Sincere-Strategy Preference-Based Approval Voting Fully Resists Constructive Control and Broadly Resists Destructive Control*

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Abstract

We study sincere-strategy preference-based approval voting (SP-AV), a system proposed by Brams and Sanver [BS06], with respect to procedural control. In such control scenarios, an external agent seeks to change the outcome of an election via actions such as adding/deleting/partitioning either candidates or voters. SP-AV combines the voters' preference rankings with their approvals of candidates, and we adapt it here so as to keep its useful features with respect to approval strategies even in the presence of control actions. We prove that this system is computationally resistant (i.e., the corresponding control problems are NP-hard) to 19 out of 22 types of constructive and destructive control. Thus, SP-AV has more resistances to control, by three, than is currently known for any other natural voting system with a polynomial-time winner problem. In particular, SP-AV is (after Copeland voting, see Faliszewski et al. [FHHR08]) the second natural voting system with an easy winner-determination procedure that is known to have full resistance to constructive control, and unlike Copeland voting it in addition displays broad resistance to destructive control.

1 Introduction

Voting provides a particularly useful method for preference aggregation and collective decision-making. While voting systems were originally used in political science, economics, and operations

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research, they are now also of central importance in various areas of computer science, such as artificial intelligence (in particular, within multiagent systems). In automated, large-scale computer settings, voting systems have been applied, e.g., for planning [ER93] and similarity search [FKS03], and have also been used in the design of recommender systems [GMHS99] and ranking algorithms [DKNS01] (where they help to lessen the spam in meta-search web-page rankings). For such applications, it is crucial to explore the computational properties of voting systems and, in particular, to study the complexity of problems related to voting (see, e.g., the survey by Faliszewski et al. [FHHR]).

The study of voting systems from a complexity-theoretic perspective was initiated by Bartholdi, Tovey, and Trick's series of seminal papers about the complexity of winner determination [BTT89b], manipulation [BTT89a], and procedural control [BTT92] in elections. This paper contributes to the study of electoral control, where an external agent—traditionally called *the chair*—seeks to influence the outcome of an election via procedural changes to the election's structure, namely via adding/deleting/partitioning either candidates or voters (see Section 2.2 for the formal definitions of our control problems). We consider both *constructive* control (introduced by Bartholdi, Tovey, and Trick [BTT92]), where the chair's goal is to make a given candidate the unique winner, and *destructive* control (introduced by Hemaspaandra, Hemaspaandra, and Rothe [HHR07a]), where the chair's goal is to prevent a given candidate from being a unique winner.

We investigate the same twenty types of constructive and destructive control that were studied for approval voting [HHR07a] and two additional control types introduced by Faliszewski et al. [FHHR07], and we do so for a voting system that was proposed by Brams and Sanver [BS06] as a combination of preference-based and approval voting. Approval voting was introduced by Brams and Fishburn ([BF78, BF83], see also [BF02]) as follows: Every voter either approves or disapproves of each candidate, and every candidate with the largest number of approvals is a winner. One of the simplest preference-based voting systems is plurality: All voters report their preference rankings of the candidates, and the winners are the candidates that are ranked first-place by the largest number of voters. The purpose of this paper is to show that Brams and Sanver's combined system (adapted here so as to keep its useful features even in the presence of control actions) combines the strengths, in terms of computational resistance to control, of plurality and approval voting.

Some voting systems are *immune* to certain types of control in the sense that it is never possible for the chair to reach his or her goal via the corresponding control action. Of course, immunity to any type of control is most desirable, as it unconditionally shields the voting system against this particular control type. Unfortunately, like most voting systems approval voting is *susceptible* (i.e., not immune) to many types of control, and plurality voting is susceptible to all types of control.[1] However, and this was Bartholdi, Tovey, and Trick's brilliant insight [BTT92], even for systems susceptible to control, the chair's task of controlling a given election may be too hard computationally (namely, NP-hard) for him or her to succeed. The voting system is then said to be *resistant* to this control type. If a voting system is susceptible to some type of control, but the chair's task can

[1] A related line of research has shown that, in principle, all natural voting systems can be manipulated by strategic voters. Most notable among such results is the classical work of Gibbard [Gib73] and Satterthwaite [Sat75]. The study of strategy-proofness is still an extremely active and interesting area in social choice theory (see, e.g., Duggan and Schwartz [DS00]) and in artificial intelligence (see, e.g., Everaere et al. [EKM07]).

Number of	Condorcet	Approval	Llull	Copeland	Plurality	SP-AV
resistances	3	4	14	15	16	19
immunities	4	9	0	0	0	0
vulnerabilities	7	9	8	7	6	3
References	[BTT92, HHR07a]	[BTT92, HHR07a]	[FHHR07, FHHR08]	[FHHR07, FHHR08]	[BTT92, HHR07a, FHHR07]	[HHR07a] and this paper

Table 1: Number of resistances, immunities, and vulnerabilities to our 22 control types.

be solved in polynomial time, the system is said to be *vulnerable* to this control type.

The quest for a natural voting system with an easy winner-determination procedure that is universally resistant to control has lasted for more than 15 years now. Among the voting systems that have been studied with respect to control are plurality, Condorcet, approval, cumulative, Llull, and (variants of) Copeland voting [BTT92, HHR07a, HHR07b, PRZ07, FHHR07, FHHR08, BU08]. Among these systems, plurality and Copeland voting (denoted Copeland$^{0.5}$ in [FHHR08]) display the broadest resistance to control, yet even they are not universally control-resistant. The only system currently known to be fully resistant—to the 20 types of constructive and destructive control studied in [HHR07a, HHR07b]—is a highly artificial system constructed via hybridization [HHR07b]. (We mention that this system was not designed for direct, real-world use as a "natural" system but rather was intended to rule out the existence of a certain impossibility theorem [HHR07b].)

While approval voting nicely distinguishes between each voter's acceptable and inacceptable candidates, it ignores the preference rankings the voters may have about their approved (or disapproved) candidates. This shortcoming motivated Brams and Sanver [BS06] to introduce a voting system that combines approval and preference-based voting, and they defined the related notions of sincere and admissible approval strategies, which are quite natural requirements. We adapt their sincere-strategy preference-based approval voting system in a natural way such that, for elections with at least two candidates, admissibility of approval strategies (see Definition 2.1) can be ensured even in the presence of control actions such as deleting candidates and partitioning candidates or voters.[2] The purpose of this paper is to study if, and to what extent, this hybrid system (where "hybrid" is not meant in the sense of [HHR07b] but refers to combining preference-based with approval voting in the sense of Brams and Sanver [BS06]) inherits the control resistances of plurality (which is perhaps the simplest preference-based system) and approval voting. Denoting this system by SP-AV, we show that SP-AV does combine all the resistances of plurality and approval voting.

More specifically, we prove that sincere-strategy preference-based approval voting is resistant to 19 and vulnerable to only three of the 22 types of control considered here. For comparison, Table 1 shows the number of resistances, immunities, and vulnerabilities to our 22 control types that are known for each of Condorcet,[3] approval, Llull, plurality, and Copeland voting (see [BTT92,

[2]Note that in control by partition of voters (see Section 2.2) the run-off may have a reduced number of candidates.

[3]As in [HHR07a], we consider two types of control by partition of candidates (namely, with and without run-off) and one type of control by partition of voters, and for each partition case we use the rules TE ("ties eliminate") and TP

HHR07a, FHHR07, FHHR08]), and for SP-AV (see Theorem 3.1 and Table 2 in Section 3.1).

This paper is organized as follows. In Section 2, we define sincere-strategy preference-based approval voting, the types of control studied in this paper, and the notions of immunity, susceptibility, vulnerability, and resistance. In Section 3, we prove our results on SP-AV. Finally, in Section 4 we give our conclusions and state some open problems.

2 Preliminaries

2.1 Preference-Based Approval Voting

An election $E = (C,V)$ is specified by a finite set C of candidates and a finite collection V of voters who express their preferences over the candidates in C, where distinct voters may of course have the same preferences. How the voter preferences are represented depends on the voting system used. In approval voting (AV, for short), every voter draws a line between his or her acceptable and inacceptable candidates (by specifying a 0-1 approval vector, where 0 represents disapproval and 1 represents approval), yet does not rank them. In contrast, many other important voting systems (e.g., Condorcet voting, Copeland voting, all scoring protocols including plurality, Borda count, veto, etc.) are based on voter preferences that are specified as tie-free linear orderings of the candidates. As is most common in the literature, votes will here be represented nonsuccinctly: one ballot per voter. Note that some papers (e.g., [FHH06, FHHR07, FHHR08]) also consider succinct input representations for elections where multiplicities of votes are given in binary.

Brams and Sanver [BS06] introduced a voting system that combines approval and preference-based voting. To distinguish this system from other systems that these authors introduced with the same purpose of combining approval and preference-based voting [BS], we call the variant considered here (including the conventions and rules to be explained below) *sincere-strategy preference-based approval voting* (SP-AV, for short).

Definition 2.1 ([BS06])**.** *Let (C,V) be an election, where the voters both indicate approvals/disapprovals of the candidates and provide a tie-free linear ordering of all candidates. For each voter $v \in V$, an* AV strategy of *v is a subset $S_v \subseteq C$ such that v approves of all candidates in S_v and disapproves of all candidates in $C - S_v$. The list of AV strategies for all voters in V is called an* AV strategy profile for (C,V). *(We sometimes also speak of V*'s AV strategy profile for *C.) For each $c \in C$, let $score_{(C,V)}(c) = \|\{v \in V \mid c \in S_v\}\|$ denote the number of c's approvals. Every candidate c with the largest $score_{(C,V)}(c)$ is a winner of election (C,V).*

An AV strategy S_v of a voter $v \in V$ is said to be admissible *if S_v contains v's most preferred candidate and does not contain v's least preferred candidate. S_v is said to be* sincere *if for each $c \in C$, if v approves of c then v also approves of each candidate ranked higher than c (i.e., there*

("ties promote") for handling ties that may occur in the corresponding subelections (see Section 2.2). However, since Condorcet winners are always unique when they exist, the distinction between TE and TP is not made for the partition cases within Condorcet voting. Note further that the two additional control types in Section 2.2.1 (namely, constructive and destructive control by adding a limited number of candidates [FHHR07]) have not been considered for Condorcet voting [BTT92, HHR07a]. That is why Table 1 lists only 14 instead of 22 types of control for Condorcet.

are no gaps allowed in sincere approval strategies). An AV strategy profile for (C,V) *is* admissible *(respectively,* sincere*) if the AV strategies of all voters in* V *are admissible (respectively, sincere).*

Admissibility and sincerity are quite natural requirements. In particular, requiring the voters to be sincere ensures that their preference rankings and their approvals/disapprovals are not contradictory. Note further that admissible AV strategies are not dominated in a game-theoretic sense [BF78], and that sincere strategies for at least two candidates are always admissible if voters are neither allowed to approve of everybody nor to disapprove of everybody (i.e., if we require voters v to have only AV strategies S_v with $\emptyset \neq S_v \neq C$), a convention adopted by Brams and Sanver [BS06] and also adopted here.[4] Henceforth, we will tacitly assume that only sincere AV strategy profiles are considered (which by the above convention, whenever there are at least two candidates,[5] necessarily are admissible), i.e., a vote with an insincere strategy will be considered void.

Preferences are represented by a left-to-right ranking (separated by a space) of the candidates (e.g., $a\ b\ c$), with the leftmost candidate being the most preferred one, and approval strategies are denoted by inserting a straight line into such a ranking, where all candidates left of this line are approved and all candidates right of this line are disapproved (e.g., "$a \mid b\ c$" means that a is approved, while both b and c are disapproved). In our constructions, we sometimes also insert a subset $B \subseteq C$ into such approval rankings, where we assume some arbitrary, fixed order of the candidates in B (e.g., "$a \mid B\ c$" means that a is approved, while all $b \in B$ and c are disapproved).

2.2 Control Problems for Preference-Based Approval Voting

The control problems considered here were introduced by Bartholdi, Tovey, and Trick [BTT92] for constructive control and by Hemaspaandra, Hemaspaandra, and Rothe [HHR07a] for destructive control. In constructive control scenarios the chair's goal is to make a favorite candidate win, and in destructive control scenarios the chair's goal is to ensure that a despised candidate does not win. As is common, the chair is assumed to have complete knowledge of the voters' preference rankings and approval strategies (see [HHR07a] for a detailed discussion of this assumption), and as in most papers on electoral control (exceptions are, e.g., [PRZ07, FHHR08]) we define the control problems in the unique-winner model, i.e., in this model the chair seeks to, via the control action, either make a designated candidate the unique winner (in the constructive case) or to prevent a designated candidate from being a unique winner (in the destructive case).

To achieve his or her goal, the chair modifies the structure of a given election via adding/deleting/partitioning either candidates or voters. Such control actions—specifically, those with respect to control via deleting or partitioning candidates or via partitioning voters—may have

[4] Brams and Sanver [BS06] actually preclude only the case $S_v = C$ for voters v. However, an AV strategy that disapproves of all candidates obviously is sincere, yet not admissible, which is why we also exclude the case of $S_v = \emptyset$.

[5] Note that an AV strategy is never admissible for less than two candidates. We mention in passing that a precursor of this paper [ENR08] specifically required for single-candidate elections that each voter must approve of this candidate. In this version of the paper, we drop this requirement for two reasons. First, it in fact is not needed because the one candidate in a single-candidate election will always win—even with zero approvals (i.e., SP-AV is a "voiced" voting system). Second, it is very well comprehensible that a voter, when given just a single candidate (think, for example, of an "election" in the Eastern bloc before 1989), can get some satisfaction from denying this candidate his or her approval, even if he or she knows that this disapproval won't prevent the candidate from winning.

an undesirable impact on the resulting election in that they might violate our conventions about admissible AV strategies. That is why we define the following rule that preserves (or re-enforces) our conventions under such control actions:

> Whenever during or after a control action it happens that we obtain an election (C,V) with $\|C\| \geq 2$ and for some voter $v \in V$ we have $S_v = \emptyset$ or $S_v = C$, then each such voter's AV strategy is changed to approve of his or her top candidate and to disapprove of his or her bottom candidate. This rule re-enforces $\emptyset \neq S_v \neq C$ for each $v \in V$.

We now formally define our control problems, where each problem is defined by stating the problem instance together with two questions, one for the constructive and one for the destructive case. These control problems are tailored to sincere-strategy preference-based approval voting by requiring every election occuring in these control problems (be it before, during, or after a control action—so, in particular, this also applies to the subelections in the partitioning cases) to have a sincere AV strategy profile and to satisfy the above conventions and rules. In particular, this means that when the number of candidates is reduced (due to deleting candidates or partitioning candidates or voters), approval lines may have to be moved in accordance with the above rules.

2.2.1 Control by Adding Candidates

In this control scenario, the chair seeks to reach his or her goal by adding to the election, which originally involves only "qualified" candidates, some new candidates who are chosen from a given pool of spoiler candidates. In their study of control for approval voting, Hemaspaandra, Hemaspaandra, and Rothe [HHR07a] considered only the case of adding an *unlimited* number of spoiler candidates (which is the original variant of this problem as defined by Bartholdi, Tovey, and Trick [BTT92]). We consider the same variant of this problem here to make our results comparable with those established in [HHR07a], but for completeness we in addition consider the case of adding a *limited* number of spoiler candidates, where the prespecified limit is part of the problem instance. This variant of this problem was introduced by Faliszewski et al. [FHHR07, FHHR08] in analogy with the definitions of control by deleting candidates and of control by adding or deleting voters. They showed that, for the election system Copeland$^{\alpha}$ they investigate, the complexity of these two problems can drastically change depending on the parameter α, see [FHHR08].

We first define the unlimited variant of control by adding candidates.

Name: Control by Adding an Unlimited Number of Candidates.

Instance: An election $(C \cup D, V)$ and a designated candidate $c \in C$, where the set C of qualified candidates and the set D of spoiler candidates are disjoint.

Question (constructive): Is it possible to choose a subset $D' \subseteq D$ such that c is the unique winner of election $(C \cup D', V)$?

Question (destructive): Is it possible to choose a subset $D' \subseteq D$ such that c is not a unique winner of election $(C \cup D', V)$?

The problem Control by Adding a Limited Number of Candidates is defined analogously, with the only difference being that the chair seeks to reach his or her goal by adding at most ℓ spoiler candidates, where ℓ is part of the problem instance.

2.2.2 Control by Deleting Candidates

In this control scenario, the chair seeks to reach his or her goal by deleting (up to a given number of) candidates. Here it may happen that our conventions are violated by the control action, but will be re-enforced by the above rules (namely, by moving the line between some voter's acceptable and inacceptable candidates to behind the top candidate or to before the bottom candidate whenever necessary).

Name: Control by Deleting Candidates.
Instance: An election (C,V), a designated candidate $c \in C$, and a nonnegative integer ℓ.
Question (constructive): Is it possible to delete up to ℓ candidates from C such that c is the unique winner of the resulting election?
Question (destructive): Is it possible to delete up to ℓ candidates (other than c) from C such that c is not a unique winner of the resulting election?

2.2.3 Control by Partition and Run-Off Partition of Candidates

There are two partition-of-candidates control scenarios. In both scenarios, the chair seeks to reach his or her goal by partitioning the candidate set C into two subsets, C_1 and C_2, after which the election is conducted in two stages. In control by partition of candidates, the election's first stage is hold within only one group, say C_1, and this group's winners that survive the tie-handling rule used (see the next paragraph) run against all members of C_2 in the second and final stage. In control by run-off partition of candidates, the election's first stage is hold separately within both groups, C_1 and C_2, and the winners of both subelections that survive the tie-handling rule used run against each other in the second and final stage.

We use the two tie-handling rules proposed by Hemaspaandra, Hemaspaandra, and Rothe [HHR07a]: ties-promote (TP) and ties-eliminate (TE). In the TP model, all the first-stage winners of a subelection, (C_1,V) or (C_2,V), are promoted to the final round. In the TE model, a first-stage winner of a subelection, (C_1,V) or (C_2,V), is promoted to the final round exactly if he or she is that subelection's unique winner.

Note that partitioning the candidate set C into C_1 and C_2 is, in some sense, similar to deleting C_2 from C to obtain subelection (C_1,V) and to deleting C_1 from C to obtain subelection (C_2,V). Also, depending on the tie-handling rule used, the final stage of the election may have a reduced number of candidates. So, in the partitioning cases, it may again happen that our conventions are violated by the control action, but will be re-enforced by the above-mentioned rules.

Name: Control by Partition of Candidates.
Instance: An election (C,V) and a designated candidate $c \in C$.
Question (constructive): Is it possible to partition C into C_1 and C_2 such that c is the unique winner of the final stage of the two-stage election in which the winners of subelection (C_1,V) that survive the tie-handling rule run against all candidates in C_2 (with respect to the votes in V)?
Question (destructive): Is it possible to partition C into C_1 and C_2 such that c is not a unique winner of the final stage of the two-stage election in which the winners of subelection (C_1,V)

that survive the tie-handling rule run against all candidates in C_2 (with respect to the votes in V)?

Name: Control by Run-Off Partition of Candidates.

Instance: An election (C,V) and a designated candidate $c \in C$.

Question (constructive): Is it possible to partition C into C_1 and C_2 such that c is the unique winner of the final stage of the two-stage election in which the winners of subelection (C_1,V) that survive the tie-handling rule run (with respect to the votes in V) against the winners of subelection (C_2,V) that survive the tie-handling rule?

Question (destructive): Is it possible to partition C into C_1 and C_2 such that c is not a unique winner of the final stage of the two-stage election in which the winners of subelection (C_1,V) that survive the tie-handling rule run (with respect to the votes in V) against the winners of subelection (C_2,V) that survive the tie-handling rule?

2.2.4 Control by Adding Voters

In this control scenario, the chair seeks to reach his or her goal by introducing new voters into a given election. These additional voters are chosen from a given pool of voters whose preferences and approval strategies over the candidates from the original election are known. Again, the number of voters that can be added is prespecified.

Name: Control by Adding Voters.

Instance: An election (C,V), a collection W of additional voters with known preferences and approval strategies over C, a designated candidate $c \in C$, and a nonnegative integer ℓ.

Question (constructive): Is it possible to choose a subset $W' \subseteq W$ with $\|W'\| \leq \ell$ such that c is the unique winner of election $(C,V \cup W')$?

Question (destructive): Is it possible to choose a subset $W' \subseteq W$ with $\|W'\| \leq \ell$ such that c is not a unique winner of election $(C,V \cup W')$?

2.2.5 Control by Deleting Voters

The chair here seeks to reach his or her goal by suppressing (up to a prespecified number of) voters.

Name: Control by Deleting Voters.

Instance: An election (C,V), a designated candidate $c \in C$, and a nonnegative integer ℓ.

Question (constructive): Is it possible to delete up to ℓ voters from V such that c is the unique winner of the resulting election?

Question (destructive): Is it possible to delete up to ℓ voters from V such that c is not a unique winner of the resulting election?

2.2.6 Control by Partition of Voters

In this scenario, the election again is conducted in two stages, and the chair now seeks to reach his or her goal by partitioning the voters V into two subcommittees, V_1 and V_2. In the first stage, the subelections (C,V_1) and (C,V_2) are held separately in parallel, and the winners of each subelection who survive the tie-handling rule move forward to the second and final stage in which they compete against each other.

Name: Control by Partition of Voters.
Instance: An election (C,V) and a designated candidate $c \in C$.
Question (constructive): Is it possible to partition V into V_1 and V_2 such that c is the unique winner of the final stage of the two-stage election in which the winners of subelection (C,V_1) that survive the tie-handling rule run (with respect to the votes in V) against the winners of subelection (C,V_2) that survive the tie-handling rule?
Question (destructive): Is it possible to partition V into V_1 and V_2 such that c is not a unique winner of the final stage of the two-stage election in which the winners of subelection (C,V_1) that survive the tie-handling rule run (with respect to the votes in V) against the winners of subelection (C,V_2) that survive the tie-handling rule?

2.3 Immunity, Susceptibility, Vulnerability, and Resistance

The following notions—which are due to Bartholdi, Tovey, and Trick [BTT92]—will be central to our complexity analysis of the control problems for preference-based approval voting.

Definition 2.2. *Let $\mathscr{E}$ be an election system and let Φ be some given type of control.*

1. *$\mathscr{E}$ is said to be* immune to Φ-control *if*

   (a) *Φ is a constructive control type and it is never possible for the chair to turn a designated candidate from being not a unique winner into being the unique winner via exerting Φ-control, or*

   (b) *Φ is a destructive control type and it is never possible for the chair to turn a designated candidate from being the unique winner into being not a unique winner via exerting Φ-control.*

2. *$\mathscr{E}$ is said to be* susceptible to Φ-control *if it is not immune to Φ-control.*

3. *$\mathscr{E}$ is said to be* vulnerable to Φ-control *if $\mathscr{E}$ is susceptible to Φ-control and the control problem associated with Φ is solvable in polynomial time.*

4. *$\mathscr{E}$ is said to be* resistant to Φ-control *if $\mathscr{E}$ is susceptible to Φ-control and the control problem associated with Φ is* NP*-hard.*

For example, approval voting is known to be immune to eight of the twelve types of candidate control considered in [HHR07a]. The proofs of these results crucially employ the links between immunity/susceptibility for various control types shown in [HHR07a] and the fact that approval voting satisfies the unique version of the Weak Axiom of Revealed Preference (denoted by Unique-WARP, see [HHR07a, BTT92]): If a candidate c is the unique winner in a set C of candidates, then

c is the unique winner in every subset of C that includes c. In contrast with approval voting, sincere-strategy preference-based approval voting does not satisfy Unique-WARP, and we will see later in Section 3.2 that it indeed is susceptible to each type of control considered here.

Proposition 2.3. *Sincere-strategy preference-based approval voting does not satisfy Unique-WARP.*

Proof. Consider the election (C,V) with candidate set $C = \{a,b,c,d\}$ and voter collection $V = \{v_1,v_2,v_3,v_4\}$. Removing candidate d changes the profile as follows according to the SP-AV rules:

$$\begin{array}{llcl} v_1: & b\ c\ a \mid d & & b\ c \mid a \\ v_2: & c \mid a\ d\ b & \text{is changed to} & c \mid a\ b \\ v_3: & a\ b\ c \mid d & \text{(by removing } d\text{):} & a\ b \mid c \\ v_4: & b\ a\ c \mid d & & b\ a \mid c \end{array}$$

Note that the approval/disapproval line has been moved in voters v_1, v_3, and v_4. Although c was the unique winner in (C,V), c is not a winner in $(\{a,b,c\},V)$ (in fact, b is the unique winner in $(\{a,b,c\},V)$). Thus, SP-AV does not satisfy Unique-WARP. ❑

3 Results for Sincere-Strategy Preference-Based Approval Voting

3.1 Overview

Theorem 3.1 below (see also Table 2) shows the complexity results regarding control of elections for SP-AV. As mentioned in the introduction, with 19 resistances and only three vulnerabilities, this system has more resistances and fewer vulnerabilities to control (for our 22 control types) than is currently known for any other natural voting system with a polynomial-time winner problem.

Theorem 3.1. *Sincere-strategy preference-based approval voting is resistant and vulnerable to the* 22 *types of control defined in Section 2.2 as shown in Table 2.*

3.2 Susceptibility

By definition, all resistance and vulnerability results in particular require susceptibility. To avoid a tedious proof covering each of the 22 types of control separately, we will use the general susceptibility results and links between susceptibility cases established by Hemaspaandra, Hemaspaandra, and Rothe [HHR07a].[6] Theorem 3.2 provides such results for "voiced" voting systems. A voting system is said to be *voiced* if in every one-candidate election, this candidate wins.

Theorem 3.2 ([HHR07a]). *1. If a voiced voting system is susceptible to destructive control by partition of voters (in model TE or TP), then it is susceptible to destructive control by deleting voters.*

[6] Although [HHR07a] does not consider the case of control by adding a limited number of candidates explicitly, it is immediate that all proofs for the "unlimited" case in [HHR07a] work also for this "limited" case.

Control by	Plurality		SP-AV		AV	
	Constr.	Destr.	Constr.	Destr.	Constr.	Destr.
Adding an Unlimited Number of Candidates	R	R	**R**	**R**	I	V
Adding a Limited Number of Candidates	R	R	**R**	**R**	I	V
Deleting Candidates	R	R	**R**	**R**	V	I
Partition of Candidates	TE: R TP: R	TE: R TP: R	**TE: R** **TP: R**	**TE: R** **TP: R**	TE: V TP: I	TE: I TP: I
Run-off Partition of Candidates	TE: R TP: R	TE: R TP: R	**TE: R** **TP: R**	**TE: R** **TP: R**	TE: V TP: I	TE: I TP: I
Adding Voters	V	V	**R**	**V**	R	V
Deleting Voters	V	V	**R**	**V**	R	V
Partition of Voters	TE: V TP: R	TE: V TP: R	**TE: R** **TP: R**	**TE: V** **TP: R**	TE: R TP: R	TE: V TP: V

Table 2: Overview of results. Key: I means immune, R means resistant, V means vulnerable, TE means ties-eliminate, and TP means ties-promote. Results for SP-AV are new; their proofs are either new or draw on proofs from [HHR07a]. Results for plurality and AV, stated here to allow comparison, are due to Bartholdi, Tovey, and Trick [BTT92] and to Hemaspaandra, Hemaspaandra, and Rothe [HHR07a]. (The results for control by adding a limited number of candidates for plurality and approval voting, though not stated explicitly in [BTT92, HHR07a], follow immediately from the proofs of the corresponding results for the "unlimited" variant of the problem.)

2. *Each voiced voting system is susceptible to constructive control by deleting candidates.*

3. *Each voiced voting system is susceptible to destructive control by adding candidates.*

Theorem 3.3 ([HHR07a]). 1. *A voting system is susceptible to constructive control by adding candidates if and only if it is susceptible to destructive control by deleting candidates.*

2. *A voting system is susceptible to constructive control by deleting candidates if and only if it is susceptible to destructive control by adding candidates.*

3. *A voting system is susceptible to constructive control by adding voters if and only if it is susceptible to destructive control by deleting voters.*

4. *A voting system is susceptible to constructive control by deleting voters if and only if it is susceptible to destructive control by adding voters.*

Theorem 3.4 ([HHR07a]). 1. *If a voting system is susceptible to constructive control by partition of voters (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*

2. *If a voting system is susceptible to constructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*

3. *If a voting system is susceptible to constructive control by partition of voters in model TE, then it is susceptible to constructive control by deleting voters.*

4. *If a voting system is susceptible to destructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to destructive control by deleting candidates.*

In the following two lemmas, we apply Theorems 3.2, 3.3, and 3.4 to prove that sincere-strategy preference-based approval voting is susceptible to the 22 types of control defined in Section 2.2.

We start with susceptibility to candidate control.

Lemma 3.5. *SP-AV is susceptible to constructive and destructive control by adding candidates (in both the "limited" and the "unlimited" variant of the problem), by deleting candidates, and by partition of candidates (with or without run-off and for each in both tie-handling models, TE and TP).*

Proof. From Theorem 3.2 and the obvious fact that SP-AV is a voiced voting system, it immediately follows that SP-AV is susceptible to constructive control by deleting candidates and to destructive control by adding candidates (in both the "limited" and the "unlimited" variant of the problem).

Now, consider the election (C,V) with candidate set $C = \{a,b,c,d,e,f\}$ and voter collection $V = \{v_1,v_2,\ldots,v_6\}$ and the following partition of C into $C_1 = \{a,c,d\}$ and $C_2 = \{b,e,f\}$:

	(C,V)	is partitioned into	(C_1,V)	and	(C_2,V)
$v_1:$	$a\ b\ c\ \mid\ d\ e\ f$		$a\ c\ \mid\ d$		$b\ \mid\ e\ f$
$v_2:$	$b\ c\ \mid\ a\ d\ e\ f$		$c\ \mid\ a\ d$		$b\ \mid\ e\ f$
$v_3:$	$a\ c\ \mid\ b\ d\ e\ f$		$a\ c\ \mid\ d$		$b\ \mid\ e\ f$
$v_4:$	$b\ a\ c\ \mid\ d\ e\ f$		$a\ c\ \mid\ d$		$b\ \mid\ e\ f$
$v_5:$	$a\ b\ d\ e\ c\ \mid\ f$		$a\ d\ \mid\ c$		$b\ e\ \mid\ f$
$v_6:$	$a\ b\ d\ f\ c\ \mid\ e$		$a\ d\ \mid\ c$		$b\ f\ \mid\ e$

With six approvals, c is the unique winner of (C,V). However, a is the unique winner of (C_1,V), which implies that c is not promoted to the final stage, regardless of whether we use the TE or TP tie-handling rule and regardless of whether we employ a partition of candidates with or without run-off. Thus, SP-AV is susceptible to destructive control by partition of candidates (with or without run-off and for each in both tie-handling models, TE and TP). By Theorem 3.4, SP-AV is also susceptible to destructive control by deleting candidates. By Theorem 3.3 in turn, SP-AV is also susceptible to constructive control by adding candidates (in both the "limited" and the "unlimited" variant of the problem).

Finally, we modify the above election as follows. Let (C,V') be identical to (C,V), except that $V' = \{v_1,v_2,\ldots,v_5,v_7\}$ and v_7 has the sincere approval strategy: $a\ e\ d\ f\ c\ \mid\ b$. Note that a is not the unique winner of (C,V'), as a loses to c by 5 to 6. However, if we partition C into $C_1 = \{a,c,d\}$ and $C_2 = \{b,e,f\}$, then a is the unique winner in (C_1,V') and b is the unique winner in (C_2,V'). Since both subelections have a unique winner, it does not matter whether the TE rule or the TP rule is applied. The final-stage election is $(\{a,b\},V')$ in the case of run-off partition of candidates, and it is $(\{a,b,e,f\},V')$ in the case of partition of candidates. Since a wins against b in the former case by 4 to 2 and in the latter case by 5 to 4 (and e and f do even worse than b in this case), a is the unique winner in both cases. Thus, SP-AV is susceptible to constructive control by partition of candidates (with or without run-off and for each in both models, TE and TP). ❑

We now turn to susceptibility to voter control.

Lemma 3.6. *SP-AV is susceptible to constructive and destructive control by adding voters, by deleting voters, and by partition of voters in both tie-handling models, TE and TP.*

Proof. Consider the election (C,V) with candidate set $C = \{a,b,c,d,e,f\}$ and voter collection $V = \{v_1, v_2, \ldots, v_8\}$ and partition V into $V_1 = \{v_1,v_2,v_3,v_4\}$ and $V_2 = \{v_5,v_6,v_7,v_8\}$. Thus, we change:

	(C,V)	into	(C,V_1)	and	(C,V_2)
$v_1:$	$a\ b\ c \mid d\ e\ f$		$a\ b\ c \mid d\ e\ f$		
$v_2:$	$a\ c \mid b\ d\ e\ f$		$a\ c \mid b\ d\ e\ f$		
$v_3:$	$c\ b\ a\ d \mid e\ f$		$c\ b\ a\ d \mid e\ f$		
$v_4:$	$a\ b \mid d\ e\ c\ f$		$a\ b \mid d\ e\ c\ f$		
$v_5:$	$a\ d\ c \mid b\ e\ f$				$a\ d\ c \mid b\ e\ f$
$v_6:$	$e\ b\ c\ d \mid a\ f$				$e\ b\ c\ d \mid a\ f$
$v_7:$	$d\ e\ c\ f \mid b\ a$				$d\ e\ c\ f \mid b\ a$
$v_8:$	$d\ f \mid b\ a\ c\ e$				$d\ f \mid b\ a\ c\ e$

With six approvals, c is the unique winner of (C,V). However, a is the unique winner of (C,V_1) and d is the unique winner of (C,V_2), which implies that c is not promoted to the final stage, regardless of whether we use the TE or TP tie-handling rule. (In the final-stage election $(\{a,d\},V)$, d wins by 5 to 3.) Thus, SP-AV is susceptible to destructive control by partition of voters in models TE and TP. By Theorem 3.2 and since SP-AV is a voiced system, SP-AV is also susceptible to destructive control by deleting voters. Finally, by Theorem 3.3, SP-AV is also susceptible to constructive control by adding voters.

Now, if we let a and c change their roles in the above election and argument, we see that SP-AV is also susceptible to constructive control by partition of voters in models TE and TP. By Theorem 3.4, susceptibility to constructive control by partition of voters in model TE implies susceptibility to constructive control by deleting voters. Again, by Theorem 3.3, SP-AV is also susceptible to destructive control by adding voters. ❑

3.3 Candidate Control

Theorems 3.7 and 3.10 below show that sincere-strategy preference-based approval voting is fully resistant to candidate control. This result should be contrasted with that of Hemaspaandra, Hemaspaandra, and Rothe [HHR07a], who proved immunity and vulnerability for all cases of candidate control within approval voting (see Table 2). In fact, SP-AV has the same resistances to candidate control as plurality, and we will show that the construction presented in [HHR07a] to prove plurality resistant also works for sincere-strategy preference-based approval voting in all cases of candidate control except one—namely, except for constructive control by deleting candidates. Theorem 3.10 establishes resistance for this one missing case.

All resistance results in this section follow via a reduction from the NP-complete problem Hitting Set (see, e.g., Garey and Johnson [GJ79]): Given a set $B = \{b_1, b_2, \ldots, b_m\}$, a collection $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ of subsets $S_i \subseteq B$, and a positive integer $k \leq m$, does $\mathscr{S}$ have a hitting set of size at most k, i.e., is there a set $B' \subseteq B$ with $\|B'\| \leq k$ such that for each i, $S_i \cap B' \neq \emptyset$?

Note that some of our proofs for SP-AV are based on constructions presented in [HHR07a] to prove the corresponding results for approval voting or plurality, whereas some other of our results require new insights to make the proof work for SP-AV. For completeness, we will present each construction here (even if the modification of a previous construction is rather straightforward), noting the differences to the related previous constructions.

Theorem 3.7. *SP-AV is resistant to all types of constructive and destructive candidate control defined in Section 2.2 except for constructive control by deleting candidates.*

Resistance of SP-AV to constructive control by deleting candidates, which is the missing case in Theorem 3.7, will be shown as Theorem 3.10 below.

The proof of Theorem 3.7 is based on a construction for plurality in [HHR07a], except that only the arguments for *destructive* candidate control are given there (simply because plurality was shown resistant to all cases of constructive candidate control already by Bartholdi, Tovey, and Trick [BTT92] via different constructions). We now provide a short proof sketch of Theorem 3.7 and the construction from [HHR07a] (slightly modified so as to be formally conform with the SP-AV voter representation) in order to (i) show that the same construction can be used to establish all but one resistances of SP-AV to *constructive* candidate control, and (ii) explain why constructive control by deleting candidates (which is missing in Theorem 3.7) does *not* follow from this construction.

Proof Sketch of Theorem 3.7. Susceptibility holds by Lemma 3.5 in each case. The resistance proofs are based on a reduction from Hitting Set and employ Construction 3.8 below, slightly modified so as to be formally conform with the SP-AV voter representation.

Construction 3.8 ([HHR07a]). *Let $(B, \mathscr{S}, k)$ be a given instance of Hitting Set, where $B = \{b_1, b_2, \ldots, b_m\}$ is a set, $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ is a collection of subsets $S_i \subseteq B$, and $k \leq m$ is a positive integer. Define the election (C,V), where $C = B \cup \{c, w\}$ is the candidate set and where V consists of the following voters:*

1. *There are $2(m-k) + 2n(k+1) + 4$ voters of the form: $c \mid w\ B$.*
2. *There are $2n(k+1) + 5$ voters of the form: $w \mid c\ B$.*
3. *For each i, $1 \leq i \leq n$, there are $2(k+1)$ voters of the form: $S_i \mid c\ w\ (B - S_i)$.*
4. *For each j, $1 \leq j \leq m$, there are two voters of the form: $b_j \mid w\ c\ (B - \{b_j\})$.*

Since $score_{(\{c,w\},V)}(c) - score_{(\{c,w\},V)}(w) = 2k(n-1) + 2n - 1$ is positive (because of $n \geq 1$), c is the unique winner of election $(\{c,w\},V)$. The key observation is the following proposition, which can be proven as in [HHR07a].

Proposition 3.9 ([HHR07a]). *1. If $\mathscr{S}$ has a hitting set B' of size k, then w is the unique SP-AV winner of election $(B' \cup \{c,w\}, V)$.*

2. Let $D \subseteq B \cup \{w\}$. If c is not the unique SP-AV winner of election $(D \cup \{c\}, V)$, then there exists a set $B' \subseteq B$ such that

(a) $D = B' \cup \{w\}$,

(b) w is the unique SP-AV winner of election $(B' \cup \{c,w\}, V)$, and

(c) B' is a hitting set of $\mathscr{S}$ of size less than or equal to k.

As an example, the resistance of SP-AV to constructive and destructive control by adding candidates (both in the limited and the unlimited version of the problem) now follows immediately from Proposition 3.9, via mapping the Hitting Set instance $(B, \mathscr{S}, k)$ to the set $\{c,w\}$ of qualified candidates and the set B of spoiler candidates, to the voter collection V, and by having c be the designated candidate in the destructive case and by having w be the designated candidate in the constructive case.

The other cases of Theorem 3.7 can be proven similarly. ❑ Theorem 3.7

Turning now to the missing case mentioned in Theorem 3.7 above: Why does Construction 3.8 not work for constructive control by deleting candidates? Informally put, the reason is that c is the only serious rival of w in the election (C,V) of Construction 3.8, so by simply deleting c the chair could make w the unique SP-AV winner, regardless of whether $\mathscr{S}$ has a hitting set of size k. However, via a different construction, we can prove resistance also in this case.

Theorem 3.10. *SP-AV is resistant to constructive control by deleting candidates.*

Proof. Susceptibility holds by Lemma 3.5. To prove resistance, we provide a reduction from Hitting Set. Let $(B, \mathscr{S}, k)$ be a given instance of Hitting Set, where $B = \{b_1, b_2, \ldots, b_m\}$ is a set, $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ is a collection of subsets $S_i \subseteq B$, and $k < m$ is a positive integer.[7]

Define the election (C,V), where $C = B \cup \{w\}$ is the candidate set and V is the collection of voters. We assume that the candidates in B are in an arbitrary but fixed order, and for each voter below, this order is also used in each subset of B. For example, if $B = \{b_1, b_2, b_3, b_4\}$ and some subset $S_i = \{b_1, b_3\}$ of B occurs in some voter then this voter prefers b_1 to b_3, and so does any other voter whose preference list contains S_i.

V consists of the following $4n(k+1) + 4m - 2k + 3$ voters:

1. For each i, $1 \leq i \leq n$, there are $2(k+1)$ voters of the form: $S_i \mid (B - S_i)\ w$.

2. For each i, $1 \leq i \leq n$, there are $2(k+1)$ voters of the form: $(B - S_i)\ w \mid S_i$.

3. For each j, $1 \leq j \leq m$, there are two voters of the form: $b_j \mid w\ (B - \{b_j\})$.

4. There are $2(m-k)$ voters of the form: $B \mid w$.

[7] Note that if $k = m$ then B is always a hitting set of size at most k (provided that $\mathscr{S}$ contains only nonempty sets—a requirement that doesn't affect the NP-completeness of the problem), and we thus may require that $k < m$.

5. There are three voters of the form: $w \mid B$.

Since for each $b_j \in B$, the difference

$$score_{(C,V)}(w) - score_{(C,V)}(b_j) = 2n(k+1)+3-(2n(k+1)+2+2(m-k)) = 1-2(m-k)$$

is negative (due to $k < m$), w loses to each member of B and so does not win election (C,V).

We claim that $\mathscr{S}$ has a hitting set B' of size k if and only if w can be made the unique SP-AV winner by deleting at most $m-k$ candidates.

From left to right: Suppose $\mathscr{S}$ has a hitting set B' of size k. Then, for each $b_j \in B'$,

$$score_{(B'\cup\{w\},V)}(w) - score_{(B'\cup\{w\},V)}(b_j) = 2n(k+1)+2(m-k)+3-(2n(k+1)+2+2(m-k)) = 1,$$

since the approval line is moved for $2(m-k)$ voters of the third group, thus transferring their approvals from members of $B-B'$ to w. So w is the unique SP-AV winner of election $(B'\cup\{w\},V)$. Since $B'\cup\{w\} = C-(B-B')$, it follows from $\|B\| = m$ and $\|B'\| = k$ that deleting $m-k$ candidates from C makes w the unique SP-AV winner.

From right to left: Let $D \subseteq B$ be any set such that $\|D\| \leq m-k$ and w is the unique SP-AV winner of election $(C-D,V)$. Let $B' = (C-D)-\{w\}$. Note that $B' \subseteq B$ and that we have the following scores in $(B'\cup\{w\},V)$:

$$\begin{aligned}
score_{(B'\cup\{w\},V)}(w) &= 2(n-\ell)(k+1)+2(m-\|B'\|)+3, \\
score_{(B'\cup\{w\},V)}(b_j) &\leq 2n(k+1)+2(k+1)\ell+2+2(m-k) \quad \text{for each } b_j \in B',
\end{aligned}$$

where ℓ is the number of sets $S_i \in \mathscr{S}$ that are not hit by B', i.e., $B' \cap S_i = \emptyset$. Recall that for each i, $1 \leq i \leq n$, all of the $2(k+1)$ voters of the form $S_i \mid (B-S_i)\ w$ in the first voter group have ranked the candidates in the same order. Thus, for each i, $1 \leq i \leq n$, whenever $B' \cap S_i = \emptyset$ one and the same candidate in B' benefits from moving the approval line, namely the candidate occurring first in our fixed ordering of B'. Call this candidate b and note that

$$score_{(B'\cup\{w\},V)}(b) = 2n(k+1)+2(k+1)\ell+2+2(m-k).$$

Since w is the unique SP-AV winner of $(B'\cup\{w\},V)$, w has more approvals than any candidate in B' and in particular more than b. Thus, we have

$$\begin{aligned}
& score_{(B'\cup\{w\},V)}(w) - score_{(B'\cup\{w\},V)}(b) \\
&= 2(n-\ell)(k+1)+2(m-\|B'\|)+3-2n(k+1)-2\ell(k+1)-2-2(m-k) \\
&= 1+2(k-\|B'\|)-4\ell(k+1) > 0.
\end{aligned}$$

Solving this inequality for ℓ, we obtain

$$0 \leq \ell < \frac{1+2(k-\|B'\|)}{4(k+1)} < \frac{4+4k}{4(k+1)} = 1.$$

Thus $\ell = 0$. It follows that $1+2(k-\|B'\|) > 0$, which implies $\|B'\| \leq k$. Thus, B' is a hitting set of size at most k. ❑

3.4 Voter Control

Turning now to control by adding and by deleting voters, it is known from [HHR07a] that approval voting is resistant to constructive control and is vulnerable to destructive control (see Table 2).[8] Their proofs can be modified so as to also apply to sincere-strategy preference-based approval voting.

Theorem 3.11. *SP-AV is resistant to constructive control by adding voters and by deleting voters and is vulnerable to destructive control by adding voters and by deleting voters.*

Proof. Susceptibility holds by Lemma 3.6 in all cases. To prove resistance to constructive control by adding voters (respectively, by deleting voters), the construction of [HHR07a, Thm. 4.43] (respectively, of [HHR07a, Thm. 4.44]) works, modified only by specifying voter preferences consistently with the voters' approval strategies (and, in the deleting-voters case, by adding a dummy candidate who is disapproved and ranked last by every voter in the construction to ensure an admissible AV strategy profile). These constructions provide polynomial-time reductions from the NP-complete problem Exact Cover by Three-Sets (denoted by X3C; see, e.g., Garey and Johnson [GJ79]), which is defined as follows: Given a set $B = \{b_1, b_2, \ldots, b_{3m}\}$, $m > 1$, and a collection $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, does $\mathscr{S}$ have an exact cover for B, i.e., is there a subcollection $\mathscr{S}' \subseteq \mathscr{S}$ such that every element of B occurs in exactly one set in $\mathscr{S}'$?

We now give proof sketches for these two resistance results. In both cases, we start from an X3C instance $(B, \mathscr{S})$ as described above.

In the case of constructive control by adding voters, for given $(B, \mathscr{S})$ we define the election (C, V), with candidate set $C = B \cup \{w\}$ and with V consisting of $m-2$ registered voters each of the form $B \mid w$. Further, we define W to consist of the following n unregistered voters: For each i, $1 \leq i \leq n$, there is one voter of the form $w \;\; S_i \mid (B - S_i)$.

We claim that $\mathscr{S}$ has an exact cover for B if and only if w can be made the unique SP-AV winner by adding at most m voters.

From left to right: Suppose $\mathscr{S}$ contains an exact cover for B. Add the m voters of W corresponding to this exact cover to V. Let $W' \subseteq W$ be the set of unregistered voters thus added. Then $score_{(C, V \cup W')}(w) = m$ and $score_{(C, V \cup W')}(b_i) = m - 1$ for all $1 \leq i \leq 3m$, so w is the unique winner.

From right to left: Let W' be any subset of W such that $\|W'\| \leq m$ and w is the unique winner of the election $(C, V \cup W')$. It follows that $\|W'\| = m$, and each $b_i \in B$ can gain only one point. Thus, the m voters in W' correspond to an exact cover for B.

In the case of constructive control by deleting voters, define the value $\ell_j = \|\{S_i \in \mathscr{S} \mid b_j \in S_i\}\|$ for each j, $1 \leq j \leq 3m$. Define the election (C, V), where $C = B \cup \{w, d\}$ is the set of candidates, w is the distinguished candidate, and V is the following collection of $2n$ voters:

1. For each i, $1 \leq i \leq n$, there is one voter of the form: $S_i \mid (B - S_i) \;\; w \;\; d$.

[8] Procaccia, Rosenschein, and Zohar [PRZ07] proved in their interesting "multi-winner" model (which generalizes Bartholdi, Tovey, and Trick's model [BTT92] by adding a utility function and some other parameters) that approval voting is resistant to constructive control by adding voters. According to Footnote 13 of [HHR07a], this resistance result immediately follows from the corresponding resistance result in [HHR05, HHR07a], essentially due to the fact that lower bounds in more flexible models are inherited from more restrictive models.

2. For each i, $1 \leq i \leq n$, there is one voter of the form: $w\ B_i \mid (B - B_i)\ d$, where $B_i = \{b_j \in B \mid i \leq n - \ell_j\}$.

Note that $score_{(C,V)}(w) = n$ and $score_{(C,V)}(b_j) = n$ for all $b_j \in B$.

We claim that $\mathscr{S}$ has an exact cover for B if and only if w can be made the unique SP-AV winner by deleting at most m voters.

From left to right: Suppose $\mathscr{S}$ contains an exact cover for B. Delete the m voters corresponding to this exact cover. Let $V' \subseteq V$ be the set of voters thus deleted. Then $score_{(C,V-V')}(w) = n$, $score_{(C,V-V')}(d) < n$, and $score_{(C,V-V')}(b_j) = n-1$ for all $b_j \in B$. Thus w is the unique winner.

From right to left: Let V' be any subset of V such that $\|V'\| \leq m$ and w is the unique winner of the election $(C, V - V')$. We can assume that the voters corresponding to V' have disapproved of the distinguished candidate w. Since each candidate $b_j \in B$ must lose at least one point and by our assumption that only voters from the first group have been deleted, it follows that the deleted voters correspond to a cover. Since the number of deleted voters is at most m, they correspond to an exact cover for B.

The polynomial-time algorithms showing that approval voting is vulnerable to destructive control by adding voters and by deleting voters [HHR07a, Thm. 4.24] can be straightforwardly adapted to also work for sincere-strategy preference-based approval voting, since no approval lines are moved in these control scenarios. For completeness, we provide these proofs.

In the case of destructive control by adding voters, the input to the algorithm is an election (C,V), a collection W of additional voters (where each voter v in $V \cup W$ has a sincere AV strategy S_v with $\emptyset \neq S_v \neq C$), a distinguished candidate $c \in C$, and a nonnegative integer ℓ. The output will be either a subset $W' \subseteq W$ of voters such that $\|W'\| \leq \ell$ and adding the voters of W' to V ensures that c is not a unique winner, or it will be "control impossible" if no such subset exists. If $C = \{c\}$ then output "control impossible" and halt, since one candidate is always the unique winner independent of the number of voters. If $\|C\| > 1$ and c is already not the unique SP-AV winner of the election (C,V) then output $W' = \emptyset$ and halt. Otherwise, for each candidate $d \neq c$ define $surplus(c,d) = score(c) - score(d)$. Among all candidates $i \neq c$ such that there exist $surplus_{(C,V)}(c,i)$ voters in W who approve of i and who disapprove of c, let j be one such candidate for which $surplus_{(C,V)}(c,j)$ is minimum. Output the $surplus_{(C,V)}(c,j)$ voters from W who approve of j and disapprove of c. If there is no such candidate j, then output "control impossible" and halt.

In the case of destructive control by deleting voters, the input to the algorithm is an election (C,V) (where each voter $v \in V$ has a sincere AV strategy S_v with $\emptyset \neq S_v \neq C$), a distinguished candidate $c \in C$, and a nonnegative integer ℓ. The output will be either a subset $V' \subseteq V$ of voters such that $\|V'\| \leq \ell$ and deleting the voters of V' from V ensures that c is not a unique winner, or it will be "control impossible" if no such subset exists. If $C = \{c\}$ then again output "control impossible" and halt. If $\|C\| > 1$ and c already is not a unique SP-AV winner of the election (C,V), then output $V' = \emptyset$ and halt. Otherwise, let $j \neq c$ be the candidate for whom $surplus_{(C,V)}(c,j)$ is minimum. If $surplus_{(C,V)}(c,j) > \ell$ then output "control impossible" and halt. Otherwise, output the $surplus_{(C,V)}(c,j)$ voters from V who approve of c and disapprove of j. ❑

We now prove that, just like plurality, sincere-strategy preference-based approval voting is resistant to constructive and destructive control by partition of voters in model TP. In fact, the proof presented in [HHR07a] for plurality in these two cases also works for SP-AV with minor modifications. In contrast, approval voting is vulnerable to the destructive variant of this control type [HHR07a].

Theorem 3.12. *SP-AV is resistant to constructive and destructive control by partition of voters in model TP.*

Proof Sketch of Theorem 3.12. The proof is again based on Construction 3.8, but the reduction is now from Restricted Hitting Set, which is defined just as Hitting Set (see Section 3.3) except that $n(k+1)+1 \leq m-k$ is required in addition. Restricted Hitting Set is also NP-complete [HHR07a]. Now, the key observation is the following proposition, which can be proven as in [HHR07a].

Proposition 3.13 ([HHR07a])**.** *Let $(B,\mathscr{S},k)$ be a given Restricted Hitting Set instance, where $B = \{b_1,b_2,\dots,b_m\}$ is a set, $\mathscr{S} = \{S_1,S_2,\dots,S_n\}$ is a collection of subsets $S_i \subseteq B$, and $k \leq m$ is a positive integer such that $n(k+1)+1 \leq m-k$. If (C,V) is the election resulting from $(B,\mathscr{S},k)$ via Construction 3.8, then the following three statements are equivalent:*

1. *$\mathscr{S}$ has a hitting set of size less than or equal to k.*
2. *V can be partitioned such that w is the unique SP-AV winner in model TP.*
3. *V can be partitioned such that c is not the unique SP-AV winner in model TP.*

The theorem now follows immediately from Proposition 3.13. ❑ Theorem 3.12

Finally, we turn to control by partition of voters in model TE. For this control type, Hemaspaandra et al. [HHR07a] proved approval voting resistant in the constructive case and vulnerable in the destructive case. We have the same results for sincere-strategy preference-based approval voting. Our resistance proof in the constructive case (see the proof of Theorem 3.14) is similar to the corresponding proof of resistance in [HHR07a]. However, while our polynomial-time algorithm showing vulnerability for SP-AV in the destructive case (see the proof of Theorem 3.15) is based on the corresponding polynomial-time algorithm for approval voting in [HHR07a], it extends their algorithm in a nontrivial way.

Theorem 3.14. *SP-AV is resistant to constructive control by partition of voters in model TE.*

Proof Sketch of Theorem 3.14. Susceptibility holds by Lemma 3.6. The proof of resistance is based on the construction of [HHR07a, Thm. 4.46] with only minor changes. Let an X3C instance $(B,\mathscr{S})$ be given, where $B = \{b_1,b_2,\dots,b_{3m}\}$, $m > 1$, is a set and $\mathscr{S} = \{S_1,S_2,\dots,S_n\}$ is a collection of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i. Without loss of generality, we may assume that $n \geq m$. Define the value $\ell_j = \|\{S_i \in \mathscr{S} \mid b_j \in S_i\}\|$ for each j, $1 \leq j \leq 3m$ as in the proof of Theorem 3.11.

Define the election (C,V), where $C = B \cup \{w,x,y\} \cup Z$ is the candidate set with the distinguished candidate w, $Z = \{z_1,z_2,\dots,z_n\}$, and where V is defined to consist of the following $4n+m$ voters:

1. For each i, $1 \leq i \leq n$, there is one voter of the form: $y\ S_i \mid w\ ((B-S_i)\cup\{x\}\cup Z)$.
2. For each i, $1 \leq i \leq n$, there is one voter of the form: $y\ z_i \mid w\ (B\cup\{x\}\cup(Z-\{z_i\}))$.

3. For each i, $1 \leq i \leq n$, there is one voter of the form: $w\ (Z - \{z_i\})\ B_i\ |\ x\ y\ z_i\ (B - B_i)$, where $B_i = \{b_j \in B \,|\, i \leq n - \ell_j\}$.

4. There are $n+m$ voters of the form: $x\ |\ y\ (B \cup \{w\} \cup Z)$.

Since the above construction is only slightly modified from the proof of [HHR07a, Thm. 4.46], so as to be formally conform with the SP-AV voter representation, the same argument as in that proof shows that $\mathscr{S}$ has an exact cover for B if and only if w can be made the unique SP-AV winner by partition of voters in model TE. Note that, in the present control scenario, approval voting and SP-AV can differ only in the run-off, but the construction ensures that they don't differ there.

From left to right, if $\mathscr{S}$ has an exact cover for B then partition the set of voters as follows: V_1 consists of the m voters of the form $y\ S_i\ |\ w\ ((B - S_i) \cup \{x\} \cup Z)$ that correspond to the sets in the exact cover, of the $n+m$ voters who approve of only x, and of the n voters who approve of y and z_i, $1 \leq i \leq n$. Let $V_2 = V - V_1$. It follows that w is the unique SP-AV winner of both subelection (C, V_2) and the run-off, simply because no candidate proceeds to the run-off from the other subelection, (C, V_1), in which x and y tie for winner with a score of $n+m$ each.

From right to left, suppose w can be made the unique SP-AV winner by partition of voters in model TE. Let (V_1, V_2) be a partition of V such that w is the unique SP-AV winner of the run-off. According to model TE, w must also be the unique SP-AV winner of one subelection, say of (C, V_1). Note that each voter of the form $y\ z_i\ |\ w\ (B \cup \{x\} \cup (Z - \{z_i\}))$ has to be in V_2 (otherwise, we would have $score_{(C,V_1)}(w) = score_{(C,V_1)}(z_i)$ for at least one i, and so w would not be the unique SP-AV winner of (C, V_1) anymore). However, if there were more than m voters of the form $y\ S_i\ |\ w\ ((B - S_i) \cup \{x\} \cup Z)$ in V_2 then $score_{(C,V_2)}(y) > n+m$, and so y would be the unique SP-AV winner of the other subelection, (C, V_2). But then, also in the SP-AV model, y would win the run-off against w because $score_{(\{w,y\},V)}(y) = 3n+m > n = score_{(\{w,y\},V)}(w)$, which contradicts the assumption that w has been made the unique SP-AV winner by the partition (V_1, V_2). Hence, there are at most m voters of the form $y\ S_i\ |\ w\ ((B - S_i) \cup \{x\} \cup Z)$ in V_2, and these m voters correspond to an exact cover of B. ❑

Theorem 3.15. *SP-AV is vulnerable to destructive control by partition of voters in model TE.*

Proof. Susceptibility holds by Lemma 3.6. To prove vulnerability, we describe a polynomial-time algorithm showing that (and how) the chair can exert destructive control by partition of voters in model TE for sincere-strategy preference-based approval voting. Our algorithm extends the polynomial-time algorithm designed by Hemaspaandra et al. [HHR07a] to prove approval voting vulnerable to this type of control. Specifically, our algorithm adds Loop 2 below to their algorithm, and we will explain below why it is necessary to add this second loop.

We adopt the following notation from [HHR07a]. Let (C, V) be an election, and for each voter $v \in V$, let $S_v \subseteq C$ denote v's AV strategy. In each iteration of Loop 1 in the algorithm below, we will consider three candidates, a, b, and c. Define the following five numbers:

$$W_c = \|\{v \in V \,|\, a \notin S_v,\ b \notin S_v,\ c \in S_v\}\|, \qquad L_c = \|\{v \in V \,|\, a \in S_v,\ b \in S_v,\ c \notin S_v\}\|,$$
$$D_a = \|\{v \in V \,|\, a \in S_v,\ b \notin S_v,\ c \notin S_v\}\|, \qquad D_b = \|\{v \in V \,|\, a \notin S_v,\ b \in S_v,\ c \notin S_v\}\|, \text{ and}$$
$$D_{ac} = \|\{v \in V \,|\, a \in S_v,\ b \notin S_v,\ c \in S_v\}\|.$$

In addition, we introduce the following notation. Given an election (C,V) and two distinct candidates $x,y \in C$, let $\mathit{diff}(x,y)$ denote the number of voters in V who prefer x to y minus the number of voters in V who prefer y to x. Define B_x to be the set of candidates $y \neq x$ in C such that $\mathit{diff}(y,x) \geq 0$.

The input to our algorithm is an election (C,V), where each voter $v \in V$ has a sincere AV strategy S_v with $\emptyset \neq S_v \neq C$ (otherwise, the input is considered malformed and outright rejected), and a distinguished candidate $c \in C$. On this input, our algorithm works as follows.

1. **Checking the trivial cases:** can be done as in the case of approval voting, see the proof of [HHR07a, Thm. 4.21]. In particular, if $C = \{c\}$ then output "control impossible" and halt, since c cannot help but win. If C contains more candidates than only c but c already is not the unique SP-AV winner in (C,V) then output the (successful) partition $(V,\emptyset)$ and halt. Otherwise, if $\|C\| = 2$ then output "control impossible" and halt, as c is the unique SP-AV winner of (C,V) in the current case and so, however the voters are partitioned, c must win—against the one rivalling candidate—at least one subelection and also the run-off.

2. **Loop 1:** For each $a,b \in C$ such that $\|\{a,b,c\}\| = 3$, check whether V can be partitioned into V_1 and V_2 such that $\mathit{score}_{(C,V_1)}(a) \geq \mathit{score}_{(C,V_1)}(c)$ and $\mathit{score}_{(C,V_2)}(b) \geq \mathit{score}_{(C,V_2)}(c)$. As shown in the proof of [HHR07a, Thm. 4.21], this is equivalent to checking

$$W_c - L_c \leq D_a + D_b. \tag{3.1}$$

   If (3.1) fails, this a and b cannot prevent c from being the unique winner of at least one subelection and thus also of the run-off, so we move on to test the next a and b in this loop. If (3.1) holds, however, output the partition (V_1,V_2) and halt, where V_1 consists of the voters contributing to D_a, of the voters contributing to D_{ac}, and of $\min(W_c,D_a)$ voters contributing to W_c, and where $V_2 = V - V_1$.

3. **Loop 2:** For each $d \in B_c$, partition V as follows. Let V_1 consist of all voters in V who approve of d, and let $V_2 = V - V_1$. If d is the unique winner of (C,V_1), then output (V_1,V_2) as a successful partition and halt. Otherwise, go to the next $d \in B_c$.

4. **Termination:** If in no iteration of either Loop 1 or Loop 2 a successful partition of V was found, then output "control impossible" and halt.

Let us give a short explanation of why Loop 2 is needed for SP-AV by stressing the difference with approval voting. As shown in the proof of [HHR07a, Thm. 4.21], if none of the trivial cases applied, then condition (3.1) holds for some $a,b \in C$ with $\|\{a,b,c\}\| = 3$ if and only if destructive control by partition of voters in model TE is possible for approval voting. Thus, for approval voting, if Loop 1 was not successful for any such a and b, we may immediately jump to the termination stage, where the algorithm outputs "control impossible" and halts. In contrast, if none of the trivial cases applied, then the existence of candidates a and b with $\|\{a,b,c\}\| = 3$ who satisfy (3.1) is *not* equivalent to destructive control by partition of voters in model TE being possible for SP-AV: it is a sufficient, yet not a necessary condition. The reason is that even if there are no candidates a and b

who can prevent c from winning one subelection (in some partition of voters) and from proceeding to the run-off, it might still be possible that c loses or ties the run-off due to our rule of moving the approval line in order to re-enforce our conventions for SP-AV in this control scenario.

Indeed, if Loop 1 was not successful, c will lose or tie the run-off exactly if there exists a candidate $d \neq c$ such that $\mathit{diff}(d,c) \geq 0$ and d can win one subelection (for some partition of voters). This is precisely what is being checked in Loop 2. Indeed, note that the partition (V_1, V_2) chosen in Loop 2 for $d \in B_c$ is the best possible partition for d in the following sense: If d is not the unique SP-AV winner of subelection (C, V_1) then, for each $W \subseteq V$, d is not the unique SP-AV winner of subelection (C, W). To see this, simply note that if d is not the unique SP-AV winner of (C, V_1), then there is some candidate x with $\mathit{score}_{(C,V_1)}(x) = \mathit{score}_{(C,V_1)}(d) = \|V_1\|$, which by our choice of V_1 implies $\mathit{score}_{(C,W)}(x) \geq \mathit{score}_{(C,W)}(d)$ for each subset $W \subseteq V$. ❑

4 Conclusions and Open Questions

We have shown that Brams and Sanver's sincere-strategy preference-based approval voting system [BS06] combines the resistances of approval and plurality voting to procedural control: SP-AV is resistant to 19 of the 22 previously studied types of control. On the one hand, like Copeland voting [FHHR08], SP-AV is fully resistant to constructive control, yet unlike Copeland it additionally is broadly resistant to destructive control. On the other hand, like plurality [BTT92, HHR07a], SP-AV is fully resistant to candidate control, yet unlike plurality it additionally is broadly resistant to voter control. Thus, for these 22 types of control, SP-AV has more resistances, by three, and fewer vulnerabilities to control than is currently known for any other natural voting system with a polynomial-time winner problem.

As a work in progress, we are currently expanding our study of SP-AV's behavior with respect to procedural control towards other areas of computational social choice. In particular, our goal is to determine the complexity of manipulation [BTT89a] and bribery [FHH06] within SP-AV, in a variety of scenarios. In addition, we propose as an interesting and extremely ambitious task for future work the study of SP-AV (and other voting systems as well) beyond the worst-case—as we have done here—and towards an appropriate typical-case complexity model; see, e.g., [MPS08, PR07, CS06, HH, EHRS07] for interesting results and discussion in this direction.

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