A Diagrammatic Reasoning System with Euler Circles

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Abstract

This paper is concerned with Euler diagrammatic reasoning. Proof-theory has traditionally been developed based on linguistic (symbolic) representations of logical proofs. Recently, however, logical reasoning based on diagrammatic or graphical representations has been investigated by many logicians. Euler diagrams were introduced in the 18th century by Leonhard Euler [1768]. But it is quite recent (more precisely, in the 1990s) that logicians started to study them from a formal logical viewpoint, and there are only few proof-theoretical investigations. Accordingly, in order to fill this gap, we formalize an Euler diagrammatic inference system and prove the soundness and completeness theorems with respect to a formal set-theoretical semantics. We further consider, from a proof-theoretical viewpoint, the structure of diagrammatic proofs and manners of their construction.

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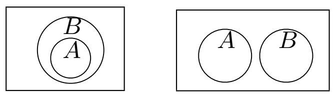
1 Introduction

Euler diagrams were introduced by Leonhard Euler [1768] to illustrate syllogistic reasoning. In Euler diagrams, logical relations among the terms of a syllogism are simply represented by topological relations among circles. [1] With Euler diagrams the universal categorical statements of the forms *All A are B* and *No A are B* are simply represented by the inclusion and the exclusion relations between circles, respectively, as seen in Fig.1.

Given two Euler diagrams which represent the premises of a syllogism, the syllogistic inference can be naturally replaced by the task of manipulating the diagrams, in particular of unifying the diagrams and extracting information from them. For example, the well-known syllogism named "Barbara," i.e., *All A are B and All B are C; therefore All A are C*, can be represented diagrammatically as in Fig.2.



All A are B *No A are B*

Fig. 1 Universal statements

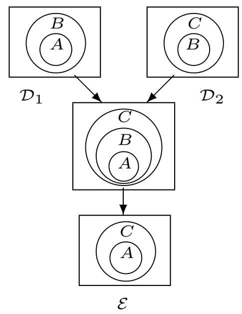


Fig. 2 Barbara with Euler diagrams

However, things become complicated when existential statements come into the picture. In Euler's original system, any region in a diagram is assumed to represent a non-empty set, and this existential import destroys the simple correspondence between Euler diagrams and categorical statements. For instance, the diagram of Fig.3 can be read as the following four categorical statements (1)–(4):

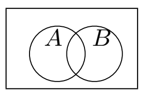


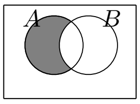
Fig. 3

1. *Some A are B*,
2. *Some B are A*,
3. *Some A are not B*,
4. *Some B are not A*.

John Venn [1881] overcame this difficulty by removing the existential import from circles. Venn fixed such a diagram of Fig.3 as a so-called "primary diagram," which does not convey any specific information about the relation between A and B. Thus Venn diagrams can represent partial, not fully specified, information between circles. Meaningful relations between circles are then expressed by specifying which regions are "empty" with the novel syntactic device of *shading*, which corresponds to logical negation. Observe that *All A are B* is equivalent to *There is nothing which is A but not B*, and the statement is expressed as the following Venn diagram by making use of the shading as in Fig.4:

In Venn diagrams, existential claims are expressed by using another syntactic device, "×," which was introduced by Charles Peirce [1897], and which represents non-emptiness of the corresponding region. Existential categorical statements of the forms *Some A are B* and *Some A are not B* are represented by using the symbol × as seen in Fig.5. Furthermore, in

[1] Throughout this paper, we mean by a circle a simple closed curve.

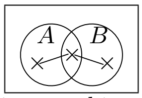


All A are B in Venn diagram
(*There is nothing which is A but not B*)
Fig. 4 Shading

order to make Venn diagrams more expressive, Peirce introduced another syntactic device: a linear symbol "$-$" which connects $\times$ symbols to represent disjunctive information as seen in Fig.6. [2]

Some A are B *Some A are not B*

Fig. 5 Existential statements



There is something A or B

Fig. 6 Disjunction

Based on Venn's and Peirce's work, Sun-Joo Shin [1994] formalized a diagrammatic reasoning system called the Venn-II system. The following devices were adopted, which with some modifications came to be regarded as the set of standard devices in subsequent studies:

1. Venn's shading (for emptiness);
2. Primary diagrams (for non-specific information);
3. Peirce's $\times$/constant symbols (for non-emptiness/for existence of particular objects);
4. Peirce's linking between $\times$'s/between constants (for disjunctive information on objects/on particular objects);
5. Linking between diagrams (for disjunctive information on diagrams).

See, for example, [Stapleton 2005, Molina 2001] for surveys of various syntactic devices.

Two Venn diagrams may be combined into another Venn diagram by accommodating the labels of circles and then by superposing the shaded regions, as illustrated in Fig.7.

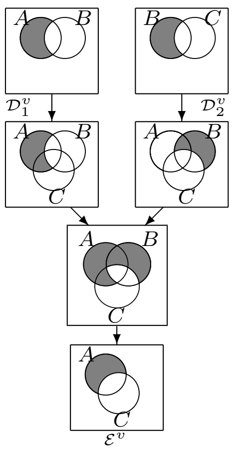


Fig. 7 Barbara with Venn diagrams

[2] Cf. [Hammer-Shin 1998] for the other device "o," which Peirce introduced in place of Venn's shading.

Because of their expressive power and their uniformity in formalizing the manipulation of combining diagrams (simply as the superposition of shadings), Venn diagrams have been very well studied; formal semantics and inference systems are given, and basic logical properties such as soundness, completeness, and decidability are shown. Moreover, automated theorem proving has been also studied. Cf. Venn-I and Venn-II systems of [Shin 1994]; Heterogeneous inference system of [Hammer 1994], Spider diagrams SD1 and SD2 of [Howse-Molina-Taylor 2000, Molina 2001], etc. For recent surveys, see [Stapleton 2005, Howse 2008].

We may summarize the underlying conception in the literature of Venn diagrammatic systems as follows:

- Region-based formalization: Logical relations among terms are represented by shading (or erasing) regions. Two diagrams are combined by superposing the shaded regions.
- Emphasis on expressive power: In order to make diagrams as expressive as possible, various syntactic devices are introduced.

Note however that the development of systems of Venn diagrams are obtained at the cost of clarity of the representations of Euler diagrams: In Venn diagrams, logical relations among terms are represented not simply by topological relations, but by the use of the shadings, which makes the translations of categorical sentences unnaturally complex as can be seen in Fig.4. Furthermore, as Venn [1881] himself already pointed out, when more than three circles are involved, Venn diagrams fail in their main purpose of providing intuitive and sensible illustration. (For some discussions on visual disadvantages of Venn diagrams, see [Hammer-Shin 1998, Gil-Howse-Tulchinsky 2002].)

Recently, *Euler diagrams with shading* were introduced to make up for the shortcomings of Venn diagrams: E.g., Euler/Venn diagrams of [Swoboda-Allwein 2004, 2005]; Spider diagrams ESD2 of [Molina 2001] and SD3 of [Howse-Stapleton-Taylor 2005]. However, their abstract syntax and semantics are still defined in terms of regions, where shaded regions of Venn diagrams are considered as "missing" regions. That is, the idea of the *region-based* Euler diagrams is essentially along the same line as Venn diagrams. Furthermore, *unification*, which plays a central role in Euler diagrammatic reasoning, is formalized by way of the combining of Venn diagrams. For example, when we unify two Euler diagrams with shading as in $\mathcal{D}_1$ and $\mathcal{D}_2$ of Fig.2, they are first transformed into Venn diagrams $\mathcal{D}_1^v$ and $\mathcal{D}_2^v$ of Fig.7, respectively; then, by superposing the shaded regions of $\mathcal{D}_1^v$ and $\mathcal{D}_2^v$, the Venn diagram $\mathcal{E}^v$ is obtained, which is transformed into the Euler diagram $\mathcal{E}$ with shading.

In contrast to the studies in the tradition of Venn diagrams, we introduce our Euler diagrammatic system based on the following conception:

- Topological-relation-based formalization: Our diagrammatic syntax and semantics are defined in terms of topological relations between two diagrammatic objects. Our *unification* of two diagrams are formalized directly in terms of topological relations without making a detour to Venn diagrams.
- Preservation of visual clarity of diagrams: In order to keep the inherent definiteness or specificity of diagrams, we avoid introducing auxiliary syntactic devices such as shading and linking, which may require arbitrary conventions. (See, for example, [Hammer-Shin 1998, Allwein-Barwise 1996, Stenning 2002] for some discussion on the

nature of diagrams.) In particular, we start our study by concentrating on the following basic syntactic devices:

1. Inclusion and exclusion relations between two diagrammatic objects;
2. Crossing relations between circles, which say nothing specific about the semantic relationship between the circles as it does in Venn diagrams;
3. Named points (constant symbols) to represent the existence of particular objects.

(We also discuss some natural extensions in Section 4.)

Compared with Shin's Venn-II system, our system lacks shading, linking between points and linking between diagrams; and hence our system is weaker than Venn-II in its expressive power. However, EUL is expressive enough to characterize basic logical reasoning such as syllogistic reasoning.

From a perspective of proof-theory, the contrast between the standpoints of Venn diagrams, i.e., the region/shading-based framework, and Euler diagrams, i.e., the topological-relation-based framework, can be understood as follows: At the level of representation, the contrast is analogous to the one between disjunctive (dually, conjunctive) normal formulas and implicational formulas; at the level of reasoning, the contrast is analogous to the one between resolution calculus style proofs and natural deduction style proofs.

The rest of this paper is organized as follows. In Section 2, we introduce a *topological-relation-based* Euler diagrammatic representation system EUL. We give a definition of an Euler diagrammatic syntax EUL in Section 2.1 and a set-theoretical semantics for it in Section 2.2. In Section 3, we formalize a diagrammatic inference system GDS. We introduce two kinds of inference rules: *unification* and *deletion*. We define in Section 3.2 the notion of *diagrammatic proof* (*d-proof*, in short), which is considered as a chain of unification and deletion steps. The inference system GDS is shown in Section 3.4 to be sound (Theorem 3.6) and complete (Theorem 3.15) with respect to our formal set-theoretical semantics. In Section 3.5, we discuss some consequences of completeness of GDS. In particular, a class of $\pm$-*normal diagrammatic proofs* of GDS is defined, and a normal form theorem (Theorem 3.19) of GDS is shown. Based on the completeness and the normal form theorems, we give a proof-theoretical analysis on structure of diagrammatic proofs. Finally, in Section 4 we discuss some possible extensions of our system and outline some future work.

2 A diagrammatic representation system (EUL) for Euler circles and its set-theoretical semantics

We introduce a diagrammatic representation system, EUL in Section 1, and its set-theoretical semantics in Section 2.2.

2.1 Diagrammatic syntax of EUL

We introduce the diagrammatic syntax of EUL. Each EUL-diagram is defined as a set of named simple closed curves and named points in a plane. We further consider some equivalence classes of concrete diagrams in terms of topological binary relations, called EUL-relations, between pairs of diagrammatic objects.

Let us start by defining the diagrams of EUL.

Definition 2.1 (EUL-diagram) An EUL-*diagram* is a plane ($\mathbb{R}^2$) with a finite number, at least two, of *named simple closed curves* [3] (denoted by $A, B, C, \dots$) and *named points* (denoted by $a, b, c, \dots$), where each named simple closed curve and named point has exactly one name, and any two distinct named simple closed curves and named points have different names. EUL-diagrams are denoted by $\mathcal{D}, \mathcal{E}, \mathcal{D}_1, \mathcal{D}_2, \dots$.

In what follows, a named simple closed curve is sometimes called a *named circle*. Moreover, named circles and named points are collectively called *objects*, and denoted by $s, t, u, \dots$. We use a rectangle to represent a plane for an EUL-diagram.

Among EUL-diagrams, there are particularly simple diagrams which consist of only two objects:

Definition 2.2 (Minimal diagram) An EUL-diagram consisting of only two objects is called a *minimal diagram*. Minimal diagrams are denoted by $\alpha, \beta, \gamma, \dots$.

We study mathematical properties of EUL-diagrams in terms of the following topological relations between two diagrammatic objects:

Definition 2.3 (EUL-relation) EUL-relations are the following binary relations between diagrammatic objects which have distinct names:

$A \sqsubset B$	"the interior[4] of A is *inside of* the interior of B,"
$A \vdash\!\dashv B$	"the interior of A is *outside of* the interior of B,"
$A \bowtie B$	"there is at least one *crossing* point between A and B,"
$b \sqsubset A$	"b is *inside of* the interior of A,"
$b \vdash\!\dashv A$	"b is *outside of* the interior of A,"
$a \vdash\!\dashv b$	"a is *outside of* b (i.e. a is not located at the point of b)."

Observe that EUL-relations $\vdash\!\dashv$ and $\bowtie$ are symmetric, while $\sqsubset$ is not. Note also that all EUL-relations are irreflexive.

Each of the EUL-relations is illustrated in the following EUL-diagrams of Fig.8:

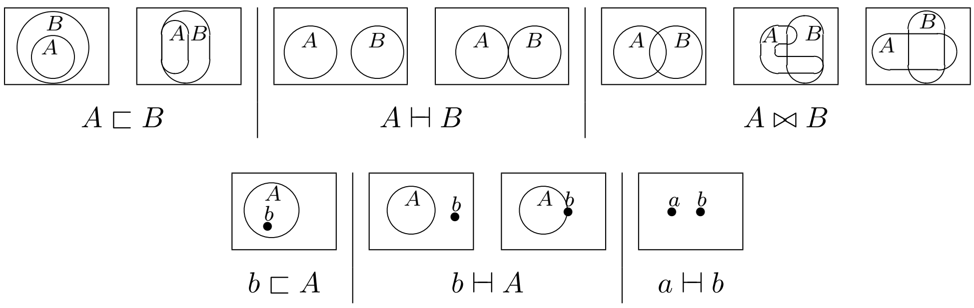


Fig. 8 EUL-relations

Proposition 2.4 *Let $\mathcal{D}$ be an* EUL*-diagram. For any distinct objects s and t of $\mathcal{D}$, exactly one of the* EUL*-relations $s \sqsubset t, t \sqsubset s, s \vdash\!\dashv t, s \bowtie t$ holds.*

[3] See [Blackett 1983] for a formal definition of simple closed curve on $\mathbb{R}^2$.

[4] Here, the interior of a named circle A means the region strictly inside of A. Cf. [Blackett 1983].

Observe that, by Proposition 2.4, for a given EUL-diagram $\mathcal{D}$, the set of EUL-relations holding on $\mathcal{D}$ is uniquely determined. We denote the set by $\mathsf{rel}(\mathcal{D})$.

The following properties, as well as Proposition 2.4, characterize EUL-diagrams.

