# A New Upper Bound for Max-2-SAT: A Graph-Theoretic Approach 

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#### Abstract

In MaxSat, we ask for an assignment which satisfies the maximum number of clauses for a boolean formula in CNF. We present an algorithm yielding a run time upper bound of $\mathcal{O}^{*}\left(2^{\frac{1}{6.2158}}{ }^{K}\right)$ for MAX-2-Sat (each clause contains at most 2 literals), where $K$ is the number of clauses. The run time has been achieved by using heuristic priorities on the choice of the variable on which we branch. The implementation of these heuristic priorities is rather simple, though they have a significant effect on the run time. The analysis is done using a tailored non-standard measure.


## 1 Introduction

Our Problems. MaxSat is an optimization version of the well-known decision problem SAT: given a boolean formula in CNF, we ask for an assignment which satisfies the maximum number of clauses. The applications for MaxSat range over such fields as combinatorial optimization, artificial intelligence and database-systems as mentioned in [6]. We put our focus on Max-2-Sat, where every formula is constrained to have at most two literals per clause, to which problems as Maximum Cut and Maximum Independent Set are reducible. Therefore Max-2-Sat is $\mathcal{N} \mathcal{P}$-complete.
Results So Far. Max-2-Sat: The best published upper bound of $\mathcal{O}^{*}\left(2^{\frac{1}{5.88}} \mathrm{~K}\right)$ has been achieved by Kulikov and Kutzov in [7] consuming only polynomial space. They build up their algorithm on the one of Kojevnikov and Kulikov [6]. They were the first who used a non-standard measure yielding a run time of $\mathcal{O}^{*}\left(2^{\frac{1}{5.5}}\right)$. If we measure the complexity in the number $n$ of variables the current fastest algorithm is the one of R. Williams [10] having run time $\mathcal{O}^{*}\left(2^{\frac{\omega}{3} n}\right)$, where $\omega<2.376$ is the matrix-multiplication exponent. A drawback of this algorithm is its requirement of exponential space. Fuerer et al. [1] presented a $\mathcal{O}^{*}\left(2^{1-\frac{1}{d-1} n}\right)$ algorithm consuming polynomial space, where $d$ is the average degree of the variable graph. MAX-2-SAT has also been studied with respect to approximation [4|8] and parameterized algorithms [23].
Our Results. The major result we present is an algorithm solving Max-2-Sat in time $\mathcal{O}^{*}\left(2^{\frac{1}{6.2158}}{ }^{K}\right)$. Basically it is a refinement of the algorithm in [6], which also in turn builds up on the results of [2]. The run time improvement is twofold.

In [6] an upper bound of $\mathcal{O}^{*}\left(1.1225^{n}\right)$ is obtained if the variable graph is cubic. Here $n$ dentoes the number of variables. We could improve this to $\mathcal{O}^{*}\left(1.11199^{n}\right)$ by a more accurate analysis. Secondly, in the case where the maximum degree of the variable graph is four, we chose a variable for branching according to some heuristic priorities. These two improvements already give a run time of $\mathcal{O}^{*}\left(2^{\frac{1}{6.137}} K\right)$. Moreover we like to point out that these heuristic priorities can be implemented such that the only consume $\mathcal{O}(n)$ time. The authors of [7] improve the algorithm of [6] by having a new branching strategy when the variable graph has maximum degree five. Now combining our improvements with the ones from [7] gives the claimed run time.
Basic Definitions and Terminology. Let $V(F)$ be the set of variables of a given boolean formula $F$. For a variable $v$ by $\bar{v}$ we denote the negation of $v$. If a variable $v$ is set, then it will be assigned the values true or false. By the word literal, we refer to a variable or its negation. A clause is a disjunction of literals. We consider formulas in conjunctive normal form ( $C N F$ ), that is a conjunction of clauses. We allow only 1- and 2-clauses, i.e., clauses with at most two literals. The weight of $v$, written $\#_{2}(v)$, refers to the number of 2clauses in which the variable $v$ or its negation occurs. For a set $U$ of variables we define $\#_{2}(U):=\sum_{u \in U} \#_{2}(u)$. If $v$ occurs in some clause $C$ we write $v \in C$. A set $A$ of literals is called assignment if for every $v \in A$ it holds that $\bar{v} \notin A$. Loosely speaking if $l \in A$ for a literal $l$, than $l$ receives the value true. We allow the formula to contain truth-clauses of the form $\{\mathcal{T}\}$ that are always satisfied. Furthermore, we consider a MAX-2-SAT instance as multiset of clauses. A variable $x$ is a neighbor of $v$, written $x \in N(v)$, if they occur in a common 2-clause. Let $N[v]:=N(v) \cup\{v\}$. The variable graph $G_{v a r}(V, E)$ is defined as follows: the vertices are represented by the variables $V(F)$ of the given formula $F$. Between two vertices is an undirected edge if they are neighbors in the formula. Observe that $G_{v a r}$ is a undirected multigraph and that it neglects clauses of size one. We will not distinguish between the words "variable" and "vertex". Every variable in a formula corresponds to a vertex in $G_{v a r}$ and vice versa. By writing $F[v]$, we mean the formula which emerges from $F$ by setting $v$ to true the following way: First, substitute all clauses containing $v$ by $\{\mathcal{T}\}$, then delete all occurrences of $\bar{v}$ from any clause and finally delete all empty clauses from $F$. $F[\bar{v}]$ is defined analogously: we set $x$ to false.

## 2 Reduction Rules \& Basic Observations

We state well-known reduction rules from previous work [26]:
RR-1 Replace any 2-clause $C$ with $l, \bar{l} \in C$, for a literal $l$, with $\{\mathcal{T}\}$.
RR-2 If for two clauses $C, D$ and a literal $l$ we have $C \backslash\{l\}=D \backslash\{\bar{l}\}$, then substitute $C$ and $D$ by $C \backslash\{l\}$ and $\{\mathcal{T}\}$.
RR-3 A literal $l$ occurring only positively (negatively, resp.) is set to true (false).
RR-4 If $\bar{l}$ does not occur in more 2-clauses than $l$ in 1-clauses, such that $l$ is a literal, then set $l$ to true.

RR-5 Let $x_{1}$ and $x_{2}$ be two variables, such that $x_{1}$ appears at most once in another clause without $x_{2}$. In this case, we call $x_{2}$ the companion of $x_{1}$. RR- $\mathbf{3}$ or RR-4 will set $x_{1}$ in $F\left[x_{2}\right]$ to $\alpha$ and in $F\left[\bar{x}_{2}\right]$ to $\beta$, where $\alpha, \beta \in\{$ true, false $\}$. Depending on $\alpha$ and $\beta$, the following actions will be carried out:

- If $\alpha=$ false, $\beta=$ false, set $x_{1}$ to false.
- If $\alpha=$ true, $\beta=$ true, set $x_{1}$ to true.
- If $\alpha=$ true, $\beta=$ false, substitute every occurrence of $x_{1}$ by $x_{2}$.
- If $\alpha=$ false, $\beta=$ true, substitute every occurrence of $x_{1}$ by $\bar{x}_{2}$.

From now on we will only consider reduced formulas $F$. This means that to a given formula $F$ we apply the following procedure: RR-i is always applied before $\mathbf{R R} \mathbf{- i}+\mathbf{1}$, each reduction rule is carried out exhaustively and after $\mathbf{R R} \mathbf{- 5}$ we start again with RR-1 if the formula changed. Concerning the reduction rules we have the following lemma:
Lemma 1. 1. If $\#_{2}(v)=1$, then $v$ will be set.
2. Let $v \in V(F)$ such that $\#_{2}(v)=3$ and $u \in V(F)$ is its companion, i.e., there are clauses $C_{1}, C_{2}, C_{3}$ with $u, v \in C_{1}, C_{2}$ and $v, z \in C_{3}$ for $z \in V(F)$, $z \neq u$. Then either $v$ is set or the common edges of $u$ and $v$ will be contracted in $G_{v a r}$.
3. After the exhaustive application of the reduction rules, we have:
(a) Any variable $u$ with $\#_{2}(u)=3$ has at least three different neighbors.
(b) If the variables $a$ and $x$ are neighbors and $\#_{2}(a)=3$, then in at least one of the formulas $F[x]$ and $F[\bar{x}]$, the reduction rules set $a$.

Proof. 1. After applying RR-2 exhaustively, either RR-3 or RR-4 applies.
2. If $v$ is not set it will be substituted by $u$ or $\bar{u}$ due to RR-5. The emerging clauses $C_{1}, C_{2}$ will be reduced either by RR-1 or become 1-clauses. Also we have an edge between $u$ and $z$ in $G_{v a r}$ as now $u, z \in C_{3}$.
3. (a) Otherwise, RR-5 can be applied. (b) See Lemma 3.2.5 in 6].

We need some auxiliary notions: A sequence of vertices $a_{1}, v_{1}, \ldots, v_{j}, a_{2}$
$(j \geq 0)$ is called lasso if $\# 2\left(v_{i}\right)=2$ for $1 \leq i \leq j, a_{1}=a_{2}, \#_{2}\left(a_{1}\right) \geq 3$ and $G_{v a r}\left[a_{1}, v_{1}, \ldots, v_{j}, a_{2}\right]$ is a cycle. A quasi-lasso is a lasso with the difference that $\#_{2}\left(v_{j}\right)=3$. A lasso is called 3-lasso (resp. 4-lasso) if $\#_{2}\left(a_{1}\right)=3\left(\#_{2}\left(a_{1}\right)=4\right.$, resp.). 3-quasi-lasso and 4-quasi-lasso are defined analogously.
Observation 1. The reduction rules delete the variables $v_{1}, \ldots, v_{j}$ of a lasso (quasi-lasso, resp.) and the weight of $a_{1}$ drops by at least two (one, resp.).
Proof. We give the proof by induction on $j$. In the lasso case for $j=0$, there must be a 2-clause $C=\left\{a_{1}, \bar{a}_{1}\right\}$, which will be deleted by RR-1, so that the initial step is shown. So now $j>0$. Then on any $v_{i}, 1 \leq i \leq j$, we can apply RR-5 with any neighbor as companion, so, w.l.o.g., it is applied to $v_{1}$ with $a_{1}$ as companion. RR-5 either sets $v_{1}$, then we are done with Lemma 111, or $v_{1}$ will be substituted by $a_{1}$. By applying RR-1, this leads to the lasso $a_{1}, v_{2}, \ldots, v_{j}, a_{2}$ in $G_{v a r}$ and the claim follows by induction. In the quasi-lasso case for $j=0$, the arguments from above hold. For $j=1$, Lemma 12 is sufficient. For $j>1$, the induction step from above also applies here.

