations R;*
I1: $|.|_i$ *is monotonic, i.e. $R \sqsubseteq S$ implies $|R|_i \leq |S|_i$ for all relations $R, S : A \to B$;*
I2: *If $U : C \to A$ and $V : C \to B$ are univalent with $U;U^{\smile} \sqcap V;V^{\smile} \sqsubseteq \mathbb{I}_C$, then*

$$|U^{\smile};V|_i = |U;U^{\smile} \sqcap V;V^{\smile}|_i.$$

$|.|_i$ is called strong iff it is surjective as a function and $|\mathbb{I}_A|_i \leq |\mathbb{I}_B|_i$ implies that there is an injection $i : A \to B$.

The first axiom has its obvious motivation in concrete relations. All versions of cardinality functions in this paper use this axiom so that we call it C0. It turns out in the next section that the second axiom actually characterizes the usage of injective functions. An immediate consequence of the last axiom (see Lemma 6(2)) is that one may compute the cardinality of a relation using its tabulation (if it exists). This idea is the motivation of Axiom (3). We will show later that the strong property makes the cardinality function unique (up to isomorphism).

The first part of the next lemma shows that an (injective) cardinality function is based on the preoredering $\lesssim_i$.

Lemma 6. *Let $|.|_i$ be a cardinality function over the allegory $\mathcal{R}$. Then:*

1. *If $i : A \to B$ is an injection, then $|\mathbb{I}_A|_i \leq |\mathbb{I}_B|_i$.*
2. *If $R : A \to B$ has a tabulation $f : C \to A$ and $g : C \to B$, then $|R|_i = |\mathbb{I}_C|_i$.*

Proof. 1. i is univalent and we have $i;i^{\smile} = i;i^{\smile} \sqcap i;i^{\smile} = \mathbb{I}_A$ since i is total and injective so that Axiom I2 shows $|\mathbb{I}_A|_i = |i^{\smile};i|_i$. The latter is less than or equal to $|\mathbb{I}_B|_i$, which follows from $i^{\smile};i \sqsubseteq \mathbb{I}_B$ by Axiom I1.

2. This is an immediately consequence of Axiom I2 since f and g are functions with $f;f^\smile \sqcap g;g^\smile = \mathbb{I}_C$ and $R = f^\smile;g$. $\square$

In order to define the canonical cardinality function on allegories for the injective case we need tabulations. Consequently, we will assume for the rest of this section that the given allegory $\mathcal{R}$ is tabular.

Let us denote by $[A]_i$ the equivalence class of an object with respect to $\sim_i$ and by $(\mathrm{Obj}_\mathcal{R}/\sim_i, \leq_i)$ the ordered class of those equivalence classes.

Definition 4. *The canonical cardinality function $|.|_i^*$ is defined by $|R|_i^* := [C]_i$ where $R : A \to B$ has a tabulation $f : C \to A$ and $g : C \to B$.*

Notice that the canonical cardinality function is well-defined since tabulations are unique up to isomorphism.

Lemma 7. *The canonical cardinality function $|.|_i^*$ is a cardinality function.*

Proof. C0: Notice that (g, f) is a tabulation of $R^\smile$ iff (f, g) is a tabulation of R. We conclude $|R|_i^* = [C]_i = |R^\smile|_i^*$.

I1: Assume $R \sqsubseteq S$, R is tabulated by $f : C \to A, g : C \to B$ and S by $h : D \to A, k : D \to B$. Then by Lemma 3(3) there is an injection $i : C \to D$. This implies $|R|_i^* = [C]_i \leq_i [D]_i = |S|_i^*$.

I2: Assume that $U : C \to A$ and $V : C \to B$ are univalent relations with $U;U^\smile \sqcap V;V^\smile \sqsubseteq \mathbb{I}_C$. Since $U;U^\smile \sqcap V;V^\smile$ is a partial identity we conclude from Lemma 3(4) that there is a function $f : D \to C$ with $U;U^\smile \sqcap V;V^\smile = f^\smile;f$ and $f;f^\smile = \mathbb{I}_D$. The relation $h := f;U$ is univalent because it is the composition of univalent relations. Furthermore, we have

$$\begin{aligned} f;U;(f;U)^\smile &= f;U;U^\smile;f^\smile \\ &\sqsupseteq f;(U;U^\smile \sqcap V;V^\smile);f^\smile \\ &= f;f^\smile;f;f^\smile && f \text{ tabulates } U;U^\smile \sqcap V;V^\smile \\ &= \mathbb{I}_D, && \text{see above} \end{aligned}$$

i.e. h is a function. Analogously, $k := f;V$ is a function. We get

$$\begin{aligned} h^\smile;k &= U^\smile;f^\smile;f;V \\ &= U^\smile;(U;U^\smile \sqcap V;V^\smile);V && f \text{ tabulates } U;U^\smile \sqcap V;V^\smile \\ &= U^\smile;V. && \text{Lemma 1(3)} \end{aligned}$$