Lemma 2.5 *Let $\mathcal{D}$ be an EUL-diagram. Then for any objects (named circles or points) $s, t, u \in ob(\mathcal{D})$, we have the following:*

1. (Transitivity) *If $s \sqsubset t, t \sqsubset u \in \mathsf{rel}(\mathcal{D})$, then $s \sqsubset u \in \mathsf{rel}(\mathcal{D})$.*
2. ($\vdash\!\dashv$-downward closedness) *If $s \vdash\!\dashv t, u \sqsubset s \in \mathsf{rel}(\mathcal{D})$, then $u \vdash\!\dashv t \in \mathsf{rel}(\mathcal{D})$.*
3. (Point determinacy) *For any point x of $\mathcal{D}$, exactly one of $x \sqsubset s$ and $x \vdash\!\dashv s$ is in $\mathsf{rel}(\mathcal{D})$.*
4. (Point minimality) *For any point x of $\mathcal{D}$, $s \sqsubset x \notin \mathsf{rel}(\mathcal{D})$.*

In order to develop our study on mathematical properties of our diagrammatic system, it is convenient to talk about *equivalence classes (or types) of diagrams* rather than drawn tokens of diagrams. We first identify objects (named circles or points) which have the same name. For example, if s is a circle named by A in one diagram and t is a circle also named by A in another diagram, then s and t are identified up to topological isomorphism. Intuitively, the circles (resp. points) s and t are intended to represent the same set (resp. element). For diagrams, we define their equivalence in terms of the EUL-relations:

Definition 2.6 (Equivalence of EUL-diagrams)

- When any two objects of the same name appear in different diagrams (planes), we identify them up to isomorphism.
- Any EUL-diagrams $\mathcal{D}$ and $\mathcal{E}$ such that $ob(\mathcal{D}) = ob(\mathcal{E})$ are *syntactically equivalent* when $\mathsf{rel}(\mathcal{D}) = \mathsf{rel}(\mathcal{E})$, that is, the following condition holds: For any objects $s, t \in ob(\mathcal{D})$ and any $* \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}$, $s * t$ holds on $\mathcal{D}$, if, and only if $s * t$ holds on $\mathcal{E}$.

For example, the following diagrams $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ of Fig.9 are equivalent since exactly the same EUL-relations $A \bowtie B, A \bowtie C, B \bowtie C, a \vdash\!\dashv A, a \sqsubset B$, and $a \vdash\!\dashv C$ hold on them. (Cf. also Section 4 (1-1) for an extension of our representation system EUL, where $\mathcal{D}_1, \mathcal{D}_2$, and $\mathcal{D}_3$ are distinguished.)

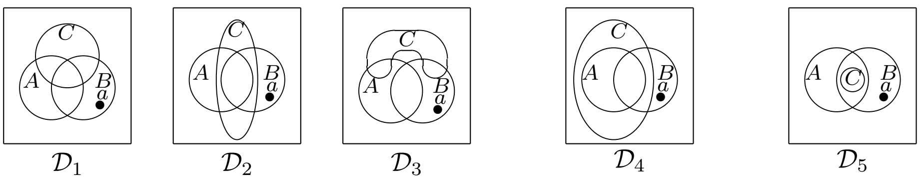


Fig. 9 Equivalence of EUL-diagrams.

On the other hand, $\mathcal{D}_1$ and $\mathcal{D}_4$ (resp. $\mathcal{D}_1$ and $\mathcal{D}_5$) are not equivalent since different EUL-relations hold on them: $A \sqsubset C$ holds on $\mathcal{D}_4$ in place of $A \bowtie C$ of $\mathcal{D}_1$ (resp. $C \sqsubset A$ and $C \sqsubset B$ hold on $\mathcal{D}_5$ in place of $A \bowtie C$ and $C \bowtie B$ of $\mathcal{D}_1$).

Our equation of diagrams may be explained in terms of a kind of "continuous transformation (deformation)" of named circles, which does not change any of the EUL-relations in a diagram. The named circle C in $\mathcal{D}_1$ of Fig.9 can be continuously transformed, without changing the EUL-relations with A, with B and with a in such a way that C covers the intersection region of A and B as it does in $\mathcal{D}_2$. Similarly, C in $\mathcal{D}_1$ can be continuously transformed, without changing the EUL-relations with A, with B and with a in such a way that C is disjoint from the intersection region of A and B as it is in $\mathcal{D}_3$.

In what follows, the diagrams which are syntactically equivalent are identified, and they are referred by a single name.

When $\mathcal{D}$ is an EUL-diagram, we denote by $pt(\mathcal{D})$ the set of named points of $\mathcal{D}$, by $cr(\mathcal{D})$ the set of named circles of $\mathcal{D}$, by $ob(\mathcal{D})$ the set of objects of $\mathcal{D}$, and by $\mathsf{rel}(\mathcal{D})$ the set of EUL-relations holding on $\mathcal{D}$.

2.2 Set-theoretical semantics of EUL

In this section, we give a formal semantics for EUL. Here, we adopt the standard set-theoretical semantics. [5] Intuitively, each circle is interpreted as a set of elements of a given domain, and each point is interpreted as an element of the domain. However, observe that each point of EUL can be considered as a special circle which does not contain, nor cross, any other objects. This observation enables us to interpret the EUL-relations $\sqsubset$ and $\vdash\!\dashv$ uniformly as the subset relation and the disjointness relation, respectively.

Definition 2.7 (Model) A *model* M is a pair (U, I), where U is a non-empty set (the domain of M), and I is an interpretation function which assigns to each diagrammatic object s a non-empty subset of U such that $I(x)$ is a singleton for any named point x, and $I(x) \neq I(y)$ for any named points x, y of distinct names.

Definition 2.8 (Truth-condition) Let $\mathcal{D}$ be an EUL-diagram. $M = (U, I)$ is a *model of* $\mathcal{D}$, written as $M \models \mathcal{D}$, if the following (1) and (2) hold: For all objects s, t of $\mathcal{D}$,

(1) $I(s) \subseteq I(t)$ if $s \sqsubset t$ holds on $\mathcal{D}$,

(2) $I(s) \cap I(t) = \emptyset$ if $s \vdash\!\dashv t$ holds on $\mathcal{D}$.

Note that when s is a named point a, for some $e \in U$, $I(a) = \{e\}$, and the above $I(a) \subseteq I(t)$ of (1) is equivalent to $e \in I(t)$. Similarly, $I(a) \cap I(t) = \emptyset$ of (2) is equivalent to $e \notin I(t)$.

Remark 2.9 (Semantic interpretation of $\bowtie$-relation) By Definition 2.8, the EUL-relation $\bowtie$ does not contribute to the truth-condition of EUL-diagrams. Informally speaking, $s \bowtie t$ may be understood as $I(s) \cap I(t) = \emptyset$ *or* $I(s) \cap I(t) \neq \emptyset$, which is true in any model.

The well-definedness of the truth-conditions in Definition 2.8 follows from Proposition 2.4, which ensures that the EUL-relations holding on a given diagram $\mathcal{D}$ are uniquely determined.

Definition 2.10 (Validity) An EUL-diagram $\mathcal{E}$ is a *semantically valid consequence* of EUL-diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$, written as $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \mathcal{E}$, when the following holds: For any model M, if $M \models \mathcal{D}_1$ and ... and $M \models \mathcal{D}_n$, then $M \models \mathcal{E}$.

Let $\mathcal{D}$ be an EUL-diagram. Let β be a minimal diagram consisting of two objects s and t which is obtained from $\mathcal{D}$ by deleting all objects other than s and t. Then, by definition, we have $\mathcal{D} \models \beta$. (See also Section 3.5.2.)

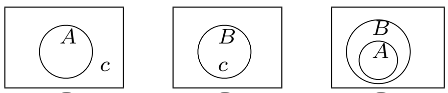
[5] For similar set-theoretical approaches to semantics of Euler diagrams, see [Hammer 1995, Hammer-Shin 1998, Swoboda-Allwein 2004, Howse-Stapleton-Taylor 2005] etc. Our semantics is distinct from theirs in that our diagrams are interpreted in terms of binary relations, and not every region in a diagram has a meaning.

3 Diagrammatic inference system GDS

In this section, we introduce Generalized Diagrammatic Syllogistic inference system GDS for the EUL-diagrams defined in Section 2.1. There are two inference rules of GDS: *unification* and *deletion*. We first give an informal explanation of our unification in Section 3.1, and we then formalize it in Section 3.2. We give an inductive definition of *diagrammatic proofs* of GDS as is usual in the study of symbolic logical systems. In Section 3.4 our GDS is shown to be sound and complete with respect to the set-theoretical semantics given in Section 2.2. In Section 3.5, we discuss some consequences of the completeness theorem of GDS. In particular, we define a class of *normal diagrammatic proofs* of GDS and we show a normal form theorem.

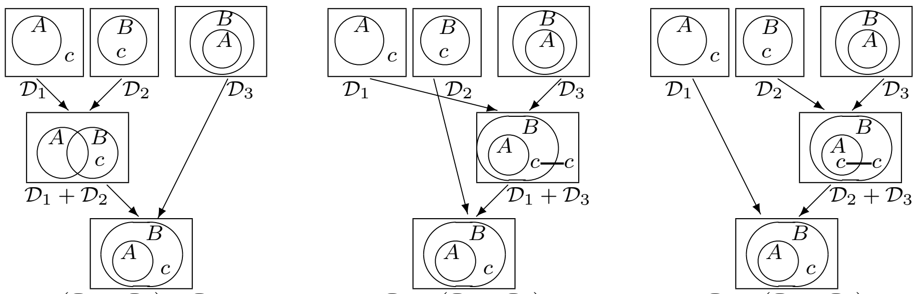
3.1 Introduction to unification

Before giving a formal description of our diagrammatic inference system, we motivate our inference rule *unification*. Let us consider the following question: Given the following diagrams $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$, what diagrammatic information on A, B and c can be obtained? (In what follows, in order to avoid notational complexity in a diagram, we express each named point, say $\overset{c}{\bullet}$, simply by its name c.)



$\mathcal{D}_1$ $\mathcal{D}_2$ $\mathcal{D}_3$

Figs.10, 11, and 12 represent the three ways of solving the question.



$(\mathcal{D}_1 + \mathcal{D}_2) + \mathcal{D}_3$ $\mathcal{D}_2 + (\mathcal{D}_1 + \mathcal{D}_3)$ $\mathcal{D}_1 + (\mathcal{D}_2 + \mathcal{D}_3)$

Fig. 10 Fig. 11 Fig. 12

In Fig.10, at the first step, two diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ are unified to obtain $\mathcal{D}_1 + \mathcal{D}_2$, where the point c in $\mathcal{D}_1$ and $\mathcal{D}_2$ are identified, and B is added to $\mathcal{D}_1$ so that c is inside of B and B overlaps with A without any implication of a relationship between A and B. Then, $\mathcal{D}_1 + \mathcal{D}_2$ is combined with another diagram $\mathcal{D}_3$ to obtain $(\mathcal{D}_1 + \mathcal{D}_2) + \mathcal{D}_3$. Note that the diagrams $\mathcal{D}_1 + \mathcal{D}_2$ and $\mathcal{D}_3$ share two circles A and B: $A \bowtie B$ holds on $\mathcal{D}_1 + \mathcal{D}_2$ and $A \sqsubset B$ holds on $\mathcal{D}_3$. Since the semantic information of $A \sqsubset B$ on $\mathcal{D}_3$ is more accurate than that of $A \bowtie B$ on $\mathcal{D}_1 + \mathcal{D}_2$, according to our semantics of EUL (recall that $A \bowtie B$ means just "true" in our semantics), one keeps the relation $A \sqsubset B$ in the unified diagram $(\mathcal{D}_1 + \mathcal{D}_2) + \mathcal{D}_3$. Observe that the unified diagram represents the information of these diagrams $\mathcal{D}_1, \mathcal{D}_2$, and $\mathcal{D}_3$, that is, their *conjunction*.

Figs.11 and 12, illustrate other procedures to solve the question. At the first step of unifying diagrams $\mathcal{D}_1$ and $\mathcal{D}_3$ in Fig.11 (and $\mathcal{D}_2$ and $\mathcal{D}_3$ in Fig.12), there are two possible positions of the point c. However, EUL-diagrams do not have syntactic devices to represent such disjunctive information about positions of a point. One solution to this problem is, as illustrated in Figs.11 and 12, to introduce Peirce's *linking* of points. However, following the conception we explained in Section 1, we keep our diagrams free from such disjunctive ambiguity. For that purpose, we impose some constraint on unification, called the constraint for *determinacy*: Any two diagrams are not permitted to be unified when the relations between each point and all circles of the two diagrams are not determined. Thus $\mathcal{D}_1$ and $\mathcal{D}_3$ of Fig.11 (respectively $\mathcal{D}_2$ and $\mathcal{D}_3$ of Fig.12) are not permitted to be unified.

We impose another constraint on unification called a constraint for *consistency*, in order to avoid complexity due to conflicting graphical information represented in a single diagram. For example, it is not permitted to unify two diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ when, as is shown in Fig.13, they share two circles C and B such that $a \sqsubset C$ and $a \sqsubset B$ hold on $\mathcal{D}_1$ and $C \vdash\!\dashv B$ holds on $\mathcal{D}_2$. Note that these relations $a \sqsubset C, a \sqsubset B$, and $C \vdash\!\dashv B$ are incompatible in the same diagram. The diagrams $\mathcal{D}_3$ and $\mathcal{D}_4$ in Fig.13 are also not permitted to be unified in our system. Recall that each circle is interpreted by non-empty set in our semantics of Definition 2.7, and hence $\mathcal{D}_3$ and $\mathcal{D}_4$ are also incompatible.

$\mathcal{D}_1$ $\mathcal{D}_2$ $\mathcal{D}_3$ $\mathcal{D}_4$

Fig. 13 Inconsistency

3.2 Generalized diagrammatic syllogistic inference system GDS

In this section, we introduce *unification* and *deletion* of GDS. We formalize our unification of two diagrams by restricting one of them to be a *minimal diagram*, except for one rule called the Point Insertion-rule. Our completeness (Theorem 3.15) ensures that any diagrams $\mathcal{D}_1, \dots, \mathcal{D}_n$ may be unified, under the constraints for determinacy and consistency, into one diagram whose semantic information is equivalent to the conjunction of that of $\mathcal{D}_1, \dots, \mathcal{D}_n$. (We will return to this issue in Section 3.5.1.)

We give a formal description of inference rules in terms of EUL-relations: Given a diagram $\mathcal{D}$ and a minimal diagram α, the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ for the unified diagram $\mathcal{D} + \alpha$ is defined. It is easily checked that the set $\mathsf{rel}(\mathcal{D} + \alpha)$ satisfies the properties of Lemma 2.5, and hence there is a concrete instance of the set. We also give a schematic diagrammatic representation and a concrete example of each rule. In the schematic representation of diagrams, to indicate the occurrence of some objects in a context on a diagram, we write the indicated objects explicitly and indicate the context by "dots" as in the diagram to the right below. For example, when we need to indicate only A and c on the left hand diagram, we could write it as shown on the right.

Definition 3.1 (Inference rules of GDS) *Axiom*, *unification*, and *deletion* of GDS are defined as follows.

Axiom:

A1: For any circles A and B, any minimal diagram where $A \bowtie B$ holds is an axiom.

A2: Any EUL-diagram which consists only of (at least two) points is an axiom.

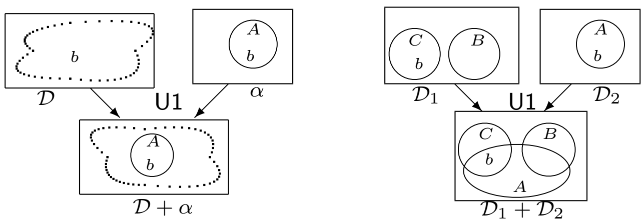
Unification: We denote by $\mathcal{D}+\alpha$ the unified diagram of $\mathcal{D}$ with a minimal diagram α. $\mathcal{D}+\alpha$ is defined when $\mathcal{D}$ and α share one or two objects. We distinguish the following two cases: (I) When $\mathcal{D}$ and α share one object, they may be unified to $\mathcal{D}+\alpha$ by rules U1–U8 according to the shared object and the relation holding on α. Each rule of (I) has a constraint for determinacy. (II) When $\mathcal{D}$ and α share two circles, if the relation which holds on α also holds on $\mathcal{D}$, $\mathcal{D}+\alpha$ is $\mathcal{D}$ itself; otherwise, they may be unified to $\mathcal{D}+\alpha$ by rules U9 or U10 according to the relation holding on α. Each rule of (II) has a constraint for consistency. Moreover, there is another unification rule called the Point Insertion-rule (III).