## 3 The Algorithm

We set $d_{i}(F):=\left|\left\{x \in V(F) \mid \#_{2}(x)=i\right\}\right|$. To measure the run time, we choose a non standard measure approach with the measure $\gamma$ defined as follows:

$$
\gamma(F)=\sum_{i=3}^{n} \omega_{i} \cdot d_{i}(F) \text { with } \omega_{3}=0.9398, \omega_{4}=1.8077, \omega_{i}=\frac{i}{2} \text { for } i \geq 5
$$

Clearly, $\gamma(F)$ never exceeds the number of clauses $K$ in the corresponding formula. So, by showing an upper bound of $c^{\gamma(F)}$ we can infer an upper bound $c^{K}$. We set $\Delta_{3}:=\omega_{3}, \Delta_{i}:=w_{i}-w_{i-1}$ for $i \geq 4$. Concerning the $\omega_{i}$ 's we have $\Delta_{i} \geq \Delta_{i+1}$ for $i \geq 3$ and $\omega_{4} \geq 2 \cdot \Delta_{4}$. The algorithm presented in this paper proceeds as follows: After applying the above-mentioned reduction rules exhaustively, it will branch on a variable $v$. That is, we will reduce the problem to the two formulas $F[v]$ and $F[\bar{v}]$. In each of the two branches, we must determine by how much the original formula $F$ will be reduced in terms of $\gamma(F)$. Reduction in $\gamma(F)$ can be due to branching on a variable or to the subsequent application of reduction rules. By an $\left(a_{1}, \ldots, a_{\ell}\right)$-branch, we mean that in the $i$-th branch $\gamma(F)$ is reduced by at least $a_{i}$. We will also speak of the $i$-th component of a branch when we mean the search tree evolving from the $i$-th branch (i.e., $a_{i}$ ). By writing $\left(\left\{a_{1}\right\}^{i_{1}}, \ldots,\left\{a_{\ell}\right\}^{i_{\ell}}\right)$-branch we mean a $\left(a_{1}^{1}, \ldots, a_{1}^{i_{1}}, \ldots, a_{\ell}^{1}, \ldots, a_{\ell}^{i_{\ell}}\right)$-branch where $a_{j}^{s}=a_{j}$ with $1 \leq s \leq i_{j}$. A $\left(a_{1}, \ldots, a_{\ell}\right)$-branch dominates a $\left(b_{1}, \ldots, b_{\ell}\right)$-branch if $a_{i} \geq b_{i}$ for $1 \leq i \leq \ell$.
Heuristic Priorities If the maximum degree of $G_{v a r}$ is four, variables $v$ with $\#_{2}(v)=4$ will be called limited if there is another variable $u$ appearing with $v$ in two 2 -clauses (i.e., we have two edges between $v$ and $u$ in $G_{v a r}$ ). We call such $u, v$ a limited pair. Note that also $u$ is limited and that at this point by RR-5 no two weight 4 variables can appear in more than two clauses together. We call $u_{1}, \ldots u_{\ell}$ a limited sequence if $\ell \geq 3$ and $u_{i}, u_{i+1}$ with $1 \leq i \leq \ell-1$ are limited pairs. A limited cycle is a limited sequence with $u_{1}=u_{\ell}$. To obtain an asymptotically fast algorithmic behavior we introduce heuristic priorities (HP), concerning the choice of the variable used for branching.

1. Choose any $v$ with $\#_{2}(v) \geq 7$.
2. Choose any $v$ with $\#_{2}(v)=6$, preferably with $\#_{2}(N(v))<36$.
3. Choose any $v$ with $\#_{2}(v)=5$, preferably with $\#_{2}(N(v))<25$.
4. Choose any unlimited $v$ with $\#_{2}(v)=4$ and a limited neighbor.
5. Choose the vertex $u_{1}$ in a limited sequence or cycle .
6. Pick a limited pair $u_{1}, u_{2}$. Let $c \in N\left(u_{1}\right) \backslash\left\{u_{2}\right\}$ with $s(c):=\mid\left(N(c) \cap N\left(u_{1}\right)\right) \backslash$ $\left\{c, u_{1}\right\} \mid$ maximal. If $s(c)>1$, then choose the unique vertex in $N\left(u_{1}\right) \backslash\left\{u_{2}, c\right\}$, else choose $u_{1}$.
7. From $Y:=\left\{v \in V(F) \mid \#_{2}(v)=4, \exists z \in N(v): \not \#_{2}(z)=3 \wedge N(z) \nsubseteq N(v)\right\}$ choose $v$, preferably such that $\#_{2}(N(v))$ is maximal.
8. Choose any $v$, with $\#_{2}(v)=4$, preferably with $\#_{2}(N(v))<16$.
9. Choose any $v$, with $\#_{2}(v)=3$, such that there is $a \in N(v)$, which forms a triangle $a, b, c$ and $b, c \notin N[v]$ (we say $v$ has pending triangle $a, b, c$ ).
10. Choose any $v$, such that we have a $\left(6 \omega_{3}, 8 \omega_{3}\right)$ - or a $\left(4 \omega_{3}, 10 \omega_{3}\right)$-branch.

From now on $v$ denotes the variable picked according to HP.
Key Ideas The main idea is to have some priorities on the choice of a weight 4 variable such that the branching behavior is beneficial. For example limited variables tend to be unstable in the following sense: If their weight is decreased due to branching they will be reduced due to Lemma 112. This means we can get an amount of $\omega_{4}$ instead of $\Delta_{4}$. In a graph lacking limited vertices we want a variable $v$ with a weight 3 neighbor $u$ such that $N(u) \nsubseteq N(v)$. In the branch on $v$ where $u$ is set (Lemma 13b) we can gain some extra reduction (at least $\Delta_{4}$ ) from $N(u) \backslash N(v)$. If we fail to find a variable according to priorities 5-7 we show that either $v$ as four weight 4 variables and that the graph is four regular, or otherwise we have two distinct situations which can be handled quite efficiently. Further, the most critical branches are when we have to chose $v$ such that all variables in $N[v]$ have weight $\omega_{i}$. Then the reduction in $\gamma(F)$ is minimal (i.e., $\left.w_{i}+i \cdot \Delta_{i}\right)$. We analyze this regular case together with its immediately preceding branch. Thereby we prove a better branching behavior compared to a separate analysis. Our algorithm proceeds as follows:

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Algorithm 1 An algorithm for solving MAX-2-SAT.
Procedure: SolMax2Sat \((F)\)
    Apply SolMax2Sat on every component of \(G_{v a r}\) separately.
    Apply the reduction rules exhaustively to \(F\).
    Search exhaustively on any sub-formula being a component of at most 9 variables.
    if \(F=\{\mathcal{T}\} \ldots\{\mathcal{T}\}\) then
        return \(|F|\)
    else
        Choose a variable \(v\) according to HP.
        return \(\max \{\operatorname{SolMax} 2 \operatorname{SAT}(F[v])\), \(\operatorname{SolMax} 2 \operatorname{Sat}(F[\bar{v}])\}\).
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## 4 The Recurrences and their Analysis

We now investigate the cases when we branch on vertices picked according to items 1-8 of HP. For each item we will derive a branching vector which upper bounds this case in terms of $K$.

### 4.1 The Case when the Maximum Degree of $G_{v a r}$ is at least four

Priority 1 Choosing a variable $v$ with $\#_{2}(v) \geq 7$, we first obtain a reduction of $\omega_{7}$ because $v$ will be deleted. Secondly, we get an amount of at least $7 \cdot \Delta_{7}=3.5$ as the weights of the neighboring variables each drops by at least one and we have $\Delta_{i} \geq \Delta_{i+1}$. Therefore, the measure $\gamma$ is reduced by at least 7 in either of the two branches (i.e., we have a $\left(\{7\}^{2}\right)$-branch).

Regular Branches We call a branch $h$-regular if we branch on a variable $v$ such that for all $u \in N[v]$ we have $\#_{2}(u)=h$. We will handle those in a separate part. The following subsections handle non-regular branches, which means that we can find a $u \in N(v)$ with $\#_{2}(u)<\#_{2}(v)$. Observe that we already handled the $h$-regular branches for $h \geq 7$.
Priorities 2 and 3 Choosing $v \in V(F)$ with $\#_{2}(v)=6$ there is a $u \in N(v)$ with $\#_{2}(u) \leq 5$ due to non-regularity. Then by deletion of $v$, there is a reduction by $\omega_{6}$ and another of at least $5 \Delta_{6}+\Delta_{5}$, resulting from the dropping weights of the neighbors. Especially, the weight of $u$ must drop by at least $\Delta_{5}$. This leads to a $\left(\{6.1923\}^{2}\right)$-branch. Choosing a variable $v$ with $\#_{2}(v)=5$, the same observations as in the last choice lead to a reduction of at least $\omega_{5}+4 \cdot \Delta_{5}+\Delta_{4}$. Thus we have a $\left(\{6.1371\}^{2}\right)$-branch.
Priority 4 Let $u_{1} \in N(v)$ be the limited variable. $u_{1}$ appears together with the weight 4 variable $u_{2}$ in two 2 -clauses. Since priorities 1-3 did not apply we have $\#_{2}\left(u_{1}\right)=4$. After branching on $v$, the variable $u_{1}$ has weight at most 3 . At this point, $u_{1}$ appears only with one other variable $z$ in a 2 -clause. Then, RR- 5 is applicable to $u_{1}$ with $u_{2}$ as its companion. According to Lemma 112, either $u_{1}$ is set or the common edges of $u_{1}$ and $u_{2}$ will be contracted. In the first case, we receive a total reduction of at least $3 \omega_{4}+2 \Delta_{4}$, in the second of $2 \omega_{4}+4 \Delta_{4}$. We can estimate this case by a $\left(\left\{2 \omega_{4}+4 \Delta_{4}\right\}^{2}\right)$-branch, i.e., a $\left(\{7.087\}^{2}\right)$-branch.
Priority 5 If $u_{1}, \ldots, u_{\ell}$ is a limited cycle, then $\ell \geq 10$. Otherwise it is a component of at most 9 variables which are solved directly beforehand. By RR-5 this yields a $\left(10 w_{4}, 10 w_{4}\right)$-branch. If $u_{1}, \ldots, u_{\ell}$ is a limited sequence, then due to priority 4 the neighbors of $u_{1}, u_{\ell}$ lying outside the sequence have weight 3 . By RR-5 the branch on $u_{1}$ is a $\left(\left\{3 \omega_{4}+2 \omega_{3}\right\}^{2}\right)$-branch. A $\left(\{7.3027\}^{2}\right)$-branch upper bounds both.
Priority 6 At this point every limited variable $u_{1}$ has two neighboring variables $y, z$ with weight 3 and a limited neighbor $u_{2}$ with the same properties (due to priorities 4 and 5). We now examine the local structures arising from this fact and by the values of $\left|N(y) \backslash N\left(u_{1}\right)\right|$ and $\left|N(z) \backslash N\left(u_{1}\right)\right|$.

1. $\left|N(y) \backslash N\left(u_{1}\right)\right|=0,\left|N(z) \backslash N\left(u_{1}\right)\right|=0: N\left(u_{1}\right)$ is a component, which will be solved directly beforehand.
2. $\left|N(y) \backslash N\left(u_{1}\right)\right|=0,\left|N(z) \backslash N\left(u_{1}\right)\right|=1$ : Then, $N(y)=\left\{u_{2}, z, u_{1}\right\}, N\left(u_{2}\right)=$ $\left\{u_{1}, y, s_{1}\right\}$ and $N(z)=\left\{u_{1}, y, s_{2}\right\}$, see Figure 1(a). In this case we branch on $z$ as $s(y)>0$ and $s(y)>s(z)$. Then due to RR-5 $y$ and $u_{1}$ disappear; either by being set or replaced. Thereafter due to RR-1 and Lemma $11 u_{2}$ will be set. Additionally we get an amount of $\min \left\{2 \Delta_{4}, \omega_{4}, \omega_{3}+\Delta_{4}\right\}$ from $s_{1}, s_{2}$. This depends on whether $u_{1} \neq u_{2}$ or $u_{1}=u_{2}$ and in the second case on the weight of $u_{1}$. In total we have at least a $\left(2 \omega_{4}+2 \omega_{3}+2 \Delta_{4}, 2 \omega_{4}+2 \omega_{3}+2 \Delta_{4}\right)$ branch. Analogously the case $\left|N(y) \backslash N\left(u_{1}\right)\right|=1,\left|N(z) \backslash N\left(u_{1}\right)\right|=0$ can be upper bounded.
3. $\left|N(y) \backslash N\left(u_{1}\right)\right|=1,\left|N(z) \backslash N\left(u_{1}\right)\right|=1$ : Here two possibilities occur:
(a) $N(y)=\left\{u_{1}, u_{2}, s_{1}\right\}, N(z)=\left\{u_{1}, u_{2}, s_{2}\right\}$ and $N\left(u_{2}\right)=\left\{u_{1}, y, z\right\}$, see Figure 1(b) Then w.l.o.g., we branch on $z$. Similarly to item 2 . we obtain a $\left(2 \omega_{4}+2 \omega_{3}+2 \Delta_{4}, 2 \omega_{4}+2 \omega_{3}+2 \Delta_{4}\right)$-branch.
(b) $N(y)=\left\{u_{1}, z, s_{1}\right\}$ and $N(z)=\left\{u_{1}, y, s_{2}\right\}$, see Figure 1(c) W.l.o.g., we branch on $z$. Basically we get a total reduction of $2 \omega_{3}+\omega_{4}+2 \Delta_{4}$. That is $2 \omega_{3}$ from $y$ and $z, \omega_{4}$ from $u_{1}$ and $2 \Delta_{4}$ from $s_{2}$ and $u_{2}$. In the branch where $y$ is set (due to Lemma 13b) we additionally get $\Delta_{4}$ from $s_{1}$ and $\omega_{4}$ from $u_{2}$ as it will disappear due to Lemma 13a. This is a $\left(2 \omega_{4}+2 \omega_{3}+2 \Delta_{4}, 2 \omega_{3}+\omega_{4}+2 \Delta_{4}\right)$-branch.
4. $\left|N(y) \backslash N\left(u_{1}\right)\right|=1,\left|N(z) \backslash N\left(u_{1}\right)\right|=2$, see Figure $1(\mathrm{~d})$ We branch on $z$ yielding a $\left(\left\{2 \omega_{4}+2 \omega_{3}+2 \Delta_{4}\right\}^{2}\right)$-branch.
5. $\left|N(y) \backslash N\left(u_{1}\right)\right|=2,\left|N(z) \backslash N\left(u_{1}\right)\right|=2$ : In this case we chose $u_{1}$ for branching. Essentially we get a reduction of $2 \omega_{4}+2 \omega_{3}$. In the branch setting $z$ we receive an extra amount of $2 \Delta_{4}$ from $z$ 's two neighbors outside $N\left(u_{1}\right)$. Hence we have a $\left(2 \omega_{4}+2 \omega_{3}+2 \Delta_{4}, 2 \omega_{4}+2 \omega_{3}\right)$-branch. (Analogous is the case where $\left.\left|N(y) \backslash N\left(u_{1}\right)\right|=2,\left|N(z) \backslash N\left(u_{1}\right)\right|=1\right)$.