We conclude that $h : D \to A$, $k : D \to B$ is a tabulation of $U^\smile;V$, and, hence, $|U^\smile;V|_i^* = [D]_i = |U;U^\smile \sqcap V;V^\smile|_i^*$. $\square$

In order to characterize the canonical cardinality function we use the category $\mathrm{Card}_i(\mathcal{R})$. The objects of this category are the cardinality functions based on $\mathcal{R}$. A morphism between two cardinality functions $|.|_i^1 : \mathrm{Mor}_\mathcal{R} \to \mathcal{C}_1$ and $|.|_i^2 : \mathrm{Mor}_\mathcal{R} \to \mathcal{C}_2$ is a monotonic function $G : \mathcal{C}_1 \to \mathcal{C}_2$ so that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{R} \arrow[dl, "|.|_i^1"'] \arrow[dr, "|.|_i^2"] & \\ \mathcal{C}_1 \arrow[rr, "G"'] & & \mathcal{C}_2 \end{tikzcd}$$

Theorem 1. *A strong cardinality function is an initial object of $Card_i(\mathcal{R})$.*

Proof. Assume $|.|_i^s : \mathrm{Mor}_{\mathcal{R}} \to \mathcal{D}$ is a strong cardinality function. First, we want to show that every element of $\mathcal{D}$ is image of an identity relation via $|.|_i^s$. Let x be an element of $\mathcal{D}$. Since $|.|_i^s$ is strong there is a relation $R : A \to B$ with $|R|_i^s = x$. Let $f : C \to A$ and $g : C \to B$ be a tabulation of R. Then by Lemma 6(2) we have $|\mathbb{I}_C|_i^s = |R|_i^s = x$.

Let $|.|_i : \mathrm{Mor}_{\mathcal{R}} \to \mathcal{C}$ be an arbitrary cardinality function, and define $G(x) := |\mathbb{I}_A|_i$ with $|\mathbb{I}_A|_i^s = x$. We have to show that G is well-defined, i.e. it is independent of the choice of $\mathbb{I}_A$. Assume $|\mathbb{I}_A|_i^s = |\mathbb{I}_B|_i^s = x$. Since $|.|_i^s$ is strong there are injections $i_1 : A \to B$ and $i_2 : B \to A$. By Lemma 6(2) we conclude $|\mathbb{I}_A|_i = |\mathbb{I}_B|_i$. A similar argument shows that G is also monotonic.

Now, let $R : A \to B$ be a relation and $f : C \to A$ and $g : C \to B$ a tabulation of R. Then we have $G(|R|_i^s) = |\mathbb{I}_C|_i = |R|_i$ again by Lemma 6(2). G is obviously the unique function with that property. $\square$

The canonical cardinality function is strong by definition so that we get the following corollary:

Corollary 1. *The canonical cardinality function is an initial object of $Card_i(\mathcal{R})$.*

A further consequence is that any initial object of $\mathrm{Card}_i(\mathcal{R})$ must be strong because it is isomorphic to the canonical cardinality function.

Corollary 2. *A cardinality function is an initial object of $Card_i(\mathcal{R})$ iff it is strong.*

5 Cardinality Function (surjective case)

We now give the definition of cardinality function motivated by the preordering $\lesssim_s$.

Definition 5. *Let $\mathcal{R}$ be an allegory, and $(\mathcal{C}, \leq)$ be an ordered class. A function $|.|_s : \mathrm{Mor}_{\mathcal{R}} \to \mathcal{C}$ mapping the morphisms of $\mathcal{R}$ to elements of $\mathcal{C}$ is called a (surjective) cardinality function iff*

C0: *$|R^{\smile}|_s = |R|_s$ for all relations R;*
S1: *If $Q;Q^{\smile} \sqcap S;S^{\smile} \sqsubseteq \mathbb{I}_B$ for relations $Q : A \to B$ and $S : A \to C$, then for all $R : B \to C$*

$$|Q;R \sqcap S|_s \leq |R \sqcap Q^{\smile};S|_s.$$

$|.|_s$ is called strong iff it is surjective as a function and $|\mathbb{I}_A|_s \leq |\mathbb{I}_B|_s$ implies that there is a surjection $s : B \to A$.