(I) The case $\mathcal{D}$ and α share one object:

U1: If $b \sqsubset A$ holds on α and $pt(\mathcal{D}) = \{b\}$, then $\mathcal{D}$ and α may be unified to a diagram $\mathcal{D}+\alpha$ such that the set $\mathsf{rel}(\mathcal{D}+\alpha)$ of relations holding on it is the following:

$$\mathsf{rel}(\mathcal{D}) \cup \{b \sqsubset A\} \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}$$

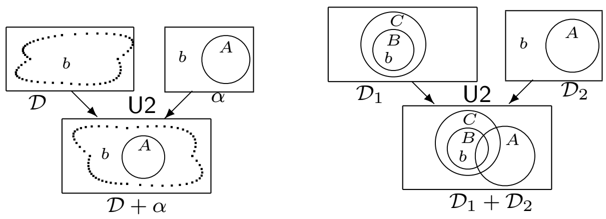
U1 is applied as follows:



U2: If $b \mathrel{\vdash\!\dashv} A$ holds on α and $pt(\mathcal{D}) = \{b\}$, then $\mathcal{D}$ and α may be unified to a diagram $\mathcal{D}+\alpha$ such that the set $\mathsf{rel}(\mathcal{D}+\alpha)$ of relations holding on it is the following:

$$\mathsf{rel}(\mathcal{D}) \cup \{b \mathrel{\vdash\!\dashv} A\} \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}$$

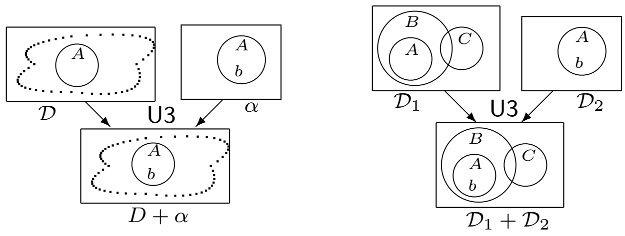
U2 is applied as follows:



U3: If $b \sqsubset A$ holds on α and $A \in cr(\mathcal{D})$, and if $A \sqsubset X$ or $A \mathrel{\vdash\!\dashv} X$ holds for any circle X in $\mathcal{D}$, then $\mathcal{D}$ and α may be unified to a diagram $\mathcal{D}+\alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D}+\alpha)$ is the following:

$$\begin{aligned}\mathsf{rel}(\mathcal{D}) \cup \{b \sqsubset A\} &\cup \{b \sqsubset X \mid A \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \\ &\cup \{b \mathrel{\vdash\!\dashv} X \mid A \mathrel{\vdash\!\dashv} X \in \mathsf{rel}(\mathcal{D})\} \\ &\cup \{b \mathrel{\vdash\!\dashv} x \mid x \in pt(\mathcal{D})\}\end{aligned}$$

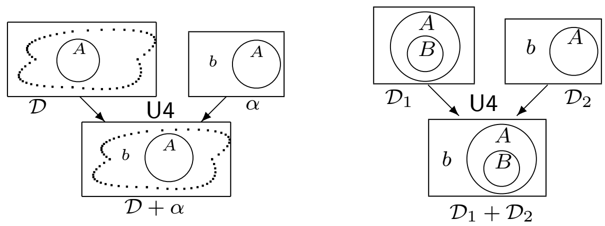
U3 is applied as follows:



U4: If $b \vdash\!\dashv A$ holds on α and $A \in cr(\mathcal{D})$, and if $X \sqsubset A$ holds for any circle X in $\mathcal{D}$, then $\mathcal{D}$ and α may be unified to a diagram $\mathcal{D} + \alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ is the following:

$$\begin{aligned} &\mathsf{rel}(\mathcal{D}) \cup \{b \vdash\!\dashv A\} \cup \{b \vdash\!\dashv X \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \\ &\qquad \cup \{b \vdash\!\dashv x \mid x \in pt(\mathcal{D})\} \end{aligned}$$

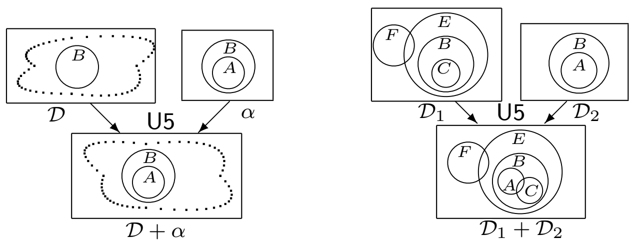
U4 is applied as follows:



U5: If $A \sqsubset B$ holds on α and $B \in cr(\mathcal{D})$, and if $x \vdash\!\dashv B$ holds for any $x \in pt(\mathcal{D})$, then $\mathcal{D}$ and α may be unified to a diagram $\mathcal{D} + \alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ is the following:

$$\begin{aligned} &\mathsf{rel}(\mathcal{D}) \cup \{A \sqsubset B\} \cup \{A \bowtie X \mid X \sqsubset B \text{ or } X \bowtie B \in \mathsf{rel}(\mathcal{D})\} \\ &\qquad \cup \{A \sqsubset X \mid B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \\ &\qquad \cup \{A \vdash\!\dashv X \mid X \vdash\!\dashv B \in \mathsf{rel}(\mathcal{D})\} \\ &\qquad \cup \{x \vdash\!\dashv A \mid x \in pt(\mathcal{D})\} \end{aligned}$$

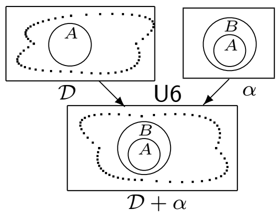
U5 is applied as follows:

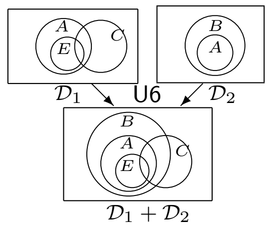


U6: If $A \sqsubset B$ holds on α and $A \in cr(\mathcal{D})$, and if $x \sqsubset A$ holds for any $x \in pt(\mathcal{D})$, then $\mathcal{D}$ and α may be unified to a diagram $\mathcal{D} + \alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ is the following:

$$\begin{aligned} &\mathsf{rel}(\mathcal{D}) \cup \{A \sqsubset B\} \cup \{X \bowtie B \mid A \sqsubset X \text{ or } A \vdash\!\dashv X \text{ or } A \bowtie X \in \mathsf{rel}(\mathcal{D})\} \\ &\qquad \cup \{X \sqsubset B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \\ &\qquad \cup \{x \sqsubset B \mid x \in pt(\mathcal{D})\} \end{aligned}$$

U6 is applied as follows:

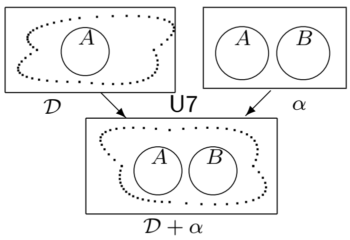


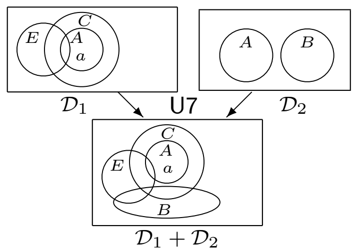


U7: If $A \mathrel{\vdash\!\dashv} B$ holds on α and $A \in cr(\mathcal{D})$, and if $x \sqsubset A$ holds for any $x \in pt(\mathcal{D})$, then $\mathcal{D}$ and α may be unified to a diagram $\mathcal{D}+\alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D}+\alpha)$ is the following:

$$\begin{aligned} \mathsf{rel}(\mathcal{D}) \cup \{A \mathrel{\vdash\!\dashv} B\} &\cup \{X \bowtie B \mid A \sqsubset X \text{ or } A \mathrel{\vdash\!\dashv} X \text{ or } A \bowtie X \in \mathsf{rel}(\mathcal{D})\} \\ &\cup \{X \mathrel{\vdash\!\dashv} B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \\ &\cup \{x \mathrel{\vdash\!\dashv} B \mid x \in pt(\mathcal{D})\} \end{aligned}$$

U7 is applied as follows:

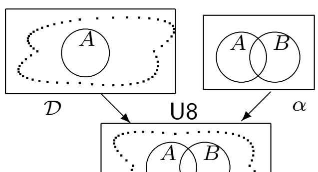


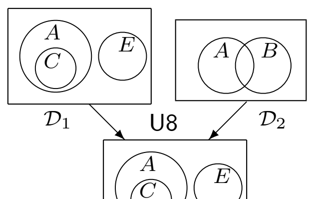


U8: If $A \bowtie B$ holds on α and $A \in cr(\mathcal{D})$, and if $pt(\mathcal{D}) = \emptyset$, then $\mathcal{D}$ and α may be unified to a diagram $\mathcal{D}+\alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D}+\alpha)$ is the following:

$$\mathsf{rel}(\mathcal{D}) \cup \{X \bowtie B \mid X \in cr(\mathcal{D})\}$$

U8 is applied as follows:



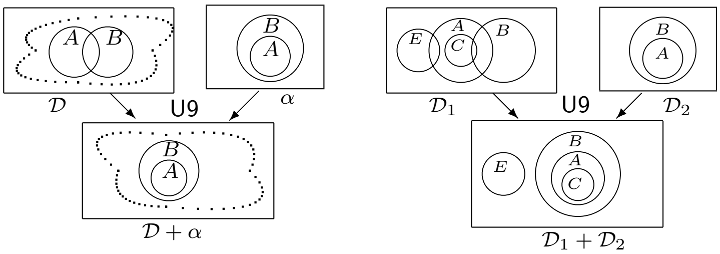


(II) When $\mathcal{D}$ and α share two circles, they may be unified to $\mathcal{D}+\alpha$ by the following U9 and U10 rules.

U9: If $A \sqsubset B$ holds on α and $A \bowtie B$ holds on $\mathcal{D}$, and if there is no object s such that $s \sqsubset A$ and $s \mathrel{\vdash\!\dashv} B$ hold on $\mathcal{D}$, then $\mathcal{D}$ and α may be unified to a diagram $\mathcal{D}+\alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D}+\alpha)$ is the following:

$$\begin{aligned}
\Big(& \mathsf{rel}(\mathcal{D}) \ \setminus \ \{A \bowtie B\} \ \setminus \ \{A \bowtie X \mid B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \ \setminus \ \{A \bowtie X \mid B \mathrel{\vdash\!\dashv} X \in \mathsf{rel}(\mathcal{D})\} \\
& \setminus \ \{X \bowtie B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \ \setminus \ \{Y \bowtie X \mid Y \sqsubset A \text{ and } B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \\
& \setminus \ \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \mathrel{\vdash\!\dashv} B \in \mathsf{rel}(\mathcal{D})\} \Big) \\
& \cup \ \{A \sqsubset B\} \ \cup \ \{A \sqsubset X \mid B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \ \cup \ \{A \mathrel{\vdash\!\dashv} X \mid B \mathrel{\vdash\!\dashv} X \in \mathsf{rel}(\mathcal{D})\} \\
& \cup \ \{X \sqsubset B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \ \cup \ \{Y \sqsubset X \mid Y \sqsubset A \text{ and } B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \\
& \cup \ \{X \mathrel{\vdash\!\dashv} Y \mid X \sqsubset A \text{ and } Y \mathrel{\vdash\!\dashv} B \in \mathsf{rel}(\mathcal{D})\}
\end{aligned}$$

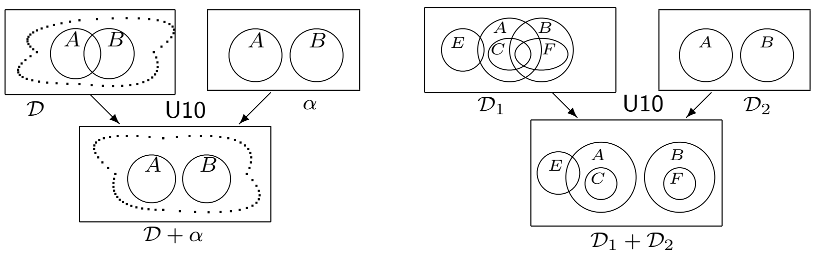
U9 is applied as follows:



U10: If $A \mathrel{\vdash\!\dashv} B$ holds on α and $A \bowtie B$ holds on $\mathcal{D}$, and if there is no object s such that $s \sqsubset A$ and $s \sqsubset B$ hold on $\mathcal{D}$, then $\mathcal{D}$ and α may be unified to a diagram $\mathcal{D}+\alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D}+\alpha)$ is the following:

$$\begin{aligned}
\Big(& \mathsf{rel}(\mathcal{D}) \ \setminus \ \{A \bowtie B\} \ \setminus \ \{X \bowtie B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \ \setminus \ \{X \bowtie A \mid X \sqsubset B \in \mathsf{rel}(\mathcal{D})\} \\
& \setminus \ \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\} \Big) \\
& \cup \ \{A \mathrel{\vdash\!\dashv} B\} \ \cup \ \{X \mathrel{\vdash\!\dashv} B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \ \cup \ \{X \mathrel{\vdash\!\dashv} A \mid X \sqsubset B \in \mathsf{rel}(\mathcal{D})\} \\
& \cup \ \{X \mathrel{\vdash\!\dashv} Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\}
\end{aligned}$$

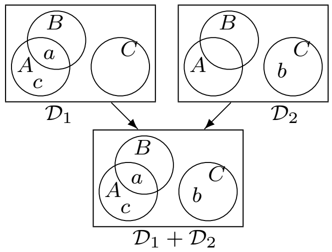
U10 is applied as follows:



(III) Point Insertion: If, for any circles X, Y and for any $\Box \in \{\sqsubset, \sqsupset, \mathrel{\vdash\!\dashv}, \bowtie\}$, $X \Box Y \in \mathsf{rel}(\mathcal{D}_1)$ *iff* $X \Box Y \in \mathsf{rel}(\mathcal{D}_2)$ holds, and if $pt(\mathcal{D}_2)$ is a singleton $\{b\}$ such that $b \notin pt(\mathcal{D}_1)$, then $\mathcal{D}_1$ and $\mathcal{D}_2$ may be unified to a diagram $\mathcal{D}_1+\mathcal{D}_2$ such that the set of relations $\mathsf{rel}(\mathcal{D}_1+\mathcal{D}_2)$ is the following:

$$\mathsf{rel}(\mathcal{D}_1) \cup \mathsf{rel}(\mathcal{D}_2) \cup \{b \mathrel{\vdash\!\dashv} x \mid x \in pt(\mathcal{D}_1)\}$$

Point Insertion is applied as follows:



Deletion: When t is an object of $\mathcal{D}$, t may be deleted from $\mathcal{D}$ to obtain a diagram $\mathcal{D} - t$ under the constraint that $\mathcal{D} - t$ has at least two objects.

