Reachability Analysis of Stochastic Hybrid Systems by Optimal Control

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Abstract. For stochastic hybrid systems, the reachability analysis is an important and difficult problem. In this paper, we prove that, under natural assumptions, reachability analysis can be characterised as an optimal stopping problem. In this way, one can apply numerical methods from optimal control to solve the reachability verification problems.

Keywords: Stochastic hybrid systems, Markov processes, reachability problem, optimal stopping.

1 Introduction

The paper addresses the reachability problem for stochastic hybrid systems, which are a class of non-linear stochastic continuous time/space hybrid dynamical systems. For a stochastic hybrid system, we show that the reach set probabilities coincide with the value functions of some particular optimal stopping problems corresponding to the indicator functions of the target sets. These optimal stopping problems are formulated in the language of the Markov process that describes the realizations of the given hybrid system. Our method is based on the (Riesz) representation of the value function for the optimal stopping problem and it was successfully used for some particular classes of Markov processes [7]. The application of this method is sketched for stochastic hybrid systems.

2 Stochastic Hybrid Systems

Stochastic Hybrid Systems can be described as an interleaving between a finite or countable family of diffusion processes and a Markov chain. We adopt the General Stochastic Hybrid System model presented in [2]. Let Q be a set of discrete states. For each $q \in Q$, we consider the Euclidean space $\mathbb{R}^{d(q)}$ with dimension $d(q)$ and we define an *invariant* as an open subset X^q of $\mathbb{R}^{d(q)}$. The hybrid state space is the set $X(Q, d, \mathcal{X}) = \bigcup_{i \in Q} \{i\} \times X^i$. The closure of $X(Q, d, \mathcal{X})$ will be $\overline{X} = X \cup \partial X$, where $\partial X = \bigcup_{i \in Q} \{i\} \times \partial X^i$. $(X, \mathcal{B}(X))$ is

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a Borel space, where $\mathcal{B}(X)$ is the Borel σ-algebra of X. Let $\mathbf{B}(X)$ be the Banach space of bounded positive measurable functions on X with the norm given by the supremum. A (General) Stochastic Hybrid System (SHS) is a collection $H = ((Q, d, \mathcal{X}), b, \sigma, Init, \lambda, R)$, where the full meaning of the constituents can be find in [2]. The realization of an SHS is built as a *Markov string* H [2]. This string is a Markov process. Denote by $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$ this Markov process. Let $\mathcal{P} = (P_t)_{t>0}$ denote the *semigroup of operators* associated to M, which maps $\mathbf{B}(X)$ into itself given by $P_t f(x) = E_x f(x_t), \forall x \in X$, where E_x is the expectation w.r.t. P_x. A nonnegative function $f \in \mathbf{B}(X)$ is called (α-)*excessive* ($\alpha \geq 0$) if $(e^{-\alpha t})P_t f \leq f$ for all $t \geq 0$ and $(e^{-\alpha t})P_t f \nearrow f$ as $t \searrow 0$. Let $\mathcal{E}_M$ be the *cone of excessive functions*. Suppose that M is *transient*[1], i.e. there is a strictly positive measurable function q such that $Uq \leq 1$. The infinitesimal generator $\mathcal{L}$ is the derivative of P_t at $t = 0$. Under the standard assumptions the realization M of an SHS is a Borel right process with cadlag property and the infinitesimal generator of an SHS is an integro-differential operator [2].

3 Stochastic Reachability as an Optimal Stopping Problem

In this section, in the framework of SHS, we prove that the stochastic reachability problem is equivalent with an optimal stopping problem.

Let us consider $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$ a (strong right) Markov process, being the realization of an SHS. For this Markov process we address the *stochastic reachability problem* as follows. Given a target set, the objective of the reachability problem is to compute the probability that the system trajectories from an arbitrary initial state will reach the target set. Formally, given a set $A \in \mathcal{B}(X)$ and a time horizon $T \in [0, \zeta]$ (where ζ is the life time of M), define $Reach_T(A) := \{\omega \in \Omega \mid \exists t \geq 0 : x_t(\omega) \in A\}$. The reachability problem consists of determining the probabilities of such a set, i.e. $P(T_A < T)$, where T_A is the first hitting time of A (i.e. $T_A = \inf\{t > 0 | x_t \in A\}$) and $(\Omega, \mathcal{F}, P)$ is the underlying probability space of M. P can be chosen to be P_x, if we want to consider the trajectories that start in x.

For any $f : X \to \mathbb{R}_+$, we denote *he réduite of* f by Rf, i.e $Rf := \inf\{u \in \mathcal{E}_M | u \geq f\}$. Rf differs from f only on a negligible set. For any $A \subset X$ and $v \in \mathcal{E}_M$, the function $R_A v = R(1_A v)$ is called *the réduite*[2] *of* v *on* A. The *balayage of the excessive function* v *on* A denoted by $B_A v$, is the $\mathcal{U}$-excessive regularization of $R_A v$ [3]. For any $x \in X$ and $A \in \mathcal{B}(X)$, we have $P_x[Reach_\infty(A)] = B_A 1(x) = P_x[T_A < \zeta]$ [3]. Since $R_A v = B_A v$ on $X \backslash A$ (see [3]), when the process starts in $x \notin A$, finding the reach set probability $P_x[Reach_\infty(A)]$ is equivalent to finding the reduite $R(1_A)(x)$. The existence of the réduite for $g \in \mathbf{B}(X)$ is based on the following equality: $Rg(x) = \sup\{E_x[g(x_S)1_{\{S<\zeta\}}]; S \text{ stopping time}\}$. The right hand side of the above equality is related with the so-called *optimal stopping problem* (OSP) associated with a Markov process. For different classes of

[1] The transience hypothesis guarantees that the cone $\mathcal{E}_M$ is rich enough to be used.

[2] We use the convention $0 \cdot (+\infty) = (+\infty) \cdot 0 = 0$.

stochastic processes[3], the fact that the optimal value function coincides with the smaller excessive majorant of the exercise payoff is a well known result. This result has been extended for right processes in [4].

Proposition 1. *If $A \in \mathcal{B}(X)$ then the reachability function $w_A : X \to [0,1]$ associated to A, defined as $w_A(x) := P_x[Reach_\infty(A)]$, coincides with the value function of the reward process $y_t = 1_A(x_t)$, i.e. $w_A(x) = \sup\{P_x(x_\tau \in A) | \tau$ stopping time$\}$, $\forall x \in X$.*

4 From Optimal Stopping to Stochastic Reachability

The realizations of SHS are (Borel) right processes, and therefore the general theory of optimal stopping developed for right processes [1] can be applied. For the OSP associated to (Borel) right processes, we propose a method based on representations of excessive functions. This method consists in establishing first an integral representation for excessive (super-harmonic) functions and then deriving information about final behaviour of paths. We show that finding the solution of such a problem is equivalent to finding the representation of the value function of the OSP in terms of the Green kernel. The support of the measure that appears in this representation is the *stopping region* for the problem.

Dealing with Optimal Stopping. For the a Markov process M and the optimal stopping problem, one can introduce: *continuation set* $C = \{x \in X | v(x) > g(x)\}$; and the *stopping set* $D = \{x \in X | v(x) = g(x)\}$.

Using the operator semigroup $\mathcal{P}$, one can define the *kernel operator* U by $Uf(x) = \int_0^\infty P_t f(x) dt$, $f \in \mathbf{B}(X)$. Uf is the solution of the equation $-\mathcal{L}\phi = f$. If in expression of U, f ranges over the indicator functions of measurable sets, we can write U as a stochastic kernel $U(x, A) = \int_0^\infty P_x(x_t \in A) dt$. For the scope of this section, we suppose that the assumptions from [6] are in force. The main assumption is related to the absolute continuity of the kernel operator U w.r.t. a σ-finite excessive measure m on $(X, \mathcal{B})$, called *reference measure.*

Assumption 1. *There exists a $\mathcal{B} \times \mathcal{B}$ measurable function $u \geq 0$ such that $U(x, dy) = u(x, y) m(dy)$, $x \in X$; and $x \longmapsto u(x, y)$ is excessive, $y \in X$.*

The *potential density* $u(x, y)$ is used to define the potential of a measure μ by setting $U\mu(x) := \int_X u(x, y) \mu(dy)$. For Borel right processes, h is harmonic iff $P^-_{K^c} h = h$, $m - a.e.$[4], for every compact K (with the complement $K^c = X \backslash K$) in an appropriate compactification of X, where $P^-_{K^c}$ is the hitting operator associated to K^c[5].

Theorem 1 (Riesz Decomposition). *[6] Let $f \in \mathcal{E}_M$. Then there exists a measure μ on $(X, \mathcal{B})$ and an harmonic function h such that $f = U\mu + h$, $m - a.e.$Moreover, μ is unique and h is unique $m - a.e.$*

[3] diffusions, Feller/Hunt/standard processes.
[4] $m - a.e.$ (m almost everywhere), i.e. outside of a set with m-measure zero.
[5] i.e. $P^-_{K^c} h = E_x[h(x_{T^-_{K^c}})]$, $T^-_{K^c} = \inf\{t | 0 < t < \zeta; x_{t-} \in K^c\}$.

Using Ass.1 and the characterization of $U\mu$, in the decomposition of Th.1, if there exists a compact set K such that the representing measure μ *does not charge* K^c, then f is harmonic on K^c, i.e. $\mu(K^c) = 0 \Longrightarrow f$ is harmonic on K^c. Then the problem of finding the maximal payoff function is equivalent to the problem of finding the representing measure μ_v of v. The continuation region C is the biggest set *not charged* by the representing measure μ_v of v, i.e. $\mu_v(C) = 0$. So, the value function v is harmonic on C.

Proposition 2. *The* measure μ_v gives the value function v, *and the support of the representation measure gives the stopping region D, i.e. $D = supp(\mu_v)$.*

Reach Set Probability Computation. Suppose that the target set A is an open set of the state space X. Define $F := X \backslash A$. Suppose that last exit time from F is finite almost surely (the process is transient), i.e. $S_F = \sup\{t \geq 0 | x_t \in F\} < \infty$. Then the reachability problem turns in an *exit time problem*, and then computing the reach set probabilities is equivalent with the computation of a dual probability $P_x[x_{S_F} \in F | S_F > 0]$.

Proposition 3. *[5] For all positive $f \in \mathbf{B}(X)$, we have $P_x[(1_F f)(x_{S_F}) | S_F > 0] = \int u(x,y)(1_F f)(y)\mu(dy)$, where μ is a measure on X.*

Therefore, for $f \equiv 1$, the reach set probabilities are $P_x[x_{S_F} \in F | S_F > 0] = \int_F u(x,y)\mu_F(dy)$, where μ_F is the *equilibrium measure* of F.

5 Conclusions

In this paper, we have characterised the reachability problem of stochastic hybrid systems as an optimal stopping problem with a discontinuous reward function. To deal with the stochastic reachability, we consider that the method based on representations of the value function of the equivalent optimal stopping problem suits best in this context.

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