S1 is also called the Dedekind inequality because of its similarity to the Dedekind formula. Notice that a weaker version was already used in [5].

The first part of the next lemma shows that a (surjective) cardinality function is based on the preordering $\lesssim_s$.

Lemma 8. *Let $|.|_s$ be a cardinality function over the allegory $\mathcal{R}$. Then:*

1. *If $s : B \to A$ is a surjection, then $|\mathbb{I}_A|_s \leq |\mathbb{I}_B|_s$.*
2. *Axiom I2 is valid.*
3. *If $R : A \to B$ has a tabulation $f : C \to A$ and $g : C \to B$, then $|R|_s = |\mathbb{I}_C|_s$.*

Proof. 1. We have $s^\smile; s = \mathbb{I}_A$ and $\mathbb{I}_B \sqsubseteq s; s^\smile$ and conclude

$$\begin{aligned}
|\mathbb{I}_A|_s &= |\mathbb{I}_A \sqcap s^\smile; s|_s && s^\smile; s = \mathbb{I}_A \\
&\leq |s \sqcap s|_s && \text{S1 since } s^\smile; s \sqcap \mathbb{I}_A; \mathbb{I}_A^\smile \sqsubseteq \mathbb{I}_A \\
&= |s^\smile \sqcap s^\smile|_s && \text{C0} \\
&\leq |s; s^\smile \sqcap \mathbb{I}_B|_s && \text{S1 since } s^\smile; s \sqsubseteq \mathbb{I}_A \\
&= |\mathbb{I}_B|_s.
\end{aligned}$$

2. Let $U : C \to A$ and $V : C \to B$ be univalent relations with $U; U^\smile \sqcap V; V^\smile \sqsubseteq \mathbb{I}_C$. Then the assertion follows from

$$\begin{aligned}
|U^\smile; V|_s &= |U^\smile; V \sqcap U^\smile; V|_s \\
&\leq |V \sqcap U; U^\smile; V|_s && \text{S1 since } U^\smile; U \sqsubseteq \mathbb{I}_A \\
&= |V^\smile; U; U^\smile \sqcap V^\smile|_s && \text{C0} \\
&\leq |U; U^\smile \sqcap V; V^\smile|_s && \text{S1 since } V^\smile; V \sqsubseteq \mathbb{I}_B \\
&\leq |U^\smile \sqcap U^\smile; V; V^\smile|_s && \text{S1 since } U; U^\smile \sqcap V; V^\smile; V; V^\smile \\
& && \qquad \sqsubseteq U; U^\smile \sqcap V; V^\smile \sqsubseteq \mathbb{I}_C \\
&= |V; V^\smile; U \sqcap U|_s && \text{C0} \\
&\leq |V^\smile; U \sqcap V^\smile; U|_s && \text{S1 since } V; V^\smile \sqcap U; U^\smile \sqsubseteq \mathbb{I}_C \\
&= |U^\smile; V|_s. && \text{C0}
\end{aligned}$$

3. This property uses the same proof as in Lemma 6(2) using (2) of the current lemma. Notice that monotonicity of the cardinality function is not used in that proof. $\square$

Again, we are just able to define the canonical cardinality function using tabulations. Therefore, we will assume for the rest of this section that the given allegory $\mathcal{R}$ is tabular.

As before, let us denote by $[A]_s$ the equivalence class of an object with respect to $\sim_i s$ and by $(\mathrm{Obj}_\mathcal{R}/\sim_s, \leq_s)$ the ordered class of those equivalence classes.

Definition 6. *The canonical cardinality function* $|.|_s^*$ *is defined by* $|R|_s^* := [C]_i$ *where* $R : A \to B$ *has a tabulation* $f : C \to A$ *and* $g : C \to B$*.*

Notice that the canonical cardinality function in the surjective case has the same definition as in the injective case. The main difference is in the ordered classes $(\mathrm{Obj}_{\mathcal{R}}/\sim_i, \leq_i)$ and $(\mathrm{Obj}_{\mathcal{R}}/\sim_s, \leq_s)$.