We have at least a $\left(2 \omega_{4}+2 \omega_{3}+2 \Delta_{4}, \omega_{4}+2 \omega_{3}+2 \Delta_{4}\right)$-branch, i.e., $(7.3332,5.5258)$ branch.
Priority 7 We need further auxiliary notions: A 3-path (4-path, resp.) for an unlimited weight 4 vertex $v$ is a sequence of vertices $u_{0} u_{1} \ldots u_{l} u_{l+1}\left(u_{0} u_{1} \ldots u_{l}\right.$, resp.) forming a path, such that $1 \leq l \leq 4\left(2 \leq l \leq 4\right.$, resp.), $u_{i} \in N(v)$ for $1 \leq i \leq l, \#_{2}\left(u_{i}\right)=3$ for $1 \leq i \leq l\left(\#_{2}\left(u_{i}\right)=3\right.$ for $1 \leq i \leq l-1, \#_{2}\left(u_{l}\right)=4$, resp.) and $u_{0}, u_{l+1} \notin N(v)\left(u_{0} \notin N(v)\right.$, resp.). Examine that due to the absence of limited vertices, every vertex $v$, chosen due to priority 7 must have a 3 - or 4 -path of length $l$.

3-path If $u_{0} \neq u_{\ell+1}$ we basically get a reduction of $\omega_{4}+\ell \omega_{3}+(4-\ell) \Delta_{4}$. In the branch where $u_{1}$ is set, $u_{2} \ldots u_{l}$ will be also set due to Lemma 11. Therefore, we gain an extra amount of at least $2 \Delta_{4}$ from $u_{0}$ and $u_{l+1}$, leading to a $\left(\omega_{4}+l \cdot \omega_{3}+(6-l) \Delta_{4}, \omega_{4}+l \cdot \omega_{3}+(4-l) \Delta_{4}\right)$-branch. If $u_{0}=u_{\ell+1}$ then in $F[v]$ and in $F[\bar{v}], u_{0} u_{1} \ldots u_{l} u_{l+1}$ is a lasso. So by Observation $11 u_{1}, \ldots, u_{l}$ are deleted and the weight of $u_{0}$ drops by 2 . If $\#_{2}\left(u_{0}\right)=4$ this yields a reduction of $l \cdot \omega_{3}+\omega_{4}$. If $\#_{2}\left(u_{0}\right)=3$ the reduction is $(l+1) \cdot \omega_{3}$ but then $v_{0}$ is set. It is not hard to see that this yields a bonus reduction of $\Delta_{4}$ (see Appendix A). Thus, we have a $\left(\left\{\omega_{4}+(l+1) \cdot \omega_{3}+(5-l) \Delta_{4}\right\}^{2}\right)$-branch.
4 -path We get an amount of $\omega_{4}+(l-1) \omega_{3}+(5-l) \Delta_{4}$ by deleting $v$. In the branch where $u_{1}$ is set we get a bonus of $\Delta_{4}$ from $u_{0}$. Further $u_{\ell}$ will be deleted completely. Hence we have a $\left(2 \omega_{4}+(l-1) \omega_{3}+(5-l) \Delta_{4}, \omega_{4}+(l-\right.$ 1) $\left.\omega_{3}+(5-l) \Delta_{4}\right)$-branch.

The first branches is worst for $\ell=1$, the second and third for $\ell=2$ (as $\ell=1$ is impossible). Thus, we have a $(7.087,5.3512)$-branch for the first and third case, and a $\left(\{7.2308\}^{2}\right)$-branch for the second.
Priority 8 Note that when we are forced to pick a variable $v$ according to priority 8 , then either $v$ has four neighbors of weight 4 or for every weight 3 neighbor $z$ we have $N(z) \subseteq N(v)$.

Lemma 2. If we have chosen a variable $v$ with $\#_{2}(v)=4$ according to priority 8, such that $\#_{2}(N(v))<16$, then we have : (a) only situations 1(e) and 1(f) of


Fig. 1.

Figure 1 can occur, (b) branching on $v$, we get at least a $\left(\left\{2 \omega_{4}+2 \omega_{3}+2 \Delta_{4}\right\}^{2}\right)$ branch.

The 4- 5- and 6-regular case The part of the algorithm when we branch on variables of weight $h \neq 4$ will be called $h$-phase. Branching according to priorities $4-8$ is the 4 -phase, according to priorities 9 and 10 the 3 -phase. Any $h$-regular branch which was preceded by a branch from the ( $h+1$ )-phase can be neglected. This situation can only occur once on each path from the root to a leaf in the search tree. Hence, the run time is only affected by a constant multiple.
We now classify $h$-regular branches: An internal $h$-regular branch is a $h$-regular branch such that another $h$-regular branch immediately follows in the search tree in at least one component. A final $h$-regular branch is a $h$-regular branch such that no $h$-regular branch immediately succeeds in either of the components. When we are forced to do an $h$-regular branch $(h \in\{4,5,6\})$, then according to HP the whole graph must be $h$-regular at this point. Hence, it follows:

Observation 2. If a branch is followed by a h-regular branch in at least one component, say in in $F[v]$, then in $F[v]$ any variable $u$ with $\#_{2}(u)<h$ will be reduced.

Due to Observation 2 every vertex in $N(v)$ must be completely deleted in $F[v]$.
Proposition 1. Any internal h-regular branch ( $h \in\{4,5,6\}$ ) can be upper bounded by $\mathcal{O}^{*}(1.1102)$.

Proof. By Observation 2 for $h=4$ this yields at least a $\left(5 \omega_{4}, \omega_{4}+4 \Delta_{4}\right)$-branch as $v$ must have 4 different weight 4 neighbors due to HP. If both components are followed by an $h$-regular branch we get a total reduction of $5 \omega_{4}$ in both cases. The same way we can analyze internal 5 - and 6 -regular branches. This yields $\left(3 \omega_{5}, \omega_{5}+5 \Delta_{5}\right)-,\left(3 \omega_{5}, 3 \omega_{5}\right)-,\left(3 \omega_{6}, \omega_{6}+6 \Delta_{6}\right)$ - and $\left(3 \omega_{6}, 3 \omega_{6}\right)$-branches as any variable has at least two neighbors.

We now analyze a final $h$-regular $\left(b_{1}, b_{2}\right)$-branch with its preceding $\left(a_{1}, a_{2}\right)$ branch. For the $\left(a_{1}, a_{2}\right)$-branch there are three possibilities: the final $h$ regular branch might follow in the first, the second or both components. So, the combined analysis would be a $\left(\left\{a_{1}+b_{1}\right\}^{2}, a_{2}\right)$, a $\left(a_{1},\left\{a_{2}+b_{2}\right\}^{2}\right)$ - and a $\left(\left\{a_{1}+b_{1}\right\}^{2},\left\{a_{2}+\right.\right.$ $\left.b_{2}\right\}^{2}$ )-branch.

Proposition 2. Any final $h$-regular branch $(h \in\{5,6\})$ considered together with its preceding branch can be upper bounded by $\mathcal{O}^{*}\left(1.1191^{K}\right)$.

Proof. We will apply a combined analysis for both branches. Due to Observation $2 N(v)$ will be deleted in the corresponding component of the preceding branch. Due to Appendices C. 1 and C. 2 the least amount we can get by deleting $N(v)$ is $\omega_{5}+\omega_{4}$ in case $h=5$ and $\omega_{6}+\omega_{4}$ in case $h=6$. Hence, we get four different branches: A $\left(\left\{3 \omega_{5}+\omega_{4}+5 \Delta_{5}\right\}^{2}, \omega_{5}+5 \Delta\right)$-, a $\left(\left\{3 \omega_{6}+\omega_{4}+6 \Delta_{6}\right\}^{2}, \omega_{6}+6 \Delta\right)$-, a $\left(\left\{3 \omega_{5}+\omega_{4}+5 \Delta_{5}\right\}^{4}\right)$ - and a $\left(\left\{3 \omega_{6}+\omega_{4}+6 \Delta_{6}\right\}^{4}\right)$-branch, respectively.

Proposition 3. Any final 4-regular branch considered with its preceding branch can be upper bounded by $\mathcal{O}^{*}\left(2^{\frac{1}{6.137} K}\right)$.

Proof. Here we must analyze a final 4-regular branch together with any possible predecessor. These are all branches derived from priorities 4-8:

Internal 4-regular branch The two corresponding branches are a $\left(\left\{6 \omega_{4}+\right.\right.$ $\left.\left.4 \Delta_{4}\right\}^{2}, \omega_{4}+4 \Delta_{4}\right)$-branch and a $\left(\left\{6 \omega_{4}+4 \Delta_{4}\right\}^{4}\right)$-branch.
Priorities 4,5 and 8 are all dominated by a $\left(\left\{2 \omega_{4}+4 \Delta_{4}\right\}^{2}\right)$-branch. Analyzing these cases together with a succeeding final 4-regular branch gives a $\left(\left\{3 \omega_{4}+\right.\right.$ $\left.\left.8 \Delta_{4}\right\}^{2}, 2 \omega_{4}+4 \Delta_{4}\right)$-branch and a $\left(\left\{3 \omega_{4}+8 \Delta_{4}\right\}^{4}\right)$-branch.
Priority 7 Let $o$ be the number of weight 4 neighbors of $v$ not lying on the 4 -path or 3 -path, respectively. If in one component a final 4 -regular branch follows then the worst case is when $o=0$ as any such vertex would be deleted completely and $\omega_{4}>\omega_{3}$. On the other hand if there is a component without an immediate 4-regular branch succeeding then the worst case appears when $o$ is maximal (i.e., $o=4$ ) as $\omega_{3} \geq \Delta_{4}$. So in the analysis we will consider for each case the particular worst case even though both together never appear. 4-path In this case we have the following branches: $a)\left(\left\{3 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{2}, \omega_{4}+\right.$ $\left.\left.\left.\omega_{3}+3 \Delta_{4}\right), b\right)\left(2 \omega_{4}+\omega_{3}+3 \Delta_{4},\left\{3 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2}\right), c\right)\left(\left\{3 \omega_{4}+4 \omega_{3}+\right.\right.$ $\left.\left.4 \Delta_{4}\right\}^{2},\left\{3 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$. The cases $\left.a\right)$ and $c$ ) are not upper bounded by $\mathcal{O}^{*}\left(2^{\frac{1}{6.137}} K\right)$ and hence need further discussion.
Suppose there is a vertex $y \in D:=N(v) \cup\left\{u_{0}, \ldots, u_{l-1}\right\}$ with weight 4 . Then by Observation 2 we have branches $\left.a^{\prime}\right)\left(\left\{4 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2}, \omega_{4}+\omega_{3}+3 \Delta_{4}\right)$ and $\left.c^{\prime}\right)\left(\left\{4 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2},\left\{3 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ which are both upperbounded by $\mathcal{O}^{*}\left(2^{\frac{1}{6.137} K}\right)$. For the remaining case we need the next proposition. For $U \subseteq V(F)$ we define $E_{3}(U)=\left\{\{u, v\} \mid u \in U, \#_{2}(u)=3, v \notin U\right\}$.
Claim. Suppose for all $y \in D$ we have $\#_{2}(y)=3$. Then there must be some $y^{\prime} \in V \backslash\left(D \cup\left\{v, u_{l}\right\}\right)$ with $\#_{2}\left(y^{\prime}\right)=3$.

Proof. Assume the contrary. Now due to the choice of $v$ and as $u_{l} \in Y$ we have that $u_{l}$ must be adjacent to $v, u_{l-1}$ and to two further weight 3 vertices in $D$. Therefore and as $D \cup\left\{v, u_{l}\right\}$ can not be a component we have $l<4$. Also for any $2 \leq l \leq 3$ we always have $E_{3}\left(D \cup\left\{u_{l}\right\}\right)=8-2 l$. There must some weight 4 vertex $r \notin D \cup\left\{v, u_{l}\right\}$ adjacent to some weight 3 vertex in $D$. Notre that $r \in Y$. Due to the choice of $v, r$ must have at least three weight 3 neighbors. Hence $l=2$. If $r$ has 4 weight 3 neighbors then
$\left(D \cup\left\{v, u_{l}, r\right\}\right)$ forms a component which is a contradiction. Because we have $E_{3}\left(D \cup\left\{v, u_{l}, r\right\}\right)=1$ we find again some $r^{\prime} \in Y \backslash\left(D \cup\left\{v, u_{l}, r\right\}\right)$ which is adjacent to at least 3 weight 3 vertices where at most one is from $D$. Thus there must be some weight 3 vertex in $V \backslash\left(D \cup\left\{v, u_{l}\right\}\right)$, a contradiction.

If for all $y \in D$ we have $\#_{2}(y)=3$ from the last claim we can derive two branches $\left.a^{\prime \prime}\right)\left(\left\{3 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2}, \omega_{4}+\omega_{3}+3 \Delta_{4}\right)$ and $\left.c^{\prime \prime}\right)\left(\left\{3 \omega_{4}+5 \omega_{3}+\right.\right.$ $\left.\left.4 \Delta_{4}\right\}^{2},\left\{3 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ which are upper bounded by $\mathcal{O}^{*}\left(2^{\frac{1}{6.137} K}\right)$.

### 4.2 The Cubic Case

Priority 9 Observe that when we have arrived at this point, the graph $G_{v a r}$ must be 3-regular and each variable has three different neighbors, due to $G_{v a r}$ being reduced and due to Lemma 113a, Also, any 3-regular graph has an even number of vertices, because we have $3 n=2 m$. Thus any branching must be of the form $\left(2 i \cdot \omega_{3}, 2 j \cdot \omega_{3}\right)$ for some $1 \leq i, j$. Also, branching on any variable will at least result in a $\left(4 \omega_{3}, 4 \omega_{3}\right)$-branch (see Lemma 113a). Note that any $u \in N(v)$ will be either set in $F[v]$ or in $F[\bar{v}]$, due to Lemma 13b,

Lemma 3. Let $v$ have a pending triangle $a, b, c$ and $N(v)=\{a, p, q\}$. Then by branching on $v$, we have an $\left(8 \omega_{3}, 6 \omega_{3}\right)$-branch.

Proof. In $F[v]$ and $F[\bar{v}]$, the variables $a, b, c$ form a 3-quasi-lasso, and by Observation 11 we know that, w.l.o.g., only $b$ remains with $\#_{2}(b)=2$. Also, in both branches, $q$ and $p$ are of weight two and therefore deleted. Note that if $N(\{q, p\}) \cup\{q, p\} \subseteq\{v, a, b, c, q, p\}$, then $q, p, v, a, b, c$ form a component of six vertices. These are handled separately by the algorithm. Therefore, w.l.o.g., there is a variable $z \in N(q)$ such that $z \notin\{v, a, b, c, q, p\}$. So, in the branch where $q$ is set, also $z$ will be deleted. Thus in this branch, seven variables will be deleted.

Priority 10 From now on due to HP $G_{v a r}$ is a triangle-free 3-regular graph. We have to show that if we are forced to choose a vertex $v$ to which none of the precedences 1-8 fits, we can choose $v$ in a way that we obtain either a $\left(6 \omega_{3}, 8 \omega_{3}\right)$ or a $\left(4 \omega_{3}, 10 \omega_{3}\right)$-branch.

Lemma 4. Let $v$ be a vertex in $G_{v a r}$ and $N(v)=\{a, b, c\}$. Suppose that, w.l.o.g. in $F[v] a, b$ and in $F[\bar{v}] c$ will be set. Then we have $a\left(6 \omega_{3}, 8 \omega_{3}\right)$-branch.

Proof. If $|(N(a) \cap N(b)) \backslash\{v\}| \leq 1$, then by setting $a$ and $b$ in $F[v]$, five variables will be reduced. Together with $v$ and $c$, this is a total of seven. If $\mid(N(a) \cap N(b)) \backslash$ $\{v\} \mid=2$, then situation 2(a) or 2(b) of Figure 2 must occur (note the absence of triangles). In case we have situation $2(\mathrm{a})$ then in $F[v]$ due to Lemma $111 d, f, e, q$ will be deleted. If we have situation $2(\mathrm{~b})$ then $d, y$ and $f, z$ will be deleted. Together with $F[\bar{v}]$ where $c$ is set, we have a $\left(6 \omega_{3}, 8 \omega_{3}\right)$-branch.


Fig. 2.

Lemma 5. If for any $v \in V\left(G_{v a r}\right)$ all its neighbors are set in one branch (say,in $F[v])$, we can perform a $\left(6 \omega_{3}, 8 \omega_{3}\right)$ - or a $\left(4 \omega_{3}, 10 \omega_{3}\right)$-branch due to cubicity.

Proof. If $|N(a, b, c) \backslash\{v\}| \geq 5$, then in $F[v], 9$ variables are deleted, so that we have a $\left(4 \omega_{3}, 10 \omega_{3}\right)$-branch. Otherwise, either one of the two following situations must occur: $a$ ) There is a variable $y \neq v$, such that $N(y)=\{a, b, c\}$, see Figure. 2(c). Then branch on $b$. In $F[\bar{b}], v, y, a, c, z$ will disappear (due to RR-5 and Lemma (12). In $F[b]$, due to setting $z$, additionally a neighbor $f \notin\{a, b, c, v, y\}$ of $z$ will be deleted as the vertices $a, b, c, v, y, z$ do not form a component.. This is a total of seven variables.
b) There are variables $p, q$, such that $|N(p) \cap\{a, b, c\}|=|N(q) \cap\{a, b, c\}|=2$, see Figure $2(\mathrm{~d})$ and $2(\mathrm{e})$ In $F[v]$, the variables $a, b, c, p, q$ will be set. Then, at least 3 additional variables will be deleted (even if there are $1 \leq i<j \leq 4$ with $h_{i}=h_{j}$ ). Theorem 4.2 of [6] contains also an alternative proof of $b$ ).

Due to the last three lemmas, branchings according to priorities 9 and 10 are upper bounded by $\mathcal{O}^{*}\left(2^{\frac{1}{6.137} K}\right)$. Especially the $\left(4 \omega_{3}, 10 \omega_{3}\right)$-branch is sharp for the given measure.

## 5 Combining Two Approaches

Kulikov and Kutzov [7] achieved a run time of $\mathcal{O}^{*}\left(2^{\frac{1}{5.88}} \mathrm{~K}\right)$. This was obtained by speeding up the 5 -phase by a concept called 'clause learning'. As in our approach the 3 - and 4 -phase was improved we will show that if we use both strategies we can even improve our previous time bound. This means that in HP we substitute priority 3 by their strategy. For the analysis we redefine the measure $\gamma(F)$ : we set $\omega_{3}=0.9521, \omega_{4}=1.8320, \omega_{5}=2.488$ and keep the other weights. Let's call this measure $\tilde{\gamma}(F)$. We will reproduce the analysis of [7] briefly with respect to $\tilde{\gamma}(F)$ to show that their derived branches for the 5 -phase are upper bounded by $\mathcal{O}^{*}\left(2^{\frac{1}{6.2158}}{ }^{K}\right)$. It also can be checked that this is also true for the branches derived for the other phases by measuring them in terms of $\tilde{\gamma}(F)$. Let $k_{i j}$ denote the number of weight $j$ variables occurring $i$ times in a 2 -clause with some $v \in V(F)$ chosen for branching. Then we must have: $k_{13}+k_{14}+k_{15}+2 k_{24}+2 k_{25}+3 k_{35}=5$. If $F^{\prime}$ is the the formula obtained by assigning a value to $v$ and by applying the reduction rules afterwards we have:

$$
\begin{aligned}
\tilde{\gamma}(F)-\tilde{\gamma}\left(F^{\prime}\right) \geq & 5 \Delta_{5}+\omega_{5}+\left(\omega_{3}-\Delta_{5}\right) k_{13}+\left(\Delta_{4}-\Delta_{5}\right) k_{14}+\left(\frac{\omega_{4}}{2}-\Delta_{5}\right) 2 k_{24} \\
& +\left(\Delta_{4}-\Delta_{5}\right) k_{25}+\left(\frac{\omega_{5}}{2}-\frac{3}{2} \Delta_{5}\right) 2 k_{35} \\
= & 5.768+0.2961 k_{13}+0.2239\left(k_{14}+k_{25}\right)+0.26 \cdot 2\left(k_{24}+k_{35}\right)
\end{aligned}
$$

Basically we reduce $\tilde{\gamma}(F)$ by at least $\omega_{5}+5 \Delta_{5}$. Now the coefficients of the $k_{i j}$ in the above equation express how the reduction grows if $k_{i j}>0$. If $k_{13}+k_{14}+$ $2 k_{24}+k_{25}+2 k_{35} \geq 2$ we are done as $\tilde{\gamma}(F)-\tilde{\gamma}\left(F^{\prime}\right) \geq 6.2158$.
If $k_{13}=1$ and $k_{15}=4$ then [7] stated a $\left(5 \Delta_{5}+\omega_{5}+\left(\omega_{3}-\Delta_{5}\right), 5 \Delta_{5}+\omega_{5}+\left(\omega_{3}-\right.\right.$ $\left.\left.\Delta_{5}\right)+2 \Delta_{5}\right)$-branch and for $k_{25}=1$ and $k_{15}=4$ a $\left(5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\Delta_{5}\right), 5 \Delta_{5}+\right.$ $\left.\omega_{5}+\left(\Delta_{4}-\Delta_{5}\right)+\omega_{3}\right)$-branch.
If $k_{14}=1$ and $k_{15}=4$ a branching of the kind $F[v], F\left[\bar{v}, v_{1}\right], F\left[\bar{v}, \bar{v}_{1}, v_{2}, v_{3}, v_{4}, v_{5}\right]$ is applied, where $v_{1}, \ldots, v_{5} \in N(v)$. From this follow a $\left(5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\right.\right.$ $\left.\left.\Delta_{5}\right), 4 \Delta_{5}+\omega_{5}+\Delta_{4}+\omega_{4}+3 \Delta_{4}+\Delta_{5}, 5 \omega_{5}+\omega_{4}+\Delta_{5}\right)$ - and a $\left(\omega_{5}+4 \Delta_{5}+\Delta_{4}, \omega_{5}+\right.$ $\left.4 \Delta_{5}+\Delta_{4}+\omega_{4}+4 \Delta_{5}, 5 \omega_{5}+\omega_{4}+3 \omega_{3}\right)$-branch. This depends whether $v_{1}$ has at least three neighbors of weight less than 5 in $F[\bar{v}]$ or not. We observed that we can get a additional reduction of $\Delta_{5}$ in the third component of the first branch as $N(v)$ cannot be a component in $V(F)$ after step 3 of Alg. For the analysis of the 5 -regular branch (i.e. $k_{15}=5$ ) we refer to Appendix E. 1 . It proceeds the same as in the simple version of the algorithm except that we have to take in account the newly introduced branches. We point out that in that case in contrast to [7] we do a simple branching on $v$. But also we prefer any other variable as a candidate for branching such that $k_{15} \leq 4$.