It is easily checked that unification preserves EUL-relations other than $s \bowtie t$ of unified diagrams:

Lemma 3.2 (Preservation of EUL-relations) *Let $\mathcal{D} + \alpha$ be an* EUL*-diagram obtained by an application of unification rules between $\mathcal{D}$ and α. For any relation $s\Box t$ with $\Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv\}$, if $s\Box t \in \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha)$, then $s\Box t \in \mathsf{rel}(\mathcal{D} + \alpha)$.*

We give an inductive definition of diagrammatic proofs (d-proofs) of GDS.

We sometimes denote by $\overrightarrow{\mathcal{D}}$ a sequence $\mathcal{D}_1, \dots, \mathcal{D}_n$ of EUL-diagrams, where by a *sequence*, we mean a set of diagrams $\{\mathcal{D}_1, \dots, \mathcal{D}_n\}$, i.e., more than one $\mathcal{D}_i$ may be identified for each $1 \leq i \leq n$. For instance, sequences $\mathcal{D}_1, \mathcal{D}_1, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_2, \mathcal{D}_3$ and $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ may be identified.

Definition 3.3 (Diagrammatic proofs of GDS) A *diagrammatic proof* (or *d-proof*) π of GDS is defined inductively as follows:

1. A diagram $\mathcal{D}$ is a d-proof from the premise $\mathcal{D}$ to the conclusion $\mathcal{D}$.

2. Let π_1 be a d-proof from $\mathcal{D}_1, \dots, \mathcal{D}_n$ to $\mathcal{F}$ and π_2 be a d-proof from $\mathcal{E}_1, \dots, \mathcal{E}_m$ to $\mathcal{E}$, respectively. If $\mathcal{D}$ is obtained by an application of unification of $\mathcal{F}$ and $\mathcal{E}$, then the following (i) is a d-proof π from $\mathcal{D}_1, \dots, \mathcal{D}_n, \mathcal{E}_1, \dots, \mathcal{E}_m$ to $\mathcal{D}$ in GDS.

3. Let π_1 be a d-proof from $\mathcal{D}_1, \dots, \mathcal{D}_n$ to $\mathcal{E}$. If $\mathcal{D}$ is obtained by an application of Deletion to $\mathcal{E}$, then the following (ii) is a d-proof π from $\mathcal{D}_1, \dots, \mathcal{D}_n$ to $\mathcal{D}$ in GDS.

(i)
$$\begin{tikzcd}[column sep=small] \overset{\pi_1}{\mathcal{F}} \arrow[dr] & & \overset{\pi_2}{\mathcal{E}} \arrow[dl] \\ & \mathcal{D} & \end{tikzcd}$$

(ii)
$$\begin{tikzcd} \overset{\pi_1}{\mathcal{E}} \arrow[d] \\ \mathcal{D} \end{tikzcd}$$

Here $\overset{\pi}{\mathcal{D}}$ means a d-proof π with $\mathcal{D}$ as the conclusion. The *length* of a d-proof is defined as the number of applications of inference rules.

Definition 3.4 (Provability) Let $\overrightarrow{\mathcal{D}}$ be a sequence of EUL-diagrams. An EUL-diagram $\mathcal{E}$ is *provable* from $\overrightarrow{\mathcal{D}}$, written as $\overrightarrow{\mathcal{D}} \vdash \mathcal{E}$, if there is a d-proof of $\mathcal{E}$ in GDS from a sequence $\mathcal{D}_1, \dots, \mathcal{D}_m$ which is a subset of $\overrightarrow{\mathcal{D}}$.

3.3 EUL-structure

In this section, we describe the unification rules of Definition 3.1 of Section 3.2 in terms of a graphical representation of EUL-diagrams, which may assist with the understanding and motivation of our unification rules.

As seen in Section 2.1, given an EUL-diagram $\mathcal{D}$, the set $\mathsf{rel}(\mathcal{D})$ of relations holding on it is uniquely determined by Proposition 2.4. $\mathsf{rel}(\mathcal{D})$ can be regarded as a kind of partially ordered structure $(D, \sqsubset, \dashv\vdash)$, called an EUL-structure, where D is the set of names of the objects of $\mathcal{D}$ and:

1. $\sqsubset$ is an irreflexive transitive ordering relation on D;

2. $\dashv\vdash$ is an irreflexive symmetric relation on D.

Furthermore, we have the following properties (cf. Lemma 2.5):

3. ($\dashv\vdash$-downward closedness) For any $s, t, u \in D$, $s \dashv\vdash t$ and $t \sqsupset u$ imply $s \dashv\vdash u$.

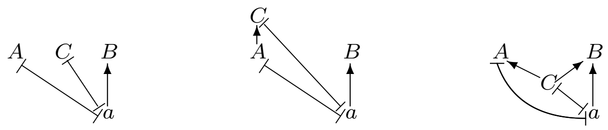
Hence the EUL-structure $(D, \sqsubset, \dashv\vdash)$ is an *event structure* of [Nielsen-Plotkin-Winskel 1980].

4. (Point minimality) For any s and point x of D, not$(s \sqsubset x)$.

5. (Point determinacy) For any s and any point x of D, $x \sqsubset s$ or $x \dashv\vdash s$.

Observe that above properties (1), (2), and (3) imply that, for any pair of elements of D, at most one of the relations $\sqsubset$ and $\dashv\vdash$ holds (cf. Proposition 2.4); because if both of them hold, say $s \sqsubset t$ and $s \dashv\vdash t$, the property (3) implies $s \dashv\vdash s$, which contradicts the irreflexivity of $\dashv\vdash$-relation.

For example, $\mathsf{rel}(\mathcal{D}_1), \mathsf{rel}(\mathcal{D}_4)$ and $\mathsf{rel}(\mathcal{D}_5)$ of Fig.9 in Section 2.1 are expressed graphically as follows: Here the ordering relations $\sqsubset$ are expressed by $\rightarrow$-edges.

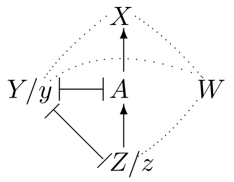


$\mathsf{rel}(\mathcal{D}_1)$ $\mathsf{rel}(\mathcal{D}_4)$ $\mathsf{rel}(\mathcal{D}_5)$

Observe that there is no edge for $\bowtie$-relation.

Remark 3.5 Several authors introduce *abstract (type) syntax*, which is defined independently of *concrete (token) syntax*. See, for example, [Molina 2001, Howse-Molina-Shin-Taylor 2002, Howse-Stapleton-Taylor 2005]. For our EUL, it is possible to define EUL-structures, as abstract syntax, independent of concrete plane diagrams of Definition 2.1 by defining them using the above properties (1)–(5). We can show that any EUL-structure has an EUL-diagram as a concrete instance. However, in this paper, we do not pursue the "abstract diagrams" separated in this way from the concrete objects.

With the graphical representation of EUL-structures, our unification of diagrams can also be expressed by a simple unification of two graphs. In order to describe graphically the unification (U1–U8 rules) of EUL-diagrams $\mathcal{D}$ and α, we focus on the shared object of $\mathcal{D}$ and α, say A, and express the EUL-structure of $\mathsf{rel}(\mathcal{D})$ as follows:



rel($\mathcal{D}$)

$\longrightarrow$-edge denotes $\sqsubset$-relation

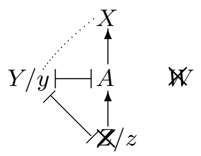
$\vdash\!\dashv$-edge denotes $\vdash\!\dashv$-relation

No edge for $\bowtie$-relation

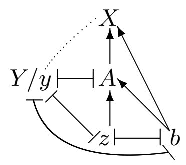
"$\cdots$" denotes one of $\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie$

The variables X, Y, Z, W (resp. y, z) are representative circles (resp. points) which are possibly related to A. When it makes no difference whether a possibly related object is circle or point, we denote the object as Y/y (instead of simply writing s). Each dotted line between objects expresses that there may be one of the relations $\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie$ between the objects. Note that there is no edge for each $\bowtie$-relation, as seen between A and W. We omit the trivial transitive edge $Z \longrightarrow X$ to avoid notational complexity. In the following description of each unification rule for $\mathcal{D}$ and α, we give a graphical representation of the EUL-structures of rel($\mathcal{D}$) in the left-hand graph, and rel($\mathcal{D} + \alpha$) in the right-hand graph. We begin with U3-rule since U1 and U2 rules are rather untypical cases:

U3 Under the constraint of U3-rule, there is no circle Z such that $Z \sqsubset A$ holds, and no circle W such that $A \bowtie W$ holds, which is expressed by $\times$ in the graph of rel($\mathcal{D}$). According to U3-rule of Definition 3.1, rel($\mathcal{D} + (b \sqsubset A)$) is represented by the graph on the right.



rel($\mathcal{D}$)

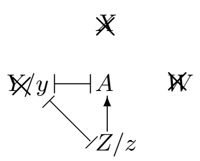


rel($\mathcal{D} + (b \sqsubset A)$)

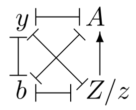
It is easily seen that rel($\mathcal{D} + (b \sqsubset A)$) is an EUL-structure: I.e., the augmented edges do not violate the properties of EUL-structure.

Note also that, without the constraint, i.e., if there is a circle Z or W as above, in order to preserve Point determinacy, we should fix a relation between b and Z (resp. W) to $\sqsubset$ or $\vdash\!\dashv$. However, neither of them is sound with respect to our formal semantics of EUL. (See the soundness theorem of GDS of Section 3.4.)

U4 Under the constraint of U4-rule, there is no circle X such that $A \sqsubset X$ holds, no circle Y such that $A \vdash\!\dashv Y$ holds, and no circle W such that $A \bowtie W$ holds, which is expressed by $\times$ in the graph of rel($\mathcal{D}$). According to U4-rule of Definition 3.1, rel($\mathcal{D} + (b \vdash\!\dashv A)$) is represented by the right hand graph below.



rel($\mathcal{D}$)

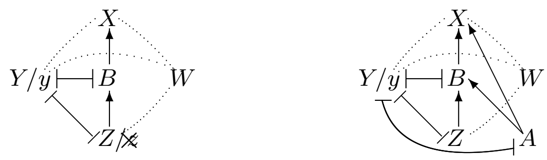


rel($\mathcal{D} + (b \vdash\!\dashv A)$)

It is easily seen that rel($\mathcal{D} + (b \vdash\!\dashv A)$) is an EUL-structure: I.e., the augmented edges do not violate the properties of EUL-structure.

Without the constraint, i.e., if there is a circle X, Y or W as above, in order to preserve Point determinacy, we should fix a relation between b and X (resp. Y, W) to $\sqsubset$ or $\vdash\!\dashv$ in rel($\mathcal{D} + (b \vdash\!\dashv A)$). However, none of them is sound with respect to our semantics of EUL.

U5 Under the constraint of U5-rule, there is no point z such that $z \sqsubset B$ holds.



$\mathsf{rel}(\mathcal{D})$ $\qquad$ $\mathsf{rel}(\mathcal{D} + (A \sqsubset B))$

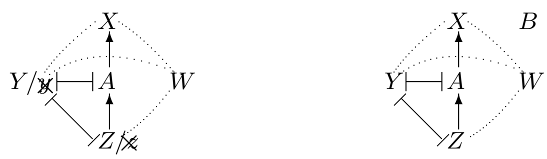
U6 Under the constraint of U6-rule, there is no point y such that $y \mathbin{\vdash\!\dashv} A$ holds.

$\mathsf{rel}(\mathcal{D})$ $\qquad$ $\mathsf{rel}(\mathcal{D} + (A \sqsubset B))$

U7 Under the constraint of U7-rule, there is no point y such that $y \mathbin{\vdash\!\dashv} A$ holds.

$\mathsf{rel}(\mathcal{D})$ $\qquad$ $\mathsf{rel}(\mathcal{D} + (A \mathbin{\vdash\!\dashv} B))$

U8 Under the constraint of U8-rule, there is no point in $\mathcal{D}$.



$\mathsf{rel}(\mathcal{D})$ $\qquad$ $\mathsf{rel}(\mathcal{D} + (A \bowtie B))$

U1 Under the constraint of U1-rule, there is no point y in $\mathcal{D}$ other than b.

$\mathsf{rel}(\mathcal{D})$ $\qquad$ $\mathsf{rel}(\mathcal{D} + (b \sqsubset A))$

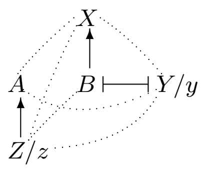
U2 Under the constraint of U2-rule, there is no point y in $\mathcal{D}$ other than b.

$\mathsf{rel}(\mathcal{D})$ $\qquad$ $\mathsf{rel}(\mathcal{D} + (b \mathbin{\vdash\!\dashv} A))$

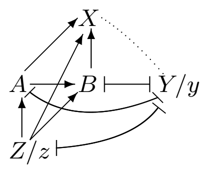
Observe that for the above U1–U8 rules, each constraint is introduced to preserve Point determinacy of EUL-structure after the unification.

In U9 and U10 rules of Definition 3.1, the unified diagrams $\mathcal{D}$ and α share two circles, which makes the graphical description of $\mathsf{rel}(\mathcal{D})$ complicated. In order to avoid notational complexity, we omit irrelevant objects and edges, which are retained after the application of U9 or U10 rule.

U9 Under the constraint of U9-rule, there is no object s such that $s \sqsubset A$ and $s \vDash B$ hold on $\mathcal{D}$, i.e., in the following description of $\mathsf{rel}(\mathcal{D})$, the dotted line between Y/y and A should not be $\rightarrow$, and the dotted line between Z/z and B should not be $\vDash$.



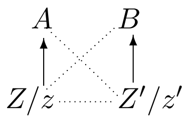
$\mathsf{rel}(\mathcal{D})$



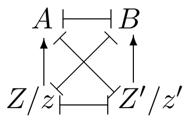
$\mathsf{rel}(\mathcal{D} + (A \sqsubset B))$

Observe that, after the unification, some of the dotted lines of $\mathsf{rel}(\mathcal{D})$ are fixed to $\rightarrow$ or $\vDash$ in $\mathsf{rel}(\mathcal{D}+(A \sqsubset B))$ according to Definition 3.1. We need to check that $\mathsf{rel}(\mathcal{D}+(A \sqsubset B))$ is an EUL-structure; for example, if the dotted line between A and X in $\mathsf{rel}(\mathcal{D})$ is $A \vDash X$ (or $A \leftarrow X$), after the application of U9-rule, there are two incompatible edges $\vDash$ (resp. $\leftarrow$) and $\rightarrow$ between A and X, which violates the irreflexivity of the $\vDash$-relation of EUL-structure. It is shown that, because of our constraint for U9-rule, the dotted line between A and X is $\bowtie$ (i.e., no edge) or $\rightarrow$. Observe that, if we have $A \vDash X$ in $\mathsf{rel}(\mathcal{D})$, by the $\vDash$-downward closedness of $\mathsf{rel}(\mathcal{D})$, we have $Z/z \vDash B$ in $\mathsf{rel}(\mathcal{D})$, which contradicts the constraint. If we have $A \leftarrow X$ in $\mathsf{rel}(\mathcal{D})$, by the transitivity of $\mathsf{rel}(\mathcal{D})$, we have $A \leftarrow B$ in $\mathsf{rel}(\mathcal{D})$, which contradicts the presupposition of U9-rule, i.e., there is no edge between A and B in $\mathsf{rel}(\mathcal{D})$. Thus the dotted line between A and X should be $\bowtie$ (i.e., no edge) or $\rightarrow$, either of which is compatible with the edge $A \rightarrow X$ in $\mathsf{rel}(\mathcal{D}+(A \sqsubset B))$. Similarly, it is shown that the other dotted lines of $\mathsf{rel}(\mathcal{D})$ are compatible with the edges of $\mathsf{rel}(\mathcal{D} + (A \sqsubset B))$. Then it is easily checked that the properties (1)–(5) of EUL-structures hold in $\mathsf{rel}(\mathcal{D} + (A \sqsubset B))$, and hence it is an EUL-structure.

U10 Under the constraint of U10-rule, there is no object s such that $s \sqsubset A$ and $s \sqsubset B$ hold on $\mathcal{D}$, i.e., in the following graph of $\mathsf{rel}(\mathcal{D})$, the dotted line between Z'/z' and A (and also between Z/z and B) should not be $\rightarrow$.



$\mathsf{rel}(\mathcal{D})$



$\mathsf{rel}(\mathcal{D} + (A \vDash B))$

We show that there are no incompatible edges in $\mathsf{rel}(\mathcal{D} + (A \vDash B))$. For the dotted line between Z/z and B, it is not $\rightarrow$ by the constraint for U10-rule. Furthermore, assume to the contrary that we have $Z/z \leftarrow B$ in $\mathsf{rel}(\mathcal{D})$. Then, by the transitivity of $\mathsf{rel}(\mathcal{D})$, we have $A \leftarrow B$ in $\mathsf{rel}(\mathcal{D})$, which contradicts the presupposition of U10-rule, i.e., there is no edge between A and B. Hence the dotted line between Z/z and B should be $\bowtie$ (i.e., no edge) or $\vDash$, either of which is compatible with the edge $Z/z \vDash B$ in $\mathsf{rel}(\mathcal{D} + (A \vDash B))$. Similarly, it is shown that the other two dotted lines of $\mathsf{rel}(\mathcal{D})$ are compatible with the edges of $\mathsf{rel}(\mathcal{D} + (A \vDash B))$. Then it is easily checked that the properties (1)–(5) of EUL-structures hold in $\mathsf{rel}(\mathcal{D} + (A \vDash B))$, and hence it is an EUL-structure.

3.4 Soundness and completeness of GDS

One of the main goals of logical analysis of an inference system $\mathcal{L}$ is to establish soundness and completeness with respect to a formal semantics of $\mathcal{L}$. In particular, the completeness theorem of $\mathcal{L}$ is formulated as follows: If a formula $\mathcal{E}$ is a semantically valid consequence of formulas $\mathcal{D}_1, \ldots, \mathcal{D}_n$ then there is a proof of $\mathcal{E}$ from $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in $\mathcal{L}$. For the completeness of the diagrammatic system GDS, we assume the following condition called semantic consistency: *There exists a model M for the premises $\mathcal{D}_1, \ldots, \mathcal{D}_n$ (i.e., $M \models \mathcal{D}_i$ for any $1 \leq i \leq n$).* Without this condition, any diagram, say $\mathcal{E}$ where $A \vdash\!\dashv C$ holds, is a valid consequence of an inconsistent set of premise diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ where $a \sqsubset B$ and $a \vdash\!\dashv B$ hold, respectively, although there is no d-proof of $\mathcal{E}$ from $\mathcal{D}_1$ and $\mathcal{D}_2$ in GDS. [6]

Under the above assumption, we prove the completeness theorem of GDS (Theorem 3.15) with respect to our formal semantics of Section 2.2. Based on the semantics of EUL, the validity of any EUL-diagram $\mathcal{E}$ is determined by the set of relations holding on it. Hence, the premise $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \mathcal{E}$ of the completeness is equivalent to saying that $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \beta$ for any minimal diagram β which corresponds to the relation holding on $\mathcal{E}$. Thus we first show atomic completeness (Proposition 3.14), which states that, for any minimal diagram β, if $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \beta$ then β is provable from $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in GDS under the assumption of semantic consistency of $\mathcal{D}_1, \ldots, \mathcal{D}_n$. Then using such provable minimal diagrams, we give a canonical way to construct a d-proof of $\mathcal{E}$.

In order to show the atomic completeness, we construct syntactic models, whose domain consists of diagrammatic objects, i.e., named points and circles. The interpretation function is defined based on the provability of diagrams in GDS so that the validity of any minimal diagram in such a model implies the provability of the minimal diagram in GDS. Recall that each named point can be regarded as a special circle which does not contain, nor cross, any other object. Then the set of relations $\mathsf{rel}(\mathcal{D})$ of an EUL-diagram $\mathcal{D}$ can be seen as an algebraic structure with $\sqsubset$-relations as ordering relations. Thus our canonical models are constructed in a similar way to the usual syntactic model, the Lindenbaum algebra, in the literature of algebraic semantics for various propositional logics.

In what follows, we sometimes refer to any minimal diagram, say α where $s \sqsubset t$ holds, by the EUL-relation holding on it, as $s \sqsubset t$. Note that this convention is harmless since there is a trivial one-to-one correspondence between the minimal diagrams and the EUL-relations under the identification of diagrams defined in Definition 2.6 of Section 2.1.

We first show the soundness theorem of GDS with respect to our formal semantics of EUL:

Theorem 3.6 (Soundness of GDS) *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{E}$ be EUL-diagrams.*

$$\text{If } \mathcal{D}_1, \ldots, \mathcal{D}_n \vdash \mathcal{E} \text{ in GDS, then } \mathcal{D}_1, \ldots, \mathcal{D}_n \models \mathcal{E}.$$

Proof. By induction on the length of the d-proof from $\mathcal{D}_1, \ldots, \mathcal{D}_n$ to $\mathcal{E}$. It is sufficient to show that each rule of U1–U10, Point Insertion, and Deletion is sound in the sense that if each premise of the rule is true in a model, then the conclusion is also true in the model. We prove it only for U10 rule since the other rules are proved similarly. Let $M = (U, I)$ be a model.

U10 Let $M \models \mathcal{D}$ and $M \models A \vdash\!\dashv B$. We show $M \models \mathcal{D} + (A \vdash\!\dashv B)$. We divide the following cases according to the relation holding on $\mathcal{D} + (A \vdash\!\dashv B)$.

[6] In place of our semantic restriction, it is possible to extend GDS by adding an inference rule corresponding to the absurdity rule of Gentzen's natural deduction system: We can infer any diagram from a pair of inconsistent diagrams. (For natural deduction systems, see, for example, [Gentzen 1969, Prawitz 1965, Troelstra-Schwichtenberg 2000].) Such a rule is introduced in, for example, [Howse-Stapleton-Taylor 2005] for spider diagrams; [Hammer-Danner 1993] for Venn diagrams; [Swoboda-Allwein 2004, Swoboda-Allwein 2005] for Euler/Venn diagrams. However, such a rule is so linguistic that we prefer our semantic restriction in our framework of a diagrammatic inference system.

1. When $R \in \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(A \mapsfrom\!\!\!\mapsto B)$ such that $R = s \mapsfrom\!\!\!\mapsto t$, we have $I(s) \cap I(t) = \emptyset$ by the induction hypothesis. The case that R is $s \sqsubset t$ is similar.
2. When $R \in \{X \mapsfrom\!\!\!\mapsto B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\}$, we have $I(X) \cap I(B) = \emptyset$ by the induction hypotheses $I(A) \cap I(B) = \emptyset$ and $I(X) \subseteq I(A)$.
3. The case $R \in \{X \mapsfrom\!\!\!\mapsto A \mid X \sqsubset B \in \mathsf{rel}(\mathcal{D})\}$ is similar.
4. When $R \in \{X \mapsfrom\!\!\!\mapsto Y \mid X \sqsubset A, Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\}$, we have $I(X) \cap I(Y) = \emptyset$ by the induction hypotheses $I(A) \cap I(B) = \emptyset$, $I(X) \subseteq I(A)$, and $I(Y) \subseteq I(B)$. ■

For completeness, let us begin by defining the notion of semantic consistency:

Definition 3.7 (Semantic consistency) A sequence of diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$ is *semantically consistent* if there is a model M such that $M \models \mathcal{D}_i$ for any $1 \leq i \leq n$.

When $\overrightarrow{\mathcal{D}}$ is a sequence $\mathcal{D}_1, \ldots, \mathcal{D}_n$ of diagrams, we sometimes write $M \models \overrightarrow{\mathcal{D}}$ for the formula $\forall_{1 \leq i \leq n}(M \models \mathcal{D}_i)$.

It is obvious that the soundness theorem (Theorem 3.6) also holds under the assumption of the semantic consistency of the premise diagrams. The following is an important consequence of semantic consistency:

Lemma 3.8 (Semantic consistency) *Let $\vec{\alpha}$ be a sequence of minimal diagrams which is semantically consistent. Then none of the following holds in* GDS *for any objects s and t:*

1. *$\vec{\alpha} \vdash s \sqsubset t$ and $\vec{\alpha} \vdash s \mapsfrom\!\!\!\mapsto t$.*
2. *There is an object u such that $\vec{\alpha} \vdash s \mapsfrom\!\!\!\mapsto t$ and $\vec{\alpha} \vdash u \sqsubset s$ and $\vec{\alpha} \vdash u \sqsubset t$.*

In order to show the completeness theorem of GDS, we construct two kinds of syntactic models, called *canonical models*. We first define the simpler one.

Definition 3.9 (Canonical model $M_{\vec{\alpha}}$) Let $\vec{\alpha}$ be a sequence $\alpha_1, \ldots, \alpha_n$ of minimal diagrams which is semantically consistent. A syntactic model $M_{\vec{\alpha}} = (M_{\vec{\alpha}}, I_{\vec{\alpha}})$, called a *canonical model*, for $\vec{\alpha}$ is defined as follows:

- The domain $M_{\vec{\alpha}}$ is the set of diagrammatic objects (named circles and points):

$$M_{\vec{\alpha}} = \{s \mid s \text{ is a diagrammatic object}\}.$$

- $I_{\vec{\alpha}}$ is an interpretation function such that, for any object t,

$$I_{\vec{\alpha}}(t) = \{s \mid \vec{\alpha} \vdash s \sqsubset t \text{ in } \mathsf{GDS}\} \cup \{t\}.$$

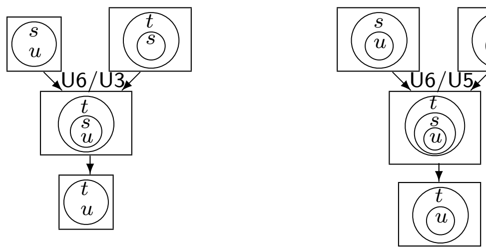
Observe that in the above definition of $I_{\vec{\alpha}}$, when t is a point, say a, its interpretation $I_{\vec{\alpha}}(a)$ is the singleton $\{a\}$ since $\vec{\alpha} \not\vdash s \sqsubset a$ for any object s by soundness (Theorem 3.6).

Lemma 3.10 (Canonical model 1) *Let $\vec{\alpha}$ be a sequence $\alpha_1, \ldots, \alpha_n$ of minimal diagrams which is semantically consistent. Then $M_{\vec{\alpha}}$ is a model of $\vec{\alpha}$.*

Proof. We show that $M_{\vec{\alpha}} \models \alpha_i$ for each $\alpha_i \in \vec{\alpha}$ $(1 \leq i \leq n)$. The case $\alpha_i = s \bowtie t$ is trivial. Otherwise, we divide the following cases according to the form of α_i:

1. When $\alpha_i \in \vec{\alpha}$ is $s \sqsubset t$, we have $\vec{\alpha} \vdash s \sqsubset t$ in GDS. We show $M_{\vec{\alpha}} \models s \sqsubset t$, i.e., $I_{\vec{\alpha}}(s) \subseteq I_{\vec{\alpha}}(t)$. Let $u \in I_{\vec{\alpha}}(s)$.
   (a) When $u \equiv s$, we immediately have $s \in I_{\vec{\alpha}}(t)$ by the fact $\vec{\alpha} \vdash s \sqsubset t$.
   (b) Otherwise, by the definition of $I_{\vec{\alpha}}(s)$, we have $\vec{\alpha} \vdash u \sqsubset s$. By composing it with $\vec{\alpha} \vdash s \sqsubset t$ as seen in the following d-proof, we have $\vec{\alpha} \vdash u \sqsubset t$ in GDS, that is, $u \in I_{\vec{\alpha}}(t)$.

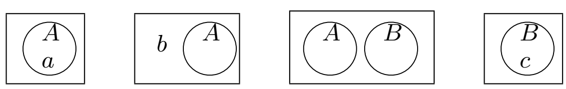
When u is a point: When u is a circle:



2. When $\alpha_i \in \vec{\alpha}$ is $s \vdash\!\dashv t$, we have $\vec{\alpha} \vdash s \vdash\!\dashv t$ in GDS. We show $M_{\vec{\alpha}} \models s \vdash\!\dashv t$, i.e., $I_{\vec{\alpha}}(s) \cap I_{\vec{\alpha}}(t) = \emptyset$. When both s and t are points, the claim is trivial. Otherwise, assume to the contrary that some $u \in I_{\vec{\alpha}}(s) \cap I_{\vec{\alpha}}(t)$.
   (a) When $u \equiv s$, we have $s \in I_{\vec{\alpha}}(t)$, i.e., $\vec{\alpha} \vdash s \sqsubset t$. This, together with $\vec{\alpha} \vdash s \vdash\!\dashv t$, is a contradiction by Lemma 3.8(1).
   (b) The same applies to the case $u \equiv t$.
   (c) Otherwise, $s \not\equiv u \not\equiv t$, and we have $\vec{\alpha} \vdash u \sqsubset s$ and $\vec{\alpha} \vdash u \sqsubset t$ by the definition of $I_{\vec{\alpha}}(s)$ and $I_{\vec{\alpha}}(t)$. They contradict $\vec{\alpha} \vdash s \vdash\!\dashv t$ by Lemma 3.8(2). ■

As an illustration of the canonical model, let us consider the following example.

Example 3.11 (Canonical model $M_{\vec{\alpha}}$) Let $\vec{\alpha}$ be the following minimal diagrams $\alpha_1, \alpha_2, \alpha_3, \alpha_4$:



α_1 α_2 α_3 α_4

Observe that we have $\vec{\alpha} \nvdash b \sqsubset B$ and $\vec{\alpha} \nvdash b \vdash\!\dashv B$. In such a case, we say that the point b is indeterminate with respect to the circle B. Let us construct a canonical model for the $\vec{\alpha}$. Following Definition 3.9, we define

$$I_{\vec{\alpha}}(A) = \{A, a\} \text{ and } I_{\vec{\alpha}}(B) = \{B, c\}.$$

Note that the indeterminate point b w.r.t. B is not contained in the interpretation $I_{\vec{\alpha}}(B)$ of B. With this interpretation, for any point $x \in I_{\vec{\alpha}}(B)$, we have $\vec{\alpha} \vdash x \sqsubset B$ (i.e., for $c \in I_{\vec{\alpha}}(B)$, $\vec{\alpha} \vdash c \sqsubset B$). In general, validity of $\sqsubset$-relation in the model $M_{\vec{\alpha}}$ imply provability of $\sqsubset$-relation.

However, $x \notin I_{\vec{\alpha}}(B)$, i.e., $I_{\vec{\alpha}}(x) \cap I_{\vec{\alpha}}(B) = \emptyset$, does not necessarily imply $\vec{\alpha} \vdash x \vdash\!\dashv B$; because we do not have $\vec{\alpha} \vdash b \vdash\!\dashv B$, while $b \notin I_{\vec{\alpha}}(B)$ in the above example. Thus, in the canonical model $M_{\vec{\alpha}}$ of Definition 3.9, validity of $\vdash\!\dashv$-relation does not imply provability of $\vdash\!\dashv$-relation, and hence the model is not enough to establish completeness.

Let us try to modify the model $M_{\vec{\alpha}}$ of Example 3.11 so that the indeterminate point b w.r.t. B is contained in the interpretation $I'_{\vec{\alpha}}(B)$ of B:

$$I'_{\vec{\alpha}}(A) = \{A, a\} \text{ and } I'_{\vec{\alpha}}(B) = \{B, c, b\}.$$

This definition also provides a model of $\vec{\alpha}$, that is, all of $\alpha_1, \alpha_2, \alpha_3$, and α_4 are true in the model. With this interpretation function $I'_{\vec{\alpha}}$, for any point $x \notin I'_{\vec{\alpha}}(B)$, we have $\vec{\alpha} \vdash x \mapsto B$ (i.e., for $a \notin I'_{\vec{\alpha}}(B)$, $\vec{\alpha} \vdash a \mapsto B$).

However, in this model, $x \in I'_{\vec{\alpha}}(B)$ does not necessarily imply $\vec{\alpha} \vdash x \sqsubset B$; because we do not have $\vec{\alpha} \vdash b \sqsubset B$, while $b \in I'_{\vec{\alpha}}(B)$.

Although the above two kinds of models alone are insufficient to establish completeness, we can obtain our completeness result in the following manner: we construct the model $M_{\vec{\alpha}}$ of Definition 3.9 for validity of $\sqsubset$-relation, which implies provability of $\sqsubset$-relation, and the model $M_{\vec{\alpha},B}$ of the following Definition 3.12 for validity of $\mapsto$-relation, which implies provability of $\mapsto$-relation.

Let us construct the second syntactic model.

Definition 3.12 (Canonical model $M_{\vec{\alpha},B}$) Let $\vec{\alpha}$ be a sequence $\alpha_1, \ldots, \alpha_n$ of minimal diagrams which is semantically consistent. Let B be a fixed named circle. A canonical model $M_{\vec{\alpha},B} = (M_{\vec{\alpha},B}, I_{\vec{\alpha},B})$ for $\vec{\alpha}$ is defined as follows:

- The domain $M_{\vec{\alpha},B}$ is the same set as $M_{\vec{\alpha}}$ of Definition 3.9.
- $I_{\vec{\alpha},B}$ is an interpretation function such that
  - for $t \equiv B$, or for any t such that $\vec{\alpha} \vdash B \sqsubset t$ in GDS,
    $I_{\vec{\alpha},B}(t) = I_{\vec{\alpha}}(t) \cup \{s \mid \vec{\alpha} \nvdash B \sqsubset s \text{ and } \vec{\alpha} \nvdash s \sqsubset B \text{ and } \vec{\alpha} \nvdash s \mapsto B\}$;
  - for any t ($\not\equiv B$) such that $\vec{\alpha} \nvdash B \sqsubset t$ in GDS,
    $I_{\vec{\alpha},B}(t) = I_{\vec{\alpha}}(t)$.

As seen in Definition 3.9, observe that $I_{\vec{\alpha},B}(a) = \{a\}$ when a is a point. Note also that $I_{\vec{\alpha},B}(t)$ is equal to $I_{\vec{\alpha}}(t)$ of Definition 3.9 when $\vec{\alpha} \nvdash B \sqsubset t$.

We sometimes writ $\vec{\alpha} \nvdash s \square t$ when none of $\vec{\alpha} \vdash s \sqsubset t, \vec{\alpha} \vdash t \sqsubset s,$ and $\vec{\alpha} \vdash s \mapsto t$ holds.

Let us show that $M_{\vec{\alpha},B}$ is a model of $\vec{\alpha}$.

Lemma 3.13 (Canonical model 2) *Let $\vec{\alpha}$ be a sequence $\alpha_1, \ldots, \alpha_n$ of minimal diagrams which is semantically consistent. Let B be a fixed named circle. Then $M_{\vec{\alpha},B}$ is a model of $\vec{\alpha}$.*

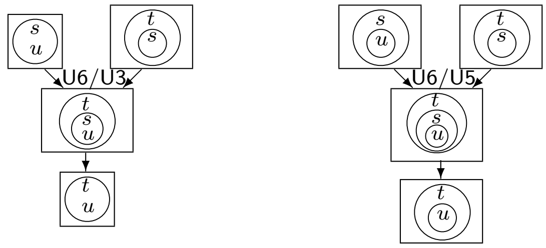
Proof. We show that, for each $\alpha_i \in \vec{\alpha}$ $(1 \leq i \leq n)$, $M_{\vec{\alpha},B} \models \alpha_i$. The case $\alpha_i = s \bowtie t$ is trivial. Otherwise, we divide the following cases according to the form of α_i:

1. When $\alpha_i \in \vec{\alpha}$ is $s \sqsubset t$, we have $\vec{\alpha} \vdash s \sqsubset t$. We show $I_{\vec{\alpha},B}(s) \subseteq I_{\vec{\alpha},B}(t)$. Let $u \in I_{\vec{\alpha},B}(s)$.
   (a) When $u \equiv s$, by the fact $\vec{\alpha} \vdash s \sqsubset t$, we immediately have $s \in I_{\vec{\alpha},B}(t)$ by the definition of $I_{\vec{\alpha},B}(t)$.

(b) Otherwise ($u \not\equiv s$), we divide the following two cases according to s and B:

(i) When $s \equiv B$ or $\vec{\alpha} \vdash B \sqsubset s$ hold, by the definition of $I_{\vec{\alpha},B}(s)$, we have (i-1) $\vec{\alpha} \vdash u \sqsubset s$ or (i-2) $\vec{\alpha} \not\vdash u \Box B$ with $\Box \in \{\sqsubset, \sqsupset, \vDash\!\!\dashv\}$. (i-1) implies, together with $\vec{\alpha} \vdash s \sqsubset t$, that $\vec{\alpha} \vdash u \sqsubset t$, i.e., $u \in I_{\vec{\alpha},B}(t)$ by the following d-proofs:

When u is a point: When u is a circle:



For (i-2), $\vec{\alpha} \vdash B \sqsubset s$ and $\vec{\alpha} \vdash s \sqsubset t$ imply $\vec{\alpha} \vdash B \sqsubset t$ by the same d-proof as above, where u is B. Hence, in conjunction with $\vec{\alpha} \not\vdash u \Box B$, we have $u \in I_{\vec{\alpha},B}(t)$ by the definition of $I_{\vec{\alpha},B}(t)$.

(ii) When $\vec{\alpha} \not\vdash B \sqsubset s$, by the definition of $I_{\vec{\alpha},B}(s)$, we have $\vec{\alpha} \vdash u \sqsubset s$. Hence this case is the same as (i-1).

2. When $\alpha_i \in \vec{\alpha}$ is $B \vDash\!\!\dashv t$, we have $\vec{\alpha} \vdash B \vDash\!\!\dashv t$. Observe that it implies $\vec{\alpha} \not\vdash B \sqsubset t$ since $\vec{\alpha}$ is semantically consistent. Hence we have $I_{\vec{\alpha},B}(t) = I_{\vec{\alpha}}(t)$. In order to show $M_{\vec{\alpha},B} \models B \vDash\!\!\dashv t$, assume to the contrary that some $u \in I_{\vec{\alpha},B}(B) \cap I_{\vec{\alpha}}(t)$.

   (a) The case $u \equiv B$ is impossible since $B \notin I_{\vec{\alpha}}(t)$.

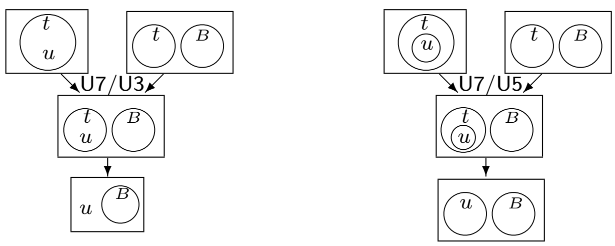
   (b) When $u \equiv t$, we have $t \in I_{\vec{\alpha},B}(B)$. That is $\vec{\alpha} \vdash t \sqsubset B$, which contradicts $\vec{\alpha} \vdash B \vDash\!\!\dashv t$ by Lemma 3.8(1).

   (c) Otherwise ($B \not\equiv u \not\equiv t$), by the definitions of $I_{\vec{\alpha},B}(B)$ and $I_{\vec{\alpha}}(t)$, there are the following two cases, each of which leads to a contradiction:

      i. $\vec{\alpha} \vdash u \sqsubset B$, and $\vec{\alpha} \vdash u \sqsubset t$;
      ii. $\vec{\alpha} \not\vdash u \Box B$ with $\Box \in \{\sqsubset, \sqsupset, \vDash\!\!\dashv\}$, and $\vec{\alpha} \vdash u \sqsubset t$.

   (i) contradicts $\vec{\alpha} \vdash B \vDash\!\!\dashv t$ by Lemma 3.8(2). For (ii), $\vec{\alpha} \vdash u \sqsubset t$ and $\vec{\alpha} \vdash B \vDash\!\!\dashv t$ imply, by the following d-proofs, that $\vec{\alpha} \vdash u \vDash\!\!\dashv B$, which contradicts $\vec{\alpha} \not\vdash u \Box B$:

   When u is a point: When u is a circle:

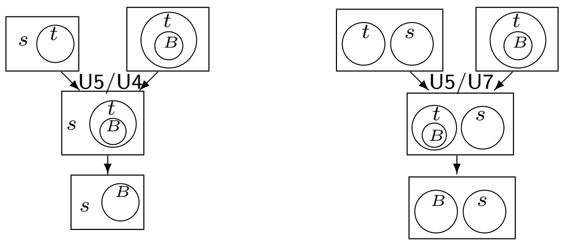


3. When $\alpha_i \in \vec{\alpha}$ is $s \vDash\!\!\dashv t$ with $s \not\equiv B \not\equiv t$, we have $\vec{\alpha} \vdash s \vDash\!\!\dashv t$. We show that $I_{\vec{\alpha},B}(s) \cap I_{\vec{\alpha},B}(t) = \emptyset$. When both s and t are points, the claim is trivial. Otherwise, assume to the contrary that some $u \in I_{\vec{\alpha},B}(s) \cap I_{\vec{\alpha},B}(t)$.

(a) When $u \equiv s$, we have $s \in I_{\vec{\alpha},B}(t)$. We divide the following two cases according to whether or not $\vec{\alpha} \vdash B \sqsubset t$ holds:

(i) When $\vec{\alpha} \vdash B \sqsubset t$ holds, by the definition of $I_{\vec{\alpha},B}(t)$, we have (i-1) $\vec{\alpha} \vdash s \sqsubset t$ or (i-2) $\vec{\alpha} \nvdash s \Box B$ with $\Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv\}$. Case (i-1) contradicts $\vec{\alpha} \vdash s \vdash\!\dashv t$ by Lemma 3.8(1). For (i-2), from $\vec{\alpha} \vdash s \vdash\!\dashv t$ and $\vec{\alpha} \vdash B \sqsubset t$, we have, by the following d-proofs, $\vec{\alpha} \vdash s \vdash\!\dashv B$, which contradicts $\vec{\alpha} \nvdash s \Box B$:

When s is a point: When s is a circle:



(ii) When $\vec{\alpha} \nvdash B \sqsubset t$, we have $\vec{\alpha} \vdash s \sqsubset t$ by the definition of $I_{\vec{\alpha},B}(t)$, which contradicts $\vec{\alpha} \vdash s \vdash\!\dashv t$ by Lemma 3.8(1).

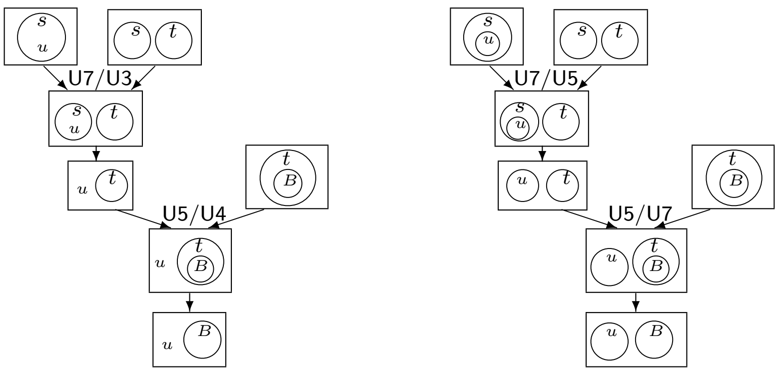
(b) The same applies to the case $u \equiv t$.

(c) Otherwise ($s \not\equiv u \not\equiv t$), we divide the following cases:

i. $\vec{\alpha} \vdash B \sqsubset s$ and $\vec{\alpha} \vdash B \sqsubset t$;
ii. $\vec{\alpha} \nvdash B \sqsubset s$ and $\vec{\alpha} \nvdash B \sqsubset t$;
iii. $\vec{\alpha} \nvdash B \sqsubset s$ and $\vec{\alpha} \vdash B \sqsubset t$;
iv. $\vec{\alpha} \vdash B \sqsubset s$ and $\vec{\alpha} \nvdash B \sqsubset t$.

(i) contradicts $\vec{\alpha} \vdash s \vdash\!\dashv t$ by Lemma 3.8(2). For (ii), by the definitions of $I_{\vec{\alpha},B}(s)$ and $I_{\vec{\alpha},B}(t)$, we have $\vec{\alpha} \vdash u \sqsubset s$ and $\vec{\alpha} \vdash u \sqsubset t$, which contradict $\vec{\alpha} \vdash s \vdash\!\dashv t$ by Lemma 3.8(2). For (iii), by the definition of $I_{\vec{\alpha},B}(s)$, we have $\vec{\alpha} \vdash u \sqsubset s$. By the definition of $I_{\vec{\alpha},B}(t)$, we have (iii-1) $\vec{\alpha} \vdash u \sqsubset t$ or (iii-2) $\vec{\alpha} \nvdash u \Box B$ with $\Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv\}$. (iii-1), together with $\vec{\alpha} \vdash u \sqsubset s$, contradicts $\vec{\alpha} \vdash s \vdash\!\dashv t$. For (iii-2), $\vec{\alpha} \vdash u \sqsubset s$, $\vec{\alpha} \vdash s \vdash\!\dashv t$, and $\vec{\alpha} \vdash B \sqsubset t$ imply, by the following d-proofs, that $\vec{\alpha} \vdash u \vdash\!\dashv B$, which contradicts $\vec{\alpha} \nvdash u \Box B$:

When u is a point: When u is a circle:



(iv) is similar to (iii). ■

Using the two kinds of canonical models introduced so far, we prove the following *atomic completeness*, from which completeness (Theorem 3.15) of GDS is derived.

Proposition 3.14 (Atomic completeness) *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be a sequence of* EUL*-diagrams which is semantically consistent. Let β be a minimal diagram. We have:*

$$\text{If } \mathcal{D}_1, \ldots, \mathcal{D}_n \models \beta, \text{ then } \mathcal{D}_1, \ldots, \mathcal{D}_n \vdash \beta \text{ in } \mathsf{GDS}.$$

Proof. We first consider the case where the premises $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are restricted to minimal diagrams $\alpha_1, \ldots, \alpha_n$. Then we extend to the general case. We denote by $\vec{\alpha}$ the sequence $\alpha_1, \ldots, \alpha_n$ of given minimal diagrams. Assume $\vec{\alpha} \models \beta$. When β is $s \bowtie t$, we immediately have $\vec{\alpha} \vdash s \bowtie t$ since it is an axiom. Otherwise, we divide the following cases according to the form of β. We use the canonical model of Definition 3.9 for the case that β is $s \sqsubset t$, and we use the canonical model of Definition 3.12 for the case that β is $s \mapsto t$.

1. When β is of the form $s \sqsubset t$, by the assumption $\vec{\alpha} \models s \sqsubset t$, we have, in particular for the canonical model of Definition 3.9, $M_{\vec{\alpha}} \models \vec{\alpha} \Rightarrow M_{\vec{\alpha}} \models s \sqsubset t$. Then, since $M_{\vec{\alpha}}$ is a model of $\vec{\alpha}$ by Lemma 3.10, we have $M_{\vec{\alpha}} \models s \sqsubset t$, i.e., $I_{\vec{\alpha}}(s) \subseteq I_{\vec{\alpha}}(t)$. Since $s \in I_{\vec{\alpha}}(s)$ by Definition 3.9, we have $s \in I_{\vec{\alpha}}(t)$, that is, $\vec{\alpha} \vdash s \sqsubset t$ in GDS.

2. When β is of the form $s \mapsto t$, observe that if s and t are both points, then the assertion is trivial since β is an axiom in that case. Otherwise, we assume, without loss of generality, that t is a circle B. By the assumption $\vec{\alpha} \models s \mapsto B$, we have, in particular for the canonical model of Definition 3.12, $M_{\vec{\alpha},B} \models \vec{\alpha} \Rightarrow M_{\vec{\alpha},B} \models s \mapsto B$. Then, since $M_{\vec{\alpha},B}$ is a model of $\vec{\alpha}$ by Lemma 3.13, we have $M_{\vec{\alpha},B} \models s \mapsto B$, i.e., $I_{\vec{\alpha},B}(s) \cap I_{\vec{\alpha},B}(B) = \emptyset$. Hence we have $s \notin I_{\vec{\alpha},B}(B)$ and $B \notin I_{\vec{\alpha},B}(s)$. Then by the definition of $I_{\vec{\alpha},B}(B)$ and $I_{\vec{\alpha},B}(s)$ of Definition 3.12, we have $\vec{\alpha} \not\vdash s \sqsubset B$, and $\vec{\alpha} \not\vdash B \sqsubset s$ and $\vec{\alpha} \vdash s \square B$ for some $\square \in \{\sqsubset, \sqsupset, \mapsto\}$. This means that, for the objects s and B, $s \sqsubset B$ or $B \sqsubset s$ or $s \mapsto B$ is provable from $\vec{\alpha}$ in GDS, but it is neither $s \sqsubset B$ nor $B \sqsubset s$. Therefore, we have $\vec{\alpha} \vdash s \mapsto B$ in GDS.

Next, we extend the premises to general diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$ instead of minimal diagrams $\vec{\alpha}$. Let $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \beta$. Then, by the definition of our semantics, it is equivalent to the fact that, for any model M, $M \models \vec{\alpha_1} \wedge \cdots \wedge M \models \vec{\alpha_n} \Rightarrow M \models \beta$, where $\vec{\alpha_i}$ is a sequence of all minimal diagrams whose relations hold on $\mathcal{D}_i$. Thus there is a sequence $\alpha_1, \ldots, \alpha_k$ of minimal diagrams such that each relation holding on α_j $(1 \leq j \leq k)$ holds on some $\mathcal{D}_i$ $(1 \leq i \leq n)$ and $\alpha_1, \ldots, \alpha_k \models \beta$. Then there is a d-proof from $\alpha_1, \ldots, \alpha_k$ to β in GDS. Thus we have the following d-proof for β from $\mathcal{D}_1, \ldots, \mathcal{D}_n$, where $\Downarrow$ indicates some applications of the Deletion rule:

$$\begin{array}{cccc}
\mathcal{D}_i & \mathcal{D}_j & & \mathcal{D}_l \\
\Downarrow & \Downarrow & & \Downarrow \\
\alpha_1 & \alpha_2 & \ldots & \alpha_k \\
 & \vdots & & \\
 & \beta & &
\end{array}$$

∎

By extending the conclusion diagram β of atomic completeness to a general (not restricted to minimal) diagram $\mathcal{E}$, we establish the completeness of GDS.

Theorem 3.15 (Completeness of GDS) *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{E}$ be* EUL*-diagrams. Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be semantically consistent.*

$$\text{If } \mathcal{D}_1, \ldots, \mathcal{D}_n \models \mathcal{E}, \text{ then } \mathcal{D}_1, \ldots, \mathcal{D}_n \vdash \mathcal{E} \text{ in } \mathsf{GDS}.$$

Proof. Using the atomic completeness theorem, we construct a d-proof of $\mathcal{E}$ from the given premise diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in a canonical way, as follows (see also Example 3.16 given after this proof):

(I) From the premise diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$, by using atomic completeness and U1, U2-rules, we first construct EUL-diagrams, called Venn-like diagrams, each of which consists of a point and all circles of $\mathcal{E}$, and in each of which $A \bowtie B$ holds for any pair of circles of the diagram.

(II) Then, by unifying all Venn-like diagrams of (I) with the Point Insertion rule, we construct a Venn-like diagram consisting of all points and circles of $\mathcal{E}$.

(III) By using atomic completeness, we construct d-proofs for all point-free minimal diagrams in each of which a relation $A \sqsubset B$ or $A \mapsfrom\!\!\mapsto B$ of $\mathcal{E}$ holds.

(IV) We then construct a diagram $\mathcal{F}$, by unifying the minimal diagrams of (III) and the Venn-like diagram of (II) with U9 and U10-rules.

(V) Finally, we check that the diagram $\mathcal{F}$ of (IV) coincides with the conclusion $\mathcal{E}$.

A diagrammatic proof is called a *canonical diagrammatic proof* when it is constructed in accordance with the above canonical construction.

We now formalize the above (I)–(V). We denote by $\overrightarrow{\mathcal{D}}$ the sequence $\mathcal{D}_1, \ldots, \mathcal{D}_n$ of the given premise diagrams.

(I) *For each point $a \in pt(\mathcal{E})$, let P_a be the set of relations holding on $\mathcal{E}$ each of which consists of the point a and a circle of $\mathcal{E}$: I.e., for $\Box \in \{\sqsubset, \vdash\!\dashv\}$, $P_a = \{a\Box X \mid a\Box X \in \mathsf{rel}(\mathcal{E})\}$. Then the set P_a gives rise to an* EUL*-diagram $\mathcal{P}_a$ such that $\overrightarrow{\mathcal{D}} \vdash \mathcal{P}_a$ in* GDS.

Proof of (I). Let $R_1, R_2, \ldots, R_n$ be an enumeration of the elements of P_a, and $\beta_1, \beta_2, \ldots, \beta_n$ be the sequence of corresponding minimal diagrams where R_i holds on β_i for $1 \leq i \leq n$. Note that all β_i share the same point a and they differ only in their circles. The assumption $\overrightarrow{\mathcal{D}} \models \mathcal{E}$ of completeness implies $\overrightarrow{\mathcal{D}} \models \beta_i$ since $R_i \in \mathsf{rel}(\mathcal{E})$. Hence we have $\overrightarrow{\mathcal{D}} \vdash \beta_i$ in GDS by atomic completeness (Proposition 3.14). Then starting from β_1, by successively applying U1-rule (when β_i is $a \sqsubset B_i$ for $1 < i \leq n$) or U2-rule (when β_i is $a \vdash\!\dashv B_i$ for $1 < i \leq n$), we have a d-proof of $\mathcal{P}_a$ from $\overrightarrow{\mathcal{D}}$ in GDS as follows:

$$
\begin{tikzcd}
\overrightarrow{\mathcal{D}} \arrow[d, dotted] & \overrightarrow{\mathcal{D}} \arrow[d, dotted] & \overrightarrow{\mathcal{D}} \arrow[d, dotted] \\
\beta_1 \arrow[dr, "\text{U1/U2}"] & \beta_2 \arrow[d] & \beta_3 \arrow[ddl] \\
& \beta_1 + \beta_2 \arrow[d, "\text{U1/U2}"] & \\
& (\beta_1 + \beta_2) + \beta_3 \arrow[d, dotted] & \\
& \mathcal{P}_a &
\end{tikzcd}
$$

$\Box$

(II) *Let $\{a_1, \ldots, a_m\} = pt(\mathcal{E})$. Let P be the union of the relations of all P_{a_i} $(1 \leq i \leq m)$ of (I): $P = \bigcup_{1 \leq i \leq m} P_{a_i}$. Then P gives rise to an* EUL*-diagram $\mathcal{P}$ such that $\overrightarrow{\mathcal{D}} \vdash \mathcal{P}$ in* GDS.

Proof of (II). Note that all diagrams $\mathcal{P}_{a_i}$ $(1 \leq i \leq m)$ of (I) share the same circles, where $X \bowtie Y$ holds for any $X, Y \in cr(\mathcal{P}_{a_i})$. The diagrams $\mathcal{P}_{a_i}$ differ only in their points. Hence by successively applying the Point Insertion rule, we have $\overrightarrow{\mathcal{D}} \vdash \mathcal{P}$ in GDS. $\Box$

Note that when $\mathcal{E}$ does not contain any point, the set $\bigcup_{1\leq i\leq m} P_{a_i}$ becomes empty. In such a case, we construct a Venn-like diagram $\mathcal{P}$ (without any point) which consists of all circles of $\mathcal{E}$. This is possible by successively applying $\mathsf{U8}$-rule to axioms of the form $X \bowtie Y$ for $X, Y \in cr(\mathcal{E})$.

(III) *Let β be a minimal diagram such that $A \sqsubset B$ or $A \vdash\dashv B$ of* $\mathsf{rel}(\mathcal{E})$ *holds. Then we have* $\overrightarrow{\mathcal{D}} \vdash \beta$ *in* GDS.

Proof of (III). Since $\overrightarrow{\mathcal{D}} \models \mathcal{E}$ and $\mathsf{rel}(\beta) \subseteq \mathsf{rel}(\mathcal{E})$, we have $\overrightarrow{\mathcal{D}} \models \beta$. Then by atomic completeness (Proposition 3.14), we have $\overrightarrow{\mathcal{D}} \vdash \beta$ in GDS. $\square$

(IV) *Let $R_1, \ldots, R_l$ be all relations of the form $A \sqsubset B$ or $A \vdash\dashv B$ holding on $\mathcal{E}$, and let $\beta_1, \ldots, \beta_l$ be the corresponding minimal diagrams, where R_i holds on β_i for $1 \leq i \leq l$. Let P be the set of relations of (II). Then the set $P \cup \{R_1, \ldots, R_l\}$ of relations gives rise to an* EUL*-diagram $(\cdots(\mathcal{P} + \beta_1) + \cdots) + \beta_l$ such that $\overrightarrow{\mathcal{D}} \vdash (\cdots(\mathcal{P} + \beta_1) + \cdots) + \beta_l$ in* GDS.

Proof of (IV). By induction on l. Let $\mathcal{P} + \mathcal{B}_l$ denote the diagram $(\cdots(\mathcal{P} + \beta_1) + \cdots) + \beta_l$. We show the induction step ($l > 1$) with the induction hypothesis $\overrightarrow{\mathcal{D}} \vdash \mathcal{P} + \mathcal{B}_{l-1}$ in GDS, since the same applies to the base step ($l = 1$).
(Induction step: $l > 1$) We divide the following two cases according to the form of β_l: (1) $A \vdash\dashv B$ holds on β_l or (2) $A \sqsubset B$ holds on β_l.
Case (1): Since $cr(\mathcal{E}) = cr(\mathcal{P} + \mathcal{B}_{l-1})$ by the construction (II) of $\mathcal{P}$ and (III), we have $A, B \in cr(\mathcal{P} + \mathcal{B}_{l-1})$. We claim that $A \bowtie B$ or $A \vdash\dashv B$ holds on the diagram $\mathcal{P} + \mathcal{B}_{l-1}$. Assume to the contrary that neither of them holds on $\mathcal{P} + \mathcal{B}_{l-1}$, that is, $A \sqsubset B$ or $B \sqsubset A$ holds by Proposition 2.4. If $A \sqsubset B$ holds on $\mathcal{P} + \mathcal{B}_{l-1}$, since $\overrightarrow{\mathcal{D}} \vdash \mathcal{P} + \mathcal{B}_{l-1}$ by the induction hypothesis, we have $\overrightarrow{\mathcal{D}} \models \mathcal{P} + \mathcal{B}_{l-1}$ by soundness of GDS (Theorem 3.6), which implies that $\overrightarrow{\mathcal{D}} \models A \sqsubset B$. This contradicts the assumption that $\overrightarrow{\mathcal{D}}$ is semantically consistent because we have $\overrightarrow{\mathcal{D}} \models A \vdash\dashv B$ from the assumptions $\overrightarrow{\mathcal{D}} \models \mathcal{E}$ and $R_l = A \vdash\dashv B \in \mathsf{rel}(\mathcal{E})$. The same applies in case $B \sqsubset A$. Thus exactly one of $A \bowtie B$ and $A \vdash\dashv B$ holds on the diagram $\mathcal{P} + \mathcal{B}_{l-1}$.

Now we prove $\overrightarrow{\mathcal{D}} \vdash (\mathcal{P}+\mathcal{B}_{l-1})+\beta_l$. When $A \vdash\dashv B$ holds on $\mathcal{P}+\mathcal{B}_{l-1}$, we obtain the assertion immediately by the induction hypothesis since $(\mathcal{P}+\mathcal{B}_{l-1})+\beta_l$ is $\mathcal{P}+\mathcal{B}_{l-1}$ itself. When $A \bowtie B$ holds on $\mathcal{P} + \mathcal{B}_{l-1}$, by applying $\mathsf{U10}$-rule to β_l and $\mathcal{P} + \mathcal{B}_{l-1}$, we have $\overrightarrow{\mathcal{D}} \vdash (\mathcal{P} + \mathcal{B}_{l-1}) + \beta_l$ in GDS. The application of $\mathsf{U10}$-rule is possible because there is no object s such that both $s \sqsubset A$ and $s \sqsubset B$ hold on $\mathcal{P}+\mathcal{B}_{l-1}$: If there were such an object s, since $\overrightarrow{\mathcal{D}} \vdash \mathcal{P}+\mathcal{B}_{l-1}$ by the induction hypothesis, we have $\overrightarrow{\mathcal{D}} \vdash s \sqsubset A$ and $\overrightarrow{\mathcal{D}} \vdash s \sqsubset B$ by applying a series of $\mathsf{Deletion}$. Then, using the soundness theorem, we would have $\overrightarrow{\mathcal{D}} \models s \sqsubset A$ and $\overrightarrow{\mathcal{D}} \models s \sqsubset B$. This contradicts the assumption that $\overrightarrow{\mathcal{D}}$ is semantically consistent because we have $\overrightarrow{\mathcal{D}} \models A \vdash\dashv B$.