Lemma 9. *The canonical cardinality function* $|.|_s^*$ *is a cardinality function.*

Proof. C0: Analogously to the injective case.

S1: Let $Q : A \to B$, $R : B \to C$ and $S : A \to C$ be relations with $Q;Q^\smile \sqcap S;S^\smile \sqsubseteq \mathbb{I}_B$. Furthermore, suppose that we have the following tabulations:

$$\begin{aligned}
&Q = f_Q^\smile; g_Q, && f_Q; f_Q^\smile \sqcap g_Q; g_Q^\smile = \mathbb{I}_X,\\
&R = f_R^\smile; g_R, && f_R; f_R^\smile \sqcap g_R; g_R^\smile = \mathbb{I}_Y,\\
&S = f_S^\smile; g_S, && f_S; f_S^\smile \sqcap g_S; g_S^\smile = \mathbb{I}_Z,\\
&Q;R = f_{Q;R}^\smile; g_{Q;R}, && f_{Q;R}; f_{Q;R}^\smile \sqcap g_{Q;R}; g_{Q;R}^\smile = \mathbb{I}_U,\\
&f_S; f_Q^\smile = h^\smile; k, && h; h^\smile \sqcap k; k^\smile = \mathbb{I}_V,\\
&g_Q; f_R^\smile = m^\smile; n, && m; m^\smile \sqcap n; n^\smile = \mathbb{I}_W.
\end{aligned}$$

By definition of the canonical cardinality function we get $|Q|_s^* = [X]_s, |R|_s^* = [Y]_s, |S|_s^* = [Z]_s$ and $|Q;R|_s^* = [U]_s$. Since $Q;Q^\smile \sqcap S;S^\smile \sqsubseteq \mathbb{I}_B$ Lemma 5 shows that $h;g_S$ and $k;g_Q$ is a tabulation of $S^\smile;Q$ so that $|Q^\smile;S|_s^* = [V]_s$ follows. Assume D is the object used in the tabulation of $Q;R \sqcap S$, i.e. $|Q;R \sqcap S|_s^* = [D]_s$. By using the construction of Lemma 4 we obtain injections $x_1 : D \to Z$ and $x_2 : D \to U$ with $x_1;f_S = x_2;f_{Q;R}$, $x_1;g_S = x_2;g_{Q;R}$ and $(x_1;f_S)^\smile;x_1;g_S = (x_2;f_{Q;R})^\smile;x_2;g_{Q;R} = Q;R \sqcap S$. Analogously, assuming that $|S;Q^\smile \sqcap R|_s^* = [E]_s$ we obtain two injection $y_1 : E \to V$ and $y_2 : E \to Y$ with $y_1;k;g_Q = y_2;f_R$, $y_1;h;g_S = y_2;g_R$ and $(y_1;k;g_Q)^\smile;y_1;h;g_S = (y_2;f_R)^\smile;y_2;g_R = S;Q^\smile \sqcap R$. The following computation

$$\begin{aligned}
k^\smile;y_1^\smile;y_2 &\sqsubseteq k^\smile;y_1^\smile;y_2;f_R;f_R^\smile && f_R \text{ total}\\
&= k^\smile;y_1^\smile;y_1;k;g_Q;f_R^\smile && y_1;k;g_Q = y_2;f_R\\
&\sqsubseteq g_Q;f_R^\smile && y_1, k \text{ univalent}
\end{aligned}$$

shows that $k^\smile;y_1^\smile;y_2$ is included in the tabulation m, n so that there is a unique function $w : E \to W$ with $w;m = y_1;k$ and $w;n = y_2$ by Lemma 3(1). Furthermore, we have $Q;R = f_Q^\smile;g_Q;f_R^\smile;g_R = f_Q^\smile;m^\smile;n;g_R$ so that there is a surjection $e : W \to U$ with $e;f_{Q;R} = m;f_Q$ and $e;g_{Q;R} = n;g_R$ by

Lemma 3(2). Finally, consider the computations

$$
\begin{aligned}
y_1; h; f_S &= y_1; k; f_Q && \text{Lemma 2 since } h, k \text{ tabulates } f_S; f_Q^{\smile} \\
&= w; m; f_Q && w; m = y_1; k \\
&= w; e; f_{Q;R}, && e; f_{Q;R} = m; f_Q \\
y_1; h; g_S &= y_2; g_R && y_1; h; g_S = y_2; g_R \\
&= w; n; g_R && w; n = y_2 \\
&= w; e; g_{Q;R}. && e; g_{Q;R} = n; g_R
\end{aligned}
$$

From Lemma 4(2) we conclude that there is a unique $s : E \to D$ with $y_1; h = s; x_1$ and $w; e = s; x_2$. The whole situation is visualized in the following diagram:

$$
\begin{tikzcd}
C & & & D \arrow[dlll, "x_1"'] \arrow[d, "x_2"] \\
Z \arrow[u, "g_S"] \arrow[r, "f_S"'] & A \arrow[ul, "S"'] \arrow[dr, "Q"] & & U \arrow[ll, "f_{Q;R}"'] \arrow[dd, "g_{Q;R}"] \\
V \arrow[u, "h"] \arrow[r, "k"'] & X \arrow[u, "f_Q"] \arrow[r, "g_Q"'] & B \arrow[dr, "R"] & \\
E \arrow[u, "y_1"'] \arrow[r, "w"'] \arrow[rr, bend right, "y_2"'] \arrow[uuurrr, bend left, "s"] & W \arrow[u, "m"] \arrow[r, "n"'] \arrow[uurr, bend left, "e"] & Y \arrow[u, "f_R"] \arrow[r, "g_R"'] & C
\end{tikzcd}
$$

It remains to show that s is surjective. First, we have

$$
\begin{aligned}
s &= s; x_1; x_1^{\smile} && x_1 \text{ injective} \\
&= y_1; h; x_1^{\smile} && y_1; h = s; x_1 \\
&= y_1; h; (f_S; f_S^{\smile} \sqcap g_S; g_S^{\smile}); x_1^{\smile} && f_s, g_S \text{ tabulation} \\
&= y_1; h; f_S; (x_1; f_S)^{\smile} \sqcap y_1; h; g_S; (x_1; g_S)^{\smile}. && \text{Lemma 1(2)}
\end{aligned}
$$

From the computation

$$
\begin{aligned}
h; f_S &= h; f_S \sqcap k; f_Q && \text{Lemma 2} \\
&\sqsubseteq h; g_S; g_S^{\smile}; f_S \sqcap k; g_Q; g_Q^{\smile}; f_Q && g_S, g_Q \text{ total} \\
&= h; g_S; S^{\smile} \sqcap k; g_Q; Q^{\smile}, \\
&= h; g_S; g_S^{\smile}; f_S \sqcap k; g_Q; g_Q^{\smile}; f_Q \\
&\sqsubseteq (h; g_S; g_S^{\smile} \sqcap k; g_Q; g_Q^{\smile}; f_Q; f_S^{\smile}); f_S \\
&= (h; g_S; g_S^{\smile} \sqcap k; g_Q; g_Q^{\smile}; k^{\smile}; h); f_S && h, k \text{ tabulates } f_S; f_Q^{\smile}
\end{aligned}
$$

$$
\begin{aligned}
&\sqsubseteq (h;g_S;g_S^\smile;h^\smile \sqcap k;g_Q;g_Q^\smile;k^\smile);h;f_S \\
&= h;f_S && \text{Lemma 5}
\end{aligned}
$$

we conclude $h;f_S = h;g_S;S^\smile \sqcap k;g_Q;Q^\smile$. In addition, from

$$
\begin{aligned}
&(y_1;h;f_S)^\smile;y_1;h;g_S \\
&= (s;x_1;f_S)^\smile;s;x_1;g_S && y_1;h = s;x_1 \\
&= (x_1;f_S)^\smile;s^\smile;s;x_1;g_S \\
&= (x_1;f_S)^\smile;x_1;g_S && s \text{ univalent} \\
&= Q;R \sqcap S && \text{tabulation} \\
&= Q;R \sqcap S \sqcap S \\
&\sqsubseteq Q;(R \sqcap Q^\smile;S) \sqcap S \\
&= Q;(y_1;k;g_Q)^\smile;y_1;h;g_S \sqcap S && \text{tabulation} \\
&= (k;g_Q;Q^\smile)^\smile;y_1^\smile;y_1;h;g_S \sqcap S \\
&= ((k;g_Q;Q^\smile)^\smile \sqcap S;(y_1^\smile;y_1;h;g_S)^\smile);y_1^\smile;y_1;h;g_S \\
&\sqsubseteq ((k;g_Q;Q^\smile)^\smile \sqcap S;(h;g_S)^\smile);y_1^\smile;y_1;h;g_S && y_1 \text{ univalent} \\
&= (k;g_Q;Q^\smile \sqcap h;g_S;S^\smile)^\smile;y_1^\smile;y_1;h;g_S \\
&= (h;f_S)^\smile;y_1^\smile;y_1;h;g_S && \text{see above} \\
&= (y_1;h;f_S)^\smile;y_1;h;g_S
\end{aligned}
$$

we obtain $(y_1;h;f_S)^\smile;y_1;h;g_S = (x_1;f_S)^\smile;x_1;g_S$. Now, we are ready to establish that s is indeed surjective.

$$
\begin{aligned}
\mathbb{I}_D &= \mathbb{I}_D \sqcap x_1;f_S;(x_1;f_S)^\smile x_1;g_S;(x_1;g_S)^\smile && x_1, f_S, g_S \text{ total} \\
&= \mathbb{I}_D \sqcap x_1;f_S;(y_1;h;f_S)^\smile;y_1;h;g_S;(x_1;g_S)^\smile && \text{see above} \\
&\sqsubseteq (x_1;f_S;(y_1;h;f_S)^\smile \sqcap x_1;g_S;(y_1;h;g_S)^\smile); && \text{Lemma 1(1)} \\
&\qquad (y_1;h;g_S;(x_1;g_S)^\smile \sqcap y_1;h;f_S;(x_1;f_S)^\smile) \\
&= s^\smile;s. && \text{see above}
\end{aligned}
$$

This completes the proof. □

As in the injective case we want to characterize the canonical cardinality function. Again we use the category of cardinality functions $\mathrm{Card}_s(\mathcal{R})$, which is defined analogously to $\mathrm{Card}_i(\mathcal{R})$.

Theorem 2. *A strong cardinality function is an initial object of $Card_s(\mathcal{R})$.*

Proof. The proof of this theorem is similar to the proof of Theorem 1 using Lemma 8(3) instead of Lemma 6(2). □

As in the injective case we get the following corollaries:

Corollary 3. *The canonical cardinality function is an initial object of $Card_s(\mathcal{R})$.*

Corollary 4. *A cardinality function is an initial object of $Card_s(\mathcal{R})$ iff it is strong.*

6 Conclusion and Outlook

In this paper we have instigated two notions of the cardinality of relations based on the preordering of object induced by the existence of injective and surjective functions, respectively. An obvious extension is to combine both notion into one concept. The abstract definition will use the Axioms C0, I1 and S1, of course. As the examples in Section 3 show a suitable definition of a a canonical cardinality function requires more structure of the underlying allegory. One may require a relational version of the Axiom of Choice:

(AC) For all relations $R : A \to B$ there is a function $f : A \to B$ with $f \sqsubseteq R$ and $\mathbb{I}_A \sqcap f; f^{\smile} = \mathbb{I}_A \sqcap R; R^{\smile}$.

Notice that the axiom above for tabular power allegories implies that the each lower semi-lattice $\mathcal{R}[A, B]$ is in fact a Boolean algebra. This is just the allegorical version of the fact that the Axiom of Choice in a topos implies that the topos is Boolean.

References

1. Berghammer, R.: Computation of Cut Completions and Concept Lattices Using Relational Algebra and RelView. JoRMiCS 1 (2004), 50-72.
2. Bird R., de Moor O.: Algebra of Programming. Prentice Hall (1997).
3. Brink C., Kahl W., Schmidt G. (eds.): Relational Methods in Computer Science, Advances in Computer Science, Springer Vienna (1997).
4. Freyd P., Scedrov A.: Categories, Allegories. North-Holland (1990).
5. Kawahara, Y.: On the Cardinality of Relations. 9th International Conference on Relational Methods in Computer Science and 4th International Workshop on Applications of Kleene Algebra. LNCS 4136 (2006), 251-265.
6. Kawahara, Y., Winter, M.: On the Tabular Closure of a Sub-Allegory of a Tabular Allegory. to appear
7. Schmidt G., Ströhlein T.: Relationen und Graphen. Springer (1989); English version: Relations and Graphs. Discrete Mathematics for Computer Scientists, EATCS Monographs on Theoret. Comput. Sci., Springer (1993).
8. Winter, M.: Goguen Categories. A Categorical Approach to L-Fuzzy Relations. Trends in Logic Vol. 25, Springer (2007).