Theorem 3. MAX-2-SAT can be solved in time $\mathcal{O}^{*}\left(2^{\frac{1}{6.2158}}{ }^{K}\right) \approx 1.118^{K}$.

## 6 Conclusion

We presented an algorithm solving Max-2-SAT in $\mathcal{O}^{*}\left(2^{\frac{1}{6.2158}}{ }^{K}\right)$, with $K$ the number of clauses of the input formula. This is currently the end of a sequence of polynomial-space algorithms each improving on the run time: beginning with $\mathcal{O}^{*}\left(2^{\frac{K}{2.88}}\right)$ which was achieved by 9 , it was subsequently improved to $\mathcal{O}^{*}\left(2^{\frac{1}{3.742} K}\right)$ by [3], to $\mathcal{O}^{*}\left(2^{\frac{1}{5} K}\right)$ by [2], to $\mathcal{O}^{*}\left(2^{\frac{1}{5.217} K}\right)$ by [5], to $\mathcal{O}^{*}\left(2^{\frac{1}{5.5} K}\right)$ by [6] and finally to the hitherto fastest upper bound of $\mathcal{O}^{*}\left(2^{\frac{1}{5.88} K}\right)$ by [7]. Our improvement has been achieved due to heuristic priorities concerning the choice of variable for branching in case of a maximum degree four variable graph. As [7] improved the case where the variable graph has maximum degree five, it seems that the only way to speed up the generic branching algorithm is to improve the maximum degree six case. Our analysis also implies that the situation when the variable graph is regular is not that harmful. The reason for this that the preceding branch must have reduced the problem size more than expected. Thus considered together these two branches balance each other.

## References

1. M. Fürer and S. Kasiviswanathan. Exact Max 2-SAT: Easier and faster. In J. van Leeuwen et al., editors, SOFSEM, volume 4362, pages 272-283. Springer, 2007.
2. J. Gramm, E. A. Hirsch, R. Niedermeier, and P. Rossmanith. Worst-case upper bounds for MAX-2-SAT with an application to MAX-CUT. Discrete Applied Mathematics, 130:139-155, 2003.
3. J. Gramm and R. Niedermeier. Faster exact solutions for MAX2SAT. In Algorithms and Complexity, 4th Italian Conference, CIAC 2000, volume 1767 of LNCS, pages 174-186. Springer, 2000.
4. T. Hofmeister. An approximation algorithm for MAX-2-SAT with cardinality constraint. In G. Di Battista and U. Zwick, editors, European Symposium on Algorithms ESA 2003, volume 2832 of LNCS, pages 301-312. Springer, 2003.
5. J. Kneis, D. Mölle, S. Richter, and P. Rossmanith. Algorithms based on the treewidth of sparse graphs. In D. Kratsch, editor, Graph-Theoretic Concepts in Computer Science, 31st International Workshop, WG 2005, volume 3787 of LNCS, pages 385-396. Springer, 2005.
6. A. Kojevnikov and A. S. Kulikov. A new approach to proving upper bounds for MAX-2-SAT. In ACM-SIAM Symposium on Discrete Algorithms SODA, pages 11-17. ACM Press, 2006.
7. A. S. Kulikov and K. Kutzkov. New bounds for max-sat by clause learning. In V. Diekert, M. V. Volkov, and A. Voronkov, editors, CSR, volume 4649 of LNCS, pages 194-204. Springer, 2007.
8. M. Lewin, D. Livnat, and U. Zwick. Improved rounding techniques for the MAX 2-SAT and MAX DI-CUT problems. In William Cook and Andreas S. Schulz, editors, Integer Programming and Combinatorial Optimization IPCO, volume 2337 of $L N C S$, pages 67-82. Springer, 2002.
9. R. Niedermeier and P. Rossmanith. New upper bounds for maximum satisfiability. Journal of Algorithms, 36:63-88, 2000.
10. R. Williams. A new algorithm for optimal 2-constraint satisfaction and its implications. Theoretical Computer Science, 348(2-3):357-365, 2005.

## A Additional Arguments concerning 3-paths in the non-regular Case

Proposition 4. Let $v \in V(F)$ be chosen due to HP such that $\#_{2}(v)=4$ and $v$ has a 3 -path of length $l$ such that $u_{0}=u_{l+1}$. Then we have at least a $\left(\left\{\omega_{4}+3 \omega_{3}+3 \Delta_{4}\right\}^{2}\right)$ branch.

Proof. In $F[v]$ and in $F[\bar{v}], u_{0} u_{1} \ldots u_{l} u_{l+1}$ is a lasso. So by Observation $1 u_{1}, \ldots, u_{l}$ are deleted and the weight of $u_{0}$ drops by 2 . If $\#_{2}\left(u_{0}\right)=4$ this yields a reduction of $l \cdot \omega_{3}+\omega_{4}$. If $\#_{2}\left(u_{0}\right)=3$ the reduction is $(l+1) \cdot \omega_{3}$ but then $u_{0}$ is set. If $N\left(u_{0}\right) \backslash N(v)$ is non-empty then we obtain a reduction of $\Delta_{4}$ in addition due to setting $u_{0}$. Otherwise there is a unique $r \in N\left(u_{0}\right) \backslash\left\{u_{1}, u_{l}\right\}$. If $\#_{2}(r)=4$ we get a $\left(\left\{2 \omega_{4}+(l+1) \omega_{3}+(3-l) \Delta_{4}\right\}^{2}\right)$-branch. If $\#_{2}\left(r_{1}\right)=3$ then $r$ is set. As $(4-l) \leq 2$ and by applying the same arguments to $r$ which previously where applied to $u_{0}$ we get at least a $\left(\left\{\omega_{4}+(l+1) \cdot \omega_{3}+(5-l) \Delta_{4}\right\}^{2}\right)$ branch. Observe that we used the fact that $\omega_{4} \geq 2 \Delta_{4}$.

## B Proof of Lemma 2

Proof. From $\#_{2}(N(v))<16$ follows that, for every weight 3 neighbor $z$, we have $N(z) \subseteq N(v)$ due to the choice of $v$ according to HP. Let $N^{4}$ (resp. $N^{3}$ ) be the set of weight four (three) neighbors of $v$. We analyze different cases induced by $k:=\left|N_{3}\right|$. Let $k=1$. If $N^{3}=\{b\}$, then there are vertices $a, c \in N^{4}$, such that $b \in N(a)$ and $b \in N(c)$. We must have $a \in N(c)$, or else $a$ would violate our assumption. Thus, we get the situation of Figure 1(e)
Let $k=2$. Then $N^{3}=\{b, c\}$ and assume that $b$ and $c$ are neighbors. If $b, c \in N(a)$ for $a \in N^{4}$, we have situation $1(\mathrm{f})$ Otherwise $b \in N(a)$ and $c \in N(d)$ for $a, d \in N^{4}$. But then, priority 7 applies to both $a$ and $d$, which is a contradiction. In the case where $b$ and $c$ are not neighbors, it can be easily observed that we must have the situation in Figure 3(a) where priority 7 applies to $a$ and $c$.
If $k=3$ it is easy to verify that we must have situation 3(b) in Figure 1 But then priority 7 applies to $a$. If $k=4$ then clearly $N[v]$ forms a component of five vertices which cannot appear after step 2 of Alg. 1
In Figure $1(\mathrm{e})$ in either branch $F[v]$ or $F[\bar{v}]$, the variables $a, b, c$ form a 3-quasi-lasso, so by Observation 1 we get a reduction of $\omega_{3}+3 \omega_{4}+\Delta_{4}=4 \omega_{4}$.
In Figure 1(f) in both branches the variables $a, b, c$ form a 3-lasso, so by Observation 1 $b, c$ are deleted and $a$ is set due to Lemma 111 We get a reduction of $\omega_{4}+2 \omega_{3}$ from this. If $d \notin N(a)$ we additionally get $2 \Delta_{4}$, otherwise $\omega_{4}$. Altogether, we reduce $\gamma(F)$ by at least $2 \omega_{4}+2 \omega_{3}+2 \Delta_{4}$.


Fig. 3.

## C Analysis of internal 5- and 6-regular branches

We will consider branches which are immediately followed by a $h$-regular branch in at least one component. In this component of the branch we can delete any variable in $N(v)$ additionally due to Observation 2. The subsections C. 1 and C. 2 will explore by how much we additionally can decrement $\gamma(F)$ in the corresponding component in case $h \in\{5,6\}$. Let $k_{i j}$ denote the number of weight $j$ variables occurring $i$ times in a 2-clause with some $v \in V(F)$ chosen for branching.

## C. 1 Internal 5-regular branch

Proposition 5. Let $v \in V(F)$ be the variable chosen due to HP such that $\#_{2}(v)=5$. If this branch is followed by a 5-regular branch in one component, then we can decrement $\gamma(F)$ by at least $\omega_{5}+\omega_{4}$ in addition to the weight of $v$ in that component.

Proof. According to [7] we must have the following relation:

$$
\begin{equation*}
k_{13}+k_{14}+k_{15}+2 k_{24}+2 k_{25}+3 k_{35}=5 \tag{1}
\end{equation*}
$$

We now have to determine an integer solution to (11) such that $\omega_{3} k_{13}+\omega_{4} k_{14}+\omega_{5} k_{15}+$ $\omega_{4} k_{24}+\omega_{5} k_{25}+\omega_{5} k_{35}$ is minimal. We can assume $k_{14}=k_{15}=0$ as we have $\omega_{3}<\omega_{4}<$ $\omega_{5}$. For any solution violating this property we can find a smaller solution by setting $k_{13}^{\prime}=k_{13}+k_{14}+k_{15}, k_{14}^{\prime}=0$ and $k_{15}^{\prime}=0$ and keeping the other coefficients. The same way we find that $k_{25}=0$ must be the case as $\omega_{4}<\omega_{5}$.
If $k_{13} \geq 2$ we set $k_{13}^{\prime}=k_{13}-2\left\lfloor\frac{k_{13}}{2}\right\rfloor, k_{24}^{\prime}=k_{24}+\left\lfloor\frac{k_{13}}{2}\right\rfloor$ and keep the other coefficients. By $2 \omega_{3}>\omega_{4}$ this is a smaller solution. Now suppose $k_{13}=1$, then we have $k_{24}=0$ in a minimal solution as $\omega_{3}+\omega_{4}>\omega_{5}$ (i.e., if $k_{24} \geq 1$ we set $k_{13}^{\prime}=0, k_{24}^{\prime}=k_{24}-1$ and $k_{35}^{\prime}=k_{35}+1$ ). But then no $k_{35}$ could satisfy (11). Thus, we have $k_{13}=0$. Then the only solution is $k_{24}=1$ and $k_{35}=1$. Hence, the minimal reduction we get from $N(v)$ is $\omega_{5}+\omega_{4}$.

## C. 2 Internal 6-regular branch

Proposition 6. Let $v \in V(F)$ be the variable chosen due to HP such that $\#_{2}(v)=6$. If this branch is followed by a 6-regular branch in one component, then we can decrement $\gamma(F)$ by at least $\omega_{6}+\omega_{4}$ in addition to the weight of $v$ in that component.

Proof. In this case the following relation holds:

$$
\begin{equation*}
k_{13}+k_{14}+k_{15}+k_{16}+2 k_{24}+2 k_{25}+2 k_{26}+3 k_{35}+3 k_{36}+4 k_{46}=6 \tag{2}
\end{equation*}
$$

We now have to determine an integer solution to (2) such that $\omega_{3} k_{13}+\omega_{4} k_{14}+\omega_{5} k_{15}+$ $\omega_{6} k_{16}+\omega_{4} k_{24}+\omega_{5} k_{25}+\omega_{6} k_{26}+\omega_{5} k_{35}+\omega_{6} k_{36}+\omega_{6} k_{46}$ is minimal. As $\omega_{3}<\omega_{4}<\omega_{5}<\omega_{6}$ we conclude that $k_{1 \ell}=0$ for $4 \leq \ell \leq 6, k_{2 \ell^{\prime}}=0$ for $5 \leq \ell^{\prime} \leq 6$ and $k_{36}=0$. We also must have $k_{13} \leq 1$ as in the section above. By $2 \omega_{4}>\omega_{6}$ we must have $k_{24} \leq 1$. By (2) we also have $k_{35} \leq 2$ and $k_{46} \leq 1$.
If $k_{13}=0$ the only solutions under the given restrictions are $k_{35}=2$ and $k_{24}=1, k_{46}=$ 1 . If $k_{13}=1$ the only solution is $k_{35}=1, k_{24}=1$. Thus, the minimal amount we get by reduction from $N(v)$ is $\omega_{6}+\omega_{4}$.

## D Omitted Cases of the Analysis of the final 4-regular Case

Proposition 7. Any final 4-regular branch considered with its preceding branch can be upper bounded by $\mathcal{O}^{*}\left(2^{\frac{1}{6.137}}{ }^{K}\right)$.

Proof. Here we must analyze a final 4-regular branch together with any possible predecessor. These are all branches derived from priorities 4-8:

Internal 4-regular branch The two corresponding branches are a $\left(\left\{6 \omega_{4}+4 \Delta_{4}\right\}^{2}, \omega_{4}+\right.$ $\left.4 \Delta_{4}\right)$-branch and a $\left(\left\{6 \omega_{4}+4 \Delta_{4}\right\}^{4}\right)$-branch.
Priorities 4,5 and 8 are all dominated by a $\left(\left\{2 \omega_{4}+4 \Delta_{4}\right\}^{2}\right)$-branch. Analyzing these cases together with a succeeding final 4-regular branch gives a $\left(\left\{3 \omega_{4}+8 \Delta_{4}\right\}^{2}, 2 \omega_{4}+\right.$ $4 \Delta_{4}$ )-branch and a $\left(\left\{3 \omega_{4}+8 \Delta_{4}\right\}^{4}\right)$-branch.
Priority 6 Cases $2,3(a)$ and 4 of our non-regular priority- 6 analysis can be analyzed similar to priorities 4,5 and 8 .
Case 1 Here we deal with small components which are directly solved without any branching. Therefore we get a $\left(\left\{3 \omega_{4}+2 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$-branch in the combined analysis. We now analyze the remaining cases.
Consider now cases 5 and $3(b)$. Let $u_{1}, u_{2}$ be the picked limited pair. Due to HP the variable $u_{2}$ has two weight 3 neighbors. Thus, if a final 4-regular branch is following in these cases we get a reduction of $2 \omega_{3}$ in addition (with respect to the component of the branch). For both cases we derived a non symmetric branch, e.g., an ( $a, b$ )-branch with $a \neq b$. Depending whether the final 4-regular branch follows in the first, the second or both components we derive three combined branches: $a$ ) $\left.\left(\left\{3 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{2}, 2 \omega_{3}+\omega_{4}+2 \Delta_{4}\right), b\right)\left(2 \omega_{3}+2 \omega_{4}+2 \Delta_{4},\left\{3 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)-$ and $c)\left(\left\{3 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{2},\left\{3 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$. As $\mathcal{O}^{*}\left(2^{\frac{1}{6.137}}{ }^{K}\right)$ not proper upperbounds $a$ ) we need a further discussion for the two cases. Remember that $a$ ) corresponds to the case where some weight 3 neighbor $t$ of $v$ is set.
Case $3 b$ First suppose that $N(z) \backslash\left(N\left(u_{1}\right) \cup N\left(u_{2}\right)\right)=\emptyset$ and $N(y) \backslash\left(N\left(u_{1}\right) \cup\right.$ $\left.N\left(u_{2}\right)\right)=\emptyset$, see Figure 4(a) Then by either branching on $y$ or $z$ we get a $\left(\left\{2 \omega_{4}+\right.\right.$ $\left.4 \omega_{3}\right\}^{2}$ )-branch. In this case the combined analysis is similar to priorities 4,5 and 8 . Secondly, w.l.o.g. we have $N(z) \backslash\left(N\left(u_{1}\right) \cup N\left(u_{2}\right)\right) \neq \emptyset$, see Figure 4(b). We might have picked $y=v$ or $z=v$. But observe that in both cases in the branch where the particular weight 3 neighbor $t$ is set such that in this component a 4 -regular branch follows we have a $\left(\left\{3 \omega_{4}+4 \omega_{3}+5 \Delta_{4}\right\}^{2}, \omega_{4}+2 \omega_{3}+2 \Delta_{4}\right)$-branch in the combined analysis instead of $a)$.
Case 5 As the vertices in $N\left(u_{1}\right) \cup N\left(u_{2}\right)$ can not form a component w.l.o.g. we have that $N(z) \backslash\left(N\left(u_{1}\right) \cup N\left(u_{2}\right)\right) \neq \emptyset$. In this case we branch on $u_{1}$. Now in the branch where we set $z$ (i.e., $z=t$ ) such that a 4 -regular branch follows in that component we have a $\left(\left\{3 \omega_{4}+4 \omega_{3}+5 \Delta_{5}\right\}^{2}, 2 \omega_{4}+2 \omega_{3}\right)$-branch in the combined analysis insted of $a$ )
Both branches replacing $a$ ) have an upper bound of $\mathcal{O}^{*}\left(2^{\frac{1}{6.137}} \mathrm{~K}\right)$.
Priority 7 Let $o$ be the number of weight 4 neighbors of $v$ not lying on the 4-path or 3-path, respectively. If in one component a final 4-regular branch follows then the worst case is when $o=0$ as any such vertex would be deleted completely and $\omega_{4}>\omega_{3}$. On the other hand if there is a component without an immediate 4 -regular branch succeeding then the worst case appears when $o$ is maximal (i.e., $o=4)$ as $\omega_{3} \geq \Delta_{4}$. So in the analysis we will consider for each case the particular


Fig. 4.
worst case even though both together never appear.
3-path In the case of 3-path such that $u_{0}=u_{\ell+1}$ the branch with $\ell=2$ is dominated by all other choices. Since this is a $\left(\{7.2308\}^{2}\right)$-branch we refer to priorities 4,5 and 8 from above.
The remaining case, a 3 -path with $u_{0} \neq u_{l+1}$, will be treated separately:
First if there is a weight 4 variable in $N(u)$ we have at least the following branches: a) $\left(\left\{3 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2}, \omega_{4}+\omega_{3}+3 \Delta_{4}\right)$, b) $\left(\omega_{4}+\omega_{3}+5 \Delta_{4},\left\{3 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ and c) $\left(\left\{3 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2},\left\{3 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$. Any of those is upper bounded by $\mathcal{O}^{*}\left(2^{\frac{1}{6.137} K}\right)$.
Now suppose for all $y \in N(V)$ we have $\#_{2}(y)=3$. Table 1 captures the derived branches for certain combinations. Here we will also consider the weights of $u_{0}$ and $u_{\ell}$.

|  | left component | right component | both components |
| :---: | :---: | :---: | :---: |
| $\#_{2}\left(u_{0}\right)=3$ | $\left(\left\{2 \omega_{4}+6 \omega_{3}+4 \Delta_{4}\right\}^{2}\right.$, | $\left(\omega_{4}+6 \omega_{3}\right.$, | $\left(\left\{2 \omega_{4}+6 \omega_{3}+4 \Delta_{4}\right\}^{2}\right.$, |
| $\#_{2}\left(u_{l+1}\right)=3$ | $\left.\omega_{4}+4 \omega_{3}\right)$ | $\left.\left\{2 \omega_{4}+6 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ | $\left.\left\{2 \omega_{4}+6 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ |
| $\#_{2}\left(u_{0}\right)=3$ | $\left(\left\{3 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2}\right.$, | $\left(\omega_{4}+5 \omega_{3}+\Delta_{4}\right.$, | $\left(\left\{3 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2}\right.$, |
| $\#_{2}\left(u_{l+1}\right)=4$ | $\left.\omega_{4}+4 \omega_{3}\right)$ | $\left.\left\{2 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ | $\left.\left\{2 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ |
| $\#_{2}\left(u_{0}\right)=4$ | $\left(\left\{4 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{2}\right.$, | $\left(\omega_{4}+4 \omega_{3}+2 \Delta_{4}\right.$, | $\left(\left\{4 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{2}\right.$, |
| $\#_{2}\left(u_{l+1}\right)=4$ | ,$\left.\omega_{4}+4 \omega_{3}\right)$ | $\left.\left\{2 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ | $\left.\left\{2 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ |

Table 1.

Any entry is upper bounded by $\mathcal{O}^{*}\left(2^{\frac{1}{6.137}} K\right)$ except $\left.\alpha\right)\left(\left\{2 \omega_{4}+6 \omega_{3}+4 \Delta_{4}\right\}^{2}, \omega_{4}+\right.$ $\left.4 \omega_{3}\right)$ the left upper entry and $\left.\beta\right)\left(\omega_{4}+4 \omega_{3}+2 \Delta_{4},\left\{2 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ the middle entry of the last row. For the further discussion of cases $\alpha$ ) and $\beta$ ) we need the next claims.
Claim. Suppose for all $y \in N(v) \cup\left\{u_{0}, u_{l+1}\right\}$ we have $\#_{2}(y)=3$. Then there must be some $y^{\prime} \in V \backslash\left(N(v) \cup\left\{u_{0}, u_{l+1}\right\}\right)$ with $\#_{2}\left(y^{\prime}\right)=3$.

Proof. Assume the contrary. For any $1 \leq l \leq 4$ we have $E_{3}\left(N(v) \cup\left\{u_{0}, u_{l+1}\right\}\right) \leq 10$. As $N(v) \cup\left\{u_{0}, u_{l+1}\right\}$ can not form a component there is a weight 4 vertex $r$ adjacent to some vertex in $N(v) \cup\left\{u_{0}, u_{l+1}\right\}$. Observe that we must have $r \in Y$. $r$ has 4 weight 3 neighbors due to the choice of $v$. Hence we must have $E_{3}(N(v) \cup$ $\left.\left\{u_{0}, u_{l+1}, r\right\}\right) \leq 6$. Using the same arguments again we find some $r^{\prime} \in Y$ with $E_{3}\left(N(v) \cup\left\{u_{0}, u_{l+1}, r, r^{\prime}\right\}\right) \leq 2$. As $N(v) \cup\left\{u_{0}, u_{l+1}, r, r^{\prime}\right\}$ can not be a component we find a $r^{\prime \prime} \in Y$ with 4 weight 3 neighbors. But due to $E_{3}\left(N(v) \cup\left\{u_{0}, u_{l+1}, r, r^{\prime}\right\}\right) \leq$ 2 there must be at least one weight 3 vertex outside $\left(N(v) \cup\left\{u_{0}, u_{l+1}\right\}\right.$ ), a contradiction.

Due to the last claim we have a $\left(\left\{2 \omega_{4}+7 \omega_{3}+4 \Delta_{4}\right\}^{2}, \omega_{4}+4 \omega_{3}\right)$-branch for case $\alpha)$. This branch is upperbounded by $\mathcal{O}^{*}\left(2^{\frac{1}{6.137}} K\right)$.

Claim. Suppose for all $y \in N(v)$ we have $\#_{2}(y)=3$ and $\#_{2}\left(u_{0}\right)=\#_{2}\left(u_{l+1}\right)=4$. Then there must be some $y^{\prime} \in V \backslash\left(N(v) \cup\left\{u_{0}, u_{l+1}\right\}\right)$ with $\#_{2}\left(y^{\prime}\right)=3$.

Proof. Assume the contrary. Observe that $u_{o}, u_{l+1} \in Y$ and due to the choice of $v$ both have 4 weight 3 neighbors which must be from $N(v)$. As we have $E_{3}(N(v)) \leq 8$ it follows that $E_{3}\left(N(v) \cup\left\{u_{0}, u_{l+1}\right\}\right)=0$. Therefore $N(v) \cup\left\{u_{0}, u_{l+1}\right\}$ forms a connected component which is a contradiction.

Due to the last claim we have a $\left(\omega_{4}+4 \omega_{3}+2 \Delta_{4},\left\{2 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$-branch for case $\beta$ ). This branch is upperbounded by $\mathcal{O}^{*}\left(2^{\frac{1}{6.137} K}\right)$.

## E Additional Analysis of the Combined Approach

## E. 1 5-regular branches in the combined approach

Internal 5-regular branches yield the same recurrences as in the simple approach. Final 5 -regular branches must be analyzed together with their immediate preceding branch. Thus they have to be analyzed together with the introduced branches of 7]. Table 2 captures some cases. For the case $k_{14}=1$ and $k_{15}=4$ there are two recurrences for

| case | one component | both components | upper bound |
| :---: | :---: | :---: | :---: |
| $k_{15}=5$ | $\left(\left\{7 \omega_{5}+5 \Delta_{5}\right\}^{2}, \omega_{5}+5 \Delta_{5}\right)$ | $\left(\left\{7 \omega_{5}+5 \Delta_{5}\right\}^{4}\right)$ | $\mathcal{O}^{*}(1.0846)$ |
| $k_{13}=1, k_{15}=4$ | $\left(\left\{6 \omega_{5}+\omega_{3}+5 \Delta_{5}\right\}^{2}, \omega_{5}+4 \Delta_{5}+\omega_{3}\right)$ | $\left.\left\{6 \omega_{5}+\omega_{3}+5 \Delta_{5}\right\}^{4}\right)$ | $\mathcal{O}^{*}(1.0878)$ |
| $k_{25}=1, k_{15}=3$ | $\left(\left\{6 \omega_{5}+5 \Delta_{5}\right\}^{2}, \omega_{5}+3 \Delta_{5}+\left(\omega_{5}-\omega_{3}\right)\right.$ | $\left(\left\{6 \omega_{5}+5 \Delta_{5}\right\}^{4}\right)$ | $\mathcal{O}^{*}(1.0914)$ |

Table 2.
the branching $F[v], F\left[\bar{v}, v_{1}\right], F\left[\bar{v}, \bar{v}_{1}, v_{2}, v_{3}, v_{4}, v_{5}\right]$. The first recurrence assumes that $v_{1}$ has at least three neighbor of weight less than five in $F[\bar{v}]$ : (A) $\left(5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\right.\right.$ $\left.\left.\Delta_{5}\right), 4 \Delta_{5}+\omega_{5}+\Delta_{4}+\omega_{4}+3 \Delta_{4}+\Delta_{5}, 5 \omega_{5}+\omega_{4}+\Delta_{5}\right)$. The other (B) $\left(\omega_{5}+4 \Delta_{5}+\Delta_{4}, \omega_{5}+\right.$ $\left.4 \Delta_{5}+\Delta_{4}+\omega_{4}+4 \Delta_{5}, 5 \omega_{5}+\omega_{4}+3 \omega_{3}\right)$ captures the remaining case. Both branches have three components. Table 3 captures the combined branches of a immediately following internal 5 -branch and branches (A) and (B). This depends whether the final 5 -regular branch follows after the first (1), the second (2) or the third (3) component or in any combination of them.

Proposition 8. Let $v \in V(F)$ be the variable chosen for branching by Alg. $\square$ such that $\#_{2}(v)=5$. Assume $v$ induces a solution to equation 1 such that it is different from $k_{13}=1, k_{15}=4 ; k_{15}=5 ; k_{25}=1, k_{15}=3 ; k_{14}=1, k_{15}=4(\star)$. If a 5 -regular branch follows in one component we have at least a $\left(\left\{3 \omega_{5}+\omega_{4}+5 \Delta_{5}\right\}^{2}, \omega_{5}+3 \Delta_{5}+2 \Delta_{4}\right)$-branch and if it follows in both a $\left(\left\{3 \omega_{5}+\omega_{4}+5 \Delta_{5}\right\}^{4}\right)$-branch

| Branch-type | components | combined branch | upper bound |
| :---: | :---: | :---: | :---: |
| (A) | (1) | $\begin{aligned} & \quad\left(\left\{6 \omega_{5}+\omega_{4}+5 \Delta_{5}\right\}^{2},\right. \\ & 5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\Delta_{5}\right)+\omega_{4}+3 \Delta_{4}+\Delta_{5}, \\ & \left.5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\Delta_{5}\right)+4 \omega_{4}+\omega_{3}+\Delta_{5}\right) \end{aligned}$ | $\mathcal{O}^{*}(1.0912)$ |
| (A) | (2) | $\begin{gathered} \left(5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\Delta_{5}\right),\left\{6 \omega_{5}+\omega_{4}+5 \Delta_{5}\right\}^{2},\right. \\ \left.5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\Delta_{5}\right)+4 \omega_{4}+\omega_{3}+\Delta_{5}\right) \end{gathered}$ | $\mathcal{O}^{*}(1.1094)$ |
| (A) | (3) | $\begin{gathered} \left(5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\Delta_{5}\right),\right. \\ 5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\Delta_{5}\right)+\omega_{4}+3 \Delta_{4}+\Delta_{5}, \\ \left.\left\{6 \omega_{5}+\omega_{4}+5 \Delta_{5}+\omega_{3}\right\}^{2}\right) \\ \hline \end{gathered}$ | $\mathcal{O}^{*}(1.1175)$ |
| (B) | (1) | $\begin{gathered} \left(\left\{6 \omega_{5}+\omega_{4}+5 \Delta_{5}\right\}^{2}\right. \\ 5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\Delta_{5}\right)+\omega_{4}+4 \Delta_{5} \\ \left.5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\Delta_{5}\right)+4 \omega_{4}+\omega_{3}+3 \omega_{3}\right) \end{gathered}$ | $\mathcal{O}^{*}(1.0894)$ |
| (B) | (2) | $\begin{gathered} \left(5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\Delta_{5}\right),\left\{6 \omega_{5}+\omega_{4}+5 \Delta_{5}\right\}^{2},\right. \\ \left.5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\Delta_{5}\right)+4 \omega_{4}+\omega_{3}+3 \omega_{3}\right) \end{gathered}$ | $\mathcal{O}^{*}(1.1052)$ |
| (B) | (3) | $\begin{gathered} \left(5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\Delta_{5}\right),\right. \\ 5 \Delta_{5}+\omega_{5}+\left(\Delta_{4}-\Delta_{5}\right)+\omega_{4}+4 \Delta_{5}, \\ \left.\left\{6 \omega_{5}+\omega_{4}+5 \Delta_{5}+3 \omega_{3}\right\}^{2}\right) \\ \hline \end{gathered}$ | $\mathcal{O}^{*}(1.1159)$ |
| $\begin{aligned} & \hline \text { (A) or } \\ & (\mathbf{B}) \\ & \hline \end{aligned}$ | $\begin{gathered} \hline(1)(2) /(1)(3) \\ /(2)(3) \\ \hline \end{gathered}$ | $\left(\left\{6 \omega_{5}+\omega_{4}+5 \Delta_{5}\right\}^{4}, \omega_{5}+4 \Delta_{5}+\Delta_{4}\right)$ | $\mathcal{O}^{*}(1.1126)$ |
| $\begin{gathered} \hline \text { (A) or } \\ (\mathrm{B}) \\ \hline \end{gathered}$ | (1)(2)(3) | $\left(\left\{6 \omega_{5}+\omega_{4}+5 \Delta_{5}\right\}^{6}\right)$ | $\mathcal{O}^{*}(1.0936)$ |

Table 3. The second column indicates after which components an final 5-regular branch immediately follows.

Proof. If a component is followed by a final 5 -regular branch the least amount we get by reduction from $N(v)$ is $\omega_{5}+\omega_{4}$. This refers to the case $k_{35}=1$ and $k_{24}=1$ which follows analogously from section C. 1 .
The least reduction from $N(v)$ without a following final 5 -regular branch can be found as follows: Consider any solution of equation (1) expect the ones in ( $\star$ ). From them find one which minimizes

$$
\begin{equation*}
\Delta_{3} k_{13}+\Delta_{4} k_{14}+\Delta_{5} k_{15}+\left(\Delta_{4}+\Delta_{3}\right) k_{24}+\left(\Delta_{5}+\Delta_{4}\right) k_{25}+\left(\Delta_{5}+\Delta_{4}+\Delta_{3}\right) k_{35} \tag{3}
\end{equation*}
$$

We can assume $k_{24}=0$ as we have $\Delta_{4}+\Delta_{3}>\Delta_{5}+\Delta_{4}$. As were excluding ( $\star$ ) we must have $k_{15} \leq 4$. If $k_{15}=4$ we conclude that either $k_{14}=1$ oder $k_{13}=1$. Both solutions are forbidden (see ( $\star$ )). Thus we must have $k_{15} \leq 3$.
If $k_{35}=1$ then there is a better solution as $\Delta_{5}+2 \Delta_{4}<\Delta_{5}+\Delta_{4}+\Delta_{3}$ : set $k_{35}^{\prime}=$ $0, k_{15}^{\prime}=k_{15}+1, k_{14}^{\prime}=k_{14}+2$ and keep the other coefficients. Note that in this case we must have $k_{15} \leq 2$ which assures that the new solutions is different from the ones in $(\star)$. Therefore it follows that $k_{35}=0$.
Now suppose $k_{25}=2$, then $k_{15}=1$ holds. But then $k_{25}^{\prime}=k_{25}-1, k_{15}^{\prime}=k_{15}+1, k_{14}^{\prime}=1$ is a no worse solution. Thus $k_{25} \leq 1$. If $k_{25}=1$ then with $k_{15}=2$ and $k_{14}=1$ this is minimal (since $k_{15}=3$ is forbidden ( $\star$ )).
Suppose $k_{25}=0$, then clearly the best solution is $k_{15}=3$ and $k_{14}=2$. Both solutions provide a reduction of $3 \Delta_{5}+2 \Delta_{4}$ which is minimal.
Now we analyze a final 5-regular branch and a branch different form ( $\star$ ) satisfying equation 3. If the final 5 -regular branch follows in only one component then we have
at least a $\left(\left\{3 \omega_{5}+\omega_{4}+5 \Delta_{5}\right\}^{2}, \omega_{5}+3 \Delta_{5}+2 \Delta_{4}\right)$-branch in the combined analysis. If it follows in both then a $\left(\left\{3 \omega_{5}+\omega_{4}+5 \Delta_{5}\right\}^{4}\right)$-branch upper bounds correctly.

Due to Proposition 8 we can upper bound the 5 -regular branches whose predecessors are different from $k_{13}=1, k_{15}=4 ; k_{15}=5 ; k_{25}=1, k_{15}=3 ; k_{14}=1, k_{15}=4$ by $\mathcal{O}^{*}(1.1171)$ in their combined analysis.

## E. 2 Additional Analysis of the combined approach

Here we provide the run times under $\tilde{\gamma}(F)$ for the cases we did not consider in section 5. The run time has been estimated with respect to $\tilde{\gamma}(F)$. Names will refer to the corresponding ones in the analysis of Alg. [1]
Non-regular Branches In Table 4 we find the derived recurrences for each priority of HP if we have a non-regular branch. You can find them together with their run times.

| Priorities | branch | upper bound |
| :---: | :---: | :---: |
| Priority 1 | $(7,7)$ | $\mathcal{O}^{*}(1.1042)$ |
| Priority 2 | $\left(\left\{\omega_{6}+5 \Delta_{6}+\Delta_{5}\right\}^{2}\right)$ | $\mathcal{O}^{*}(1.118)$ |
| Priority 3 | $\left(\left\{\omega_{5}+4 \Delta_{5}+\Delta_{4}\right\}^{2}\right)$ | $\mathcal{O}^{*}(1.123)$ |
| Priority 4 | $\left(\left\{2 \omega_{4}+4 \Delta_{4}\right\}^{2}\right)$ | $\mathcal{O}^{*}(1.102)$ |
| Priority 5 | $\left(\left\{3 \omega_{4}+2 \omega_{3}\right\}^{2}\right)$ | $\mathcal{O}^{*}(1.1099)$ |
| Priority 6 | $\left(2 \omega_{4}+2 \omega_{3}+2 \Delta_{4}, \omega_{4}+2 \omega_{3}+2 \Delta_{4}\right)$ | $\mathcal{O}^{*}(1.1143)$ |
| Priority 7 | $\left(\omega_{4}+\omega_{3}+5 \Delta_{4}, \omega_{4}+\omega_{3}+3 \Delta_{4}\right)$ | $\mathcal{O}^{*}(1.1172)$ |
|  | $\left(\left\{\omega_{4}+3 \omega_{3}+3 \Delta_{4}\right\}^{2}\right)$ | $\mathcal{O}^{*}(1.1)$ |
|  | $\left(2 \omega_{4}+\omega_{3}+3 \Delta_{4}, \omega_{4}+\omega_{3}+3 \Delta_{4}\right)$ | $\mathcal{O}^{*}(1.1165)$ |
| Priority 8 | $\left(\left\{2 \omega_{4}+2 \omega_{3}+2 \Delta_{4}\right\}^{2}\right)$ | $\mathcal{O}^{*}(1.1)$ |
| Priority 9 | $\left(8 \omega_{3}, 6 \omega_{3}\right)$ | $\mathcal{O}^{*}(1.1105)$ |
| Priority 10 | $\left(8 \omega_{3}, 6 \omega_{3}\right)$ | $\mathcal{O}^{*}(1.1105)$ |
|  | $\left(4 \omega_{3}, 10 \omega_{3}\right)$ | $\mathcal{O}^{*}(1.118)$ |

Table 4. The non-regular cases

Regular Branches Table 5 captures the run times of any internal 6,5 or 4 -regular branch. Table 6 consider 4 or 6 -final branches together with their preceding branches. The case where we have chosen $v$ due to priority 7 such that $v$ has a 3-path with $u_{0} \neq u_{l}$ is treated separately.

3-path Finally we consider the case when a variable chosen to priority 7 has a 3-path with $u_{0} \neq u_{l+1}$. The cases $\left.\left.a\right)\left(\left\{3 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2}, \omega_{4}+\omega_{3}+3 \Delta_{4}\right), b\right)\left(\omega_{4}+\omega_{3}+\right.$ $\left.5 \Delta_{4},\left\{3 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ and $\left.c\right)\left(\left\{3 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2},\left\{3 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ are upper bounded by $\mathcal{O}^{*}(1.1152), \mathcal{O}^{*}(1.1159)$ and $\mathcal{O}^{*}(1.1147)$. Table 7 captures the branches of Table 1 together with their run times in the combined algorithm.
We have a $\left(\left\{2 \omega_{4}+7 \omega_{3}+4 \Delta_{4}\right\}^{2}, \omega_{4}+4 \omega_{3}\right)$-branch for case $\alpha$ ) which $\mathcal{O}^{*}(1.1132)$ properly upper bounds. We also have a $\left(\omega_{4}+4 \omega_{3}+2 \Delta_{4},\left\{2 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$-branch for case $\beta$ ) such that it is upper bounded by $\mathcal{O}^{*}(1.1142)$.

| case | branch | upper bound |
| :---: | :---: | :---: |
| Internal 6-regular | $\left(3 \omega_{6}, \omega_{6}+6 \Delta_{6}\right)$ | $\mathcal{O}^{*}(1.0978)$ |
|  | $\left(\left\{3 \omega_{6}\right\}^{2}\right)$ | $\mathcal{O}^{*}(1.0802)$ |
| Internal 5-regular | $\left(3 \omega_{5}, \omega_{5}+5 \Delta_{5}\right)$ | $\mathcal{O}^{*}(1.1112)$ |
|  | $\left(\left\{3 \omega_{5}\right\}^{2}\right)$ | $\mathcal{O}^{*}(1.0974)$ |
| Internal 4-regular | $\left(5 \omega_{4}, \omega_{4}+4 \Delta_{4}\right)$ | $\mathcal{O}^{*}(1.103)$ |
|  | $\left(\left\{5 \omega_{4}\right\}^{2}\right)$ | $\mathcal{O}^{*}(1.079)$ |

Table 5. Internal $h$-regular cases $(h \in\{4,5,6\})$ an their upper bounds.

| Preceding branch | branch | upper bound |
| :---: | :---: | :---: |
| Final 6-regular Branch |  |  |
| Any 6-phase branch | $\begin{gathered} \left(\left\{3 \omega_{6}+\omega_{4}+6 \Delta_{6}\right\}^{2}, \omega_{6}+6 \Delta_{6}\right) \\ \left(\left\{3 \omega_{6}+\omega_{4}+6 \Delta_{6}\right\}^{4}\right) \end{gathered}$ | $\begin{gathered} \hline \mathcal{O}^{*}(1.11) \\ \mathcal{O}^{*}(1.105) \\ \hline \end{gathered}$ |
| Final 4-regular Branch |  |  |
| Internal 4-regular | $\begin{gathered} \left(\left\{6 \omega_{4}+4 \Delta_{4}\right\}^{2}, \omega_{4}+4 \Delta_{4}\right) \\ \left(\left\{6 \omega_{4}+4 \Delta_{4}\right\}^{4}\right) \\ \hline \end{gathered}$ | $\begin{aligned} & \mathcal{O}^{*}(1.1115) \\ & \mathcal{O}^{*}(1.1003) \end{aligned}$ |
| Priorities 4,5 and 8 Cases $2,3(a), 4$ of Priority 6 | $\begin{gathered} \left(\left\{3 \omega_{4}+8 \Delta_{4}\right\}^{2}, 2 \omega_{4}+4 \Delta_{4}\right) \\ \left(\left\{3 \omega_{4}+8 \Delta_{4}\right\}^{4}\right) \\ \hline \end{gathered}$ | $\begin{gathered} \mathcal{O}^{*}(1.1115) \\ \mathcal{O}^{*}(1.117) \\ \hline \end{gathered}$ |
| Priority 6 Case 1 Case $5,3(b)$ $b$ ) and $c$ ) of the analysis Case $3 b$ ), case $a$ ) of the analysis Case 5 , case $a$ ) of the analysis | $\begin{gathered} \left(\left\{3 \omega_{4}+2 \omega_{3}+4 \Delta_{4}\right\}^{2}\right) \\ \left(\left\{2 \omega_{4}+2 \omega_{3}+2 \Delta_{4},\left\{3 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{2}\right) \quad b\right) \\ \left.\left(\left\{3 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{4}\right) c\right) \\ \left(\left\{3 \omega_{4}+4 \omega_{3}+5 \Delta_{4}\right\}^{2}, \omega_{4}+2 \omega_{3}+2 \Delta_{4}\right) \\ \left(\left\{3 \omega_{4}+4 \omega_{3}+5 \Delta_{4}\right\}^{2}, 2 \omega_{4}+2 \omega_{3}\right) \\ \hline \end{gathered}$ | $\begin{gathered} \mathcal{O}^{*}(1.07) \\ \mathcal{O}^{*}(1.109) \\ \mathcal{O}^{*}(1.1143) \\ \mathcal{O}^{*}(1.1151) \\ \mathcal{O}^{*}(1.1145) \\ \hline \end{gathered}$ |
| Priority 7 Case of a 4-path Case $b$ ) Case $a^{\prime}$ ) Case $c^{\prime}$ ) Case $a^{\prime \prime}$ ) Case $c^{\prime \prime}$ ) Case of a 3-path with $u_{0}=u_{l+1}$ | $\begin{gathered} \left(2 \omega_{4}+\omega_{3}+3 \Delta_{4},\left\{3 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2}\right) \\ \left(\left\{4 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2}, \omega_{4}+\omega_{3}+3 \Delta_{4}\right) \\ \left.\left\{4 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2},\left\{3 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2}\right) \\ \left(\left\{3 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2}, \omega_{4}+\omega_{3}+3 \Delta_{4}\right) \\ \left(\left\{3 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2},\left(\left\{3 \omega_{4}+3 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)\right. \\ \text { similar to priorities 4,5 and } \mathbf{8} \end{gathered}$ | $\begin{aligned} & \mathcal{O}^{*}(1.1155) \\ & \mathcal{O}^{*}(1.1156) \\ & \mathcal{O}^{*}(1.115) \\ & \mathcal{O}^{*}(1.1152) \\ & \mathcal{O}^{*}(1.1147) \end{aligned}$ |

Table 6. The final $h$-regular cases $(h \in\{4,6\})$ and their combined analysis

|  | left component | right component | both components |
| :---: | :---: | :---: | :---: |
| $\#_{2}\left(u_{0}\right)=3$ |  | $\left(\omega_{4}+6 \omega_{3}\right.$, | $\left(\left\{2 \omega_{4}+6 \omega_{3}+4 \Delta_{4}\right\}^{2}\right.$, |
| $\#_{2}\left(u_{l+1}\right)=3$ |  |  |  |
| upper bounds |  | $\left.\left\{2 \omega_{4}+6 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ | $\left.\left\{2 \omega_{4}+6 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ |
| $\#_{2}\left(u_{0}\right)=3$ |  |  |  |
| $\#_{2}\left(u_{l+1}\right)=4$ | $\left(\left\{3 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2}\right.$, | $\left(\mathcal{O}_{4}(1.1075)\right.$ | $\mathcal{O}^{*}(1.1136)$ |
| upper bounds | $\omega_{4}+4 \omega_{3}+\Delta_{4}$, | $\left(\left\{3 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2}\right.$, |  |
| $\mathcal{O}^{*}(1.1136)$ | $\left.\left\{2 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ | $\left.\left\{2 \omega_{4}+5 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)^{2}$ |  |
| $\#_{2}\left(u_{0}\right)=4$ | $\mathcal{O}^{*}(1.1138)$ | $\mathcal{O}^{*}(1.1143)$ |  |
| $\#_{2}\left(u_{l+1}\right)=4$ | $\left\{4 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{2}$, |  | $\left(\left\{4 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{2}\right.$, |
| upper bounds | ,$\left.\omega_{4}+4 \omega_{3}\right)$ | $\mathcal{O}^{*}(1.1088)$ |  |
| $\left.\left\{2 \omega_{4}+4 \omega_{3}+4 \Delta_{4}\right\}^{2}\right)$ |  |  |  |

Table 7.