Case (2) where $A \sqsubset B$ holds on β_l is similar. $\square$

(V) *For any* EUL*-relation R, $R \in$* $\mathsf{rel}((\cdots(\mathcal{P} + \beta_1) + \cdots) + \beta_l)$ *if and only if $R \in$* $\mathsf{rel}(\mathcal{E})$.
Proof of (V). We denote by $\mathcal{P} + \mathcal{B}_l$ the diagram $(\cdots(\mathcal{P} + \beta_1) + \cdots) + \beta_l$.
$\Leftarrow$) By the constructions (II) and (IV), all minimal diagrams of $\mathcal{E}$ are unified to obtain $\mathcal{P}+\mathcal{B}_l$. Hence by Lemma 3.2, we obtain $\mathsf{rel}(\mathcal{E}) \subseteq \mathsf{rel}(\mathcal{P} + \mathcal{B}_l)$.

$\Rightarrow$) Let $R \in \mathsf{rel}(\mathcal{P}+\mathcal{B}_l)$. We divide the following argument in two cases depending on whether or not R is of the form $s \bowtie t$:
(1) When $R = s \bowtie t$, assume to the contrary that $s \bowtie t \notin \mathsf{rel}(\mathcal{E})$. Since $\mathcal{E}$ is a diagram, for some $\Box \in \{\sqsubset, \sqsupset, \vdash\dashv\}$, $s \Box t \in \mathsf{rel}(\mathcal{E})$ by Proposition 2.4. Then, by definition, for some j, β_j is

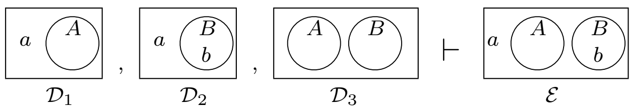
of the form $s\Box t$, which implies that $s\Box t \in \mathsf{rel}(\mathcal{P}+\mathcal{B}_l)$. This contradicts Proposition 2.4 for the diagram $\mathcal{P}+\mathcal{B}_l$, since $s \bowtie t \in \mathsf{rel}(\mathcal{P}+\mathcal{B}_l)$ by the assumption.

(2) In case, $R \neq s \bowtie t$, we show that $R \in \mathsf{rel}(\mathcal{P}+\mathcal{B}_l) \Rightarrow R \in \mathsf{rel}(\mathcal{E})$ by induction on l. We prove the induction step since the same applies to the base step.

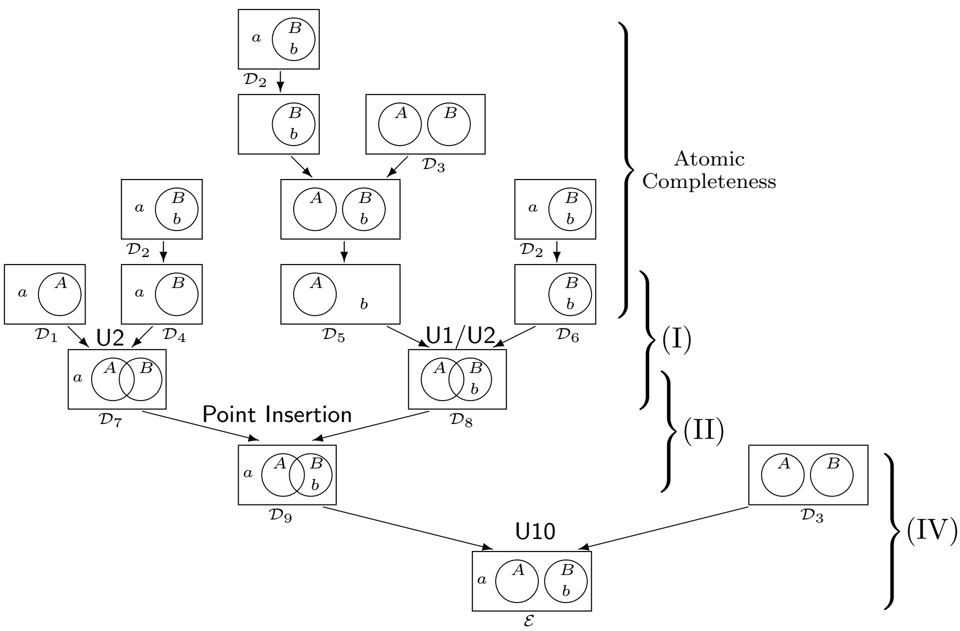
(Induction step: $l > 1$) Assume to the contrary that $R \notin \mathsf{rel}(\mathcal{E})$. Then, since $\mathsf{rel}(\mathcal{P}) \setminus \{X \bowtie Y \mid X, Y \in cr(\mathcal{P})\} \subseteq \mathsf{rel}(\mathcal{E})$ by the construction (II), R should be a relation between circles (not points), and $R \neq \beta_i$ for any i. Hence, there is some $1 \leq i \leq l$ such that $R \notin \mathsf{rel}(\mathcal{P}+\mathcal{B}_{i-1})$ but $R \in \mathsf{rel}((\mathcal{P}+\mathcal{B}_{i-1})+\beta_i)$. We show the case $(\mathcal{P}+\mathcal{B}_{i-1})+\beta_i$ is obtained by U10-rule. (The case of U9-rule is shown similarly.) Assume $A \vdash\!\dashv B$ holds on β_i. By the definition of U10-rule, there are the following three cases according to the form of R: (i) $R = X \vdash\!\dashv B$ such that $X \sqsubset A \in \mathsf{rel}(\mathcal{P}+\mathcal{B}_{i-1})$; (ii) $R = X \vdash\!\dashv A$ such that $X \sqsubset B \in \mathsf{rel}(\mathcal{P}+\mathcal{B}_{i-1})$; (iii) $R = X \vdash\!\dashv Y$ such that $X \sqsubset A, Y \sqsubset B \in \mathsf{rel}(\mathcal{P}+\mathcal{B}_{i-1})$. For case (i), by the induction hypothesis, we have $X \sqsubset A \in \mathsf{rel}(\mathcal{E})$. Then, since $A \vdash\!\dashv B \in \mathsf{rel}(\mathcal{E})$, we have $X \vdash\!\dashv B \in \mathsf{rel}(\mathcal{E})$, contrary to the assumption $R = X \vdash\!\dashv B \notin \mathsf{rel}(\mathcal{E})$. Similarly, cases (ii) and (iii) also lead to contradictions. Therefore, we have $R \in \mathsf{rel}(\mathcal{E})$. □

■

Example 3.16 (Canonical d-proof of GDS) As an illustration of the canonical construction of d-proofs given in our proof of completeness (Theorem 3.15), let us consider the following diagrams $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, and $\mathcal{E}$:



We have the following canonical d-proof of $\mathcal{E}$ from $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$:



We first derive, by using atomic completeness, all pointed minimal diagrams $\mathcal{D}_1, \mathcal{D}_4, \mathcal{D}_5$, and $\mathcal{D}_6$ each of which corresponds to an EUL-relation holding on the conclusion $\mathcal{E}$. Next, following the construction (I) with $\mathsf{U1}$ and $\mathsf{U2}$ rules, we construct two Venn-like diagrams $\mathcal{D}_7$ and $\mathcal{D}_8$ each of which consists of a point a (resp. b) and all circles A and B of $\mathcal{E}$. Then, following the construction (II) with Point Insertion rule, we unify the Venn-like diagrams $\mathcal{D}_7$ and $\mathcal{D}_8$ to obtain a Venn-like diagram $\mathcal{D}_9$ consisting of all points a and b and all circles A and B of $\mathcal{E}$. Finally, following the construction (IV) with $\mathsf{U10}$ rule, we obtain the conclusion $\mathcal{E}$.

3.5 Some consequences of completeness of GDS

In this section, we discuss some consequences of our completeness (Theorem 3.15) of GDS.

3.5.1 Unification of any (two) diagrams

Let $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{E}$ be EUL-diagrams such that for any model M, $M \models \mathcal{E}$ if and only if $M \models \mathcal{D}_1$ and $M \models \mathcal{D}_2$, that is, $\mathcal{E}$ is semantically equivalent to the conjunction of $\mathcal{D}_1$ and $\mathcal{D}_2$. We may write such $\mathcal{E}$ as $\mathcal{D}_1 + \mathcal{D}_2$. Our completeness (Theorem 3.15) ensures that $\mathcal{D}_1, \mathcal{D}_2 \vdash \mathcal{D}_1 + \mathcal{D}_2$ in GDS. This shows that the general notion of unification of two diagrams (cf. [Hammer-Shin 1998]) is completely characterized by our formalization of unification of two diagrams, where one of them is restricted to a minimal diagram.

3.5.2 Decomposition set of an EUL-diagram

Given an EUL-diagram $\mathcal{D}$ and two objects, say s and t, on $\mathcal{D}$, a minimal diagram obtained from $\mathcal{D}$ by deleting all objects other than s and t is called a *component minimal diagram* of $\mathcal{D}$. By Proposition 2.4, the set of component minimal diagrams of $\mathcal{D}$ is uniquely determined, and we call it the *decomposition set* (*decomp*($\mathcal{D}$)) of $\mathcal{D}$. According to our semantics, *decomp*($\mathcal{D}$) and $\mathcal{D}$ are semantically equivalent in the following sense: for any model M, for all $\alpha \in$ *decomp*($\mathcal{D}$), $M \models \alpha$ if and only if $M \models \mathcal{D}$. Hence, by completeness (Theorem 3.15), *decomp*($\mathcal{D}$) and $\mathcal{D}$ are also *provably equivalent*.

In particular, the canonical construction of d-proofs of the completeness shows that any EUL-diagram $\mathcal{D}$ can be constructed from the component minimal diagrams of $\mathcal{D}$.

3.5.3 Normal diagrammatic proofs

In order to prove a normal form theorem of GDS, we shall modify the semantic method introduced in our completeness proof, by adopting a semantic normal form proof for the linguistic proofs found in, for example, [Okada 1999, Okada 2002].

Let us define a class of normal diagrammatic proofs of GDS, called the $\pm$-normal d-proofs:

Definition 3.17 ($\pm$-normal d-proofs) A d-proof π is in $\pm$-*normal form* if a unification $(+)$ and a deletion $(-)$ appear alternately in π.

In Definition 3.9 and 3.12 of our canonical models, it is possible to modify the interpretation of each object by restricting the provability *with a $\pm$-normal d-proof* as follows:

- For Definition 3.9,

  $I'_{\vec{\alpha}}(t) = \{s \mid \vec{\alpha} \vdash s \sqsubset t \text{ with a } \pm\text{-normal d-proof } \} \cup \{t\}$

- For Definition 3.12, for any t such that $\vec{\alpha} \vdash B \sqsubset t$ with a $\pm$-normal d-proof,

  $I'_{\vec{\alpha},B}(t) = I'_{\vec{\alpha}}(t) \cup \{s \mid \vec{\alpha} \nvdash B \sqsubset s \text{ and } \vec{\alpha} \nvdash s \sqsubset B \text{ and } \vec{\alpha} \nvdash s \vdash\!\dashv B\}$

This slight modification of canonical models also enables us to prove the essential part of atomic completeness (Proposition 3.14) where the premise diagrams are restricted to minimal diagrams; because any d-proof appearing in our proof of Lemma 3.10 and 3.13 is in $\pm$-normal form. Hence we obtain the following version of atomic completeness:

Corollary 3.18 *Let $\vec{\alpha}$ be a sequence $\alpha_1, \dots, \alpha_n$ of minimal diagrams which is semantically consistent. Let β be a minimal diagram. Then we have:*

$$\textit{If } \vec{\alpha} \models \beta \text{ , } \textit{then } \vec{\alpha} \vdash \beta \;\; \textit{in } \mathsf{GDS} \textit{ with a } \pm\textit{-normal d-proof.}$$

Then, together with soundness (Theorem 3.6) of GDS, we obtain the following normal form theorem:

Theorem 3.19 ($\pm$-normal form for minimal diagrams) *Let $\vec{\alpha}$ be a sequence $\alpha_1, \dots, \alpha_n$ of minimal diagrams which is semantically consistent. Let β be a minimal diagram. Then we have:*

$$\textit{If } \vec{\alpha} \vdash \beta \;\; \textit{in } \mathsf{GDS}, \textit{ then } \vec{\alpha} \vdash \beta \;\; \textit{in } \mathsf{GDS} \textit{ with a } \pm\textit{-normal d-proof.}$$

Proof. Let $\vec{\alpha} \vdash \beta$ in GDS. Then, by soundness (Theorem 3.6) of GDS, we have $\vec{\alpha} \models \beta$, which implies that $\vec{\alpha} \vdash \beta$ in GDS with a $\pm$-normal d-proof by Corollary 3.18. ■

Although the above normal form theorem states only the existence of normal d-proofs, by defining a procedure to rewrite d-proofs, the theorem can be extended to a *normalization theorem of d-proofs*: *Any d-proof is rewritten into a $\pm$-normal d-proof in a finite number of steps.*

Observe that EUL-relations can be described linguistically, using usual first order formulas: $A \sqsubset B$ by $\forall x(A(x) \to B(x))$; $a \sqsubset B$ by $B(a)$; $A \vdash\!\dashv B$ by $\forall x(A(x) \to \neg B(x))$; $a \vdash\!\dashv B$ by $\neg B(a)$. Then any EUL-diagram $\mathcal{D}$ can be also described, through $\mathsf{rel}(\mathcal{D})$, as a conjunction of the above types of formulas. Thus, our diagrammatic inference rules can be described in terms of the linguistic inference rules of a version of Gentzen's natural deduction: Deletion-rule corresponds to a parallel application of $\wedge$-elimination rule; each Unification-rule essentially corresponds to a parallel application of $\wedge$-introduction rule to the relations described in Definition 3.1. (See, e.g., [Gentzen 1969, Prawitz 1965, Troelstra-Schwichtenberg 2000] for natural deduction.) A natural deduction proof thus obtained is not in general in normal form, and it may contain some redexes (certain pairs of $\wedge$-introduction and $\wedge$-elimination rules). By normalizing such a natural deduction proof with the well-known rewriting procedure, it is possible to obtain a normal proof. For such a normal natural deduction proof, we can define a translation into the normal diagrammatic proof of GDS. (For the correspondence between normal natural deduction proofs and normal diagrammatic proofs, see also the discussion following Proposition 3.21.)

3.5.4 Structure of canonical diagrammatic proofs

In order further to investigate structure of canonical d-proofs of completeness (Theorem 3.15), we give a proposition, which is proved in a way similar to that of $\pm$-normal form Theorem 3.19.

In our Definitions 3.9 and 3.12 of canonical models, it is possible to modify the interpretation of each object by restricting the provability *using only* U3–U7 *and* Deletion *rules* as follows:

- For Definition 3.9,
  $I''_{\vec{\alpha}}(t) = \{s \mid \vec{\alpha} \vdash s \sqsubset t$ with U3–U7 and Deletion rules$\} \cup \{t\}$
- For Definition 3.12, for any t such that $\vec{\alpha} \vdash B \sqsubset t$ with U3–U7 and Deletion rules,
  $I''_{\vec{\alpha},B}(t) = I''_{\vec{\alpha}}(t) \cup \{s \mid \vec{\alpha} \nvdash B \sqsubset s$ and $\vec{\alpha} \nvdash s \sqsubset B$ and $\vec{\alpha} \nvdash s \vdash\!\dashv B\}$.

Recall that U3–U7 rules are unification where exactly one circle is shared between the two premise diagrams.

These slight modification of canonical models also enables us, in a way similar to that in Corollary 3.18, to prove atomic completeness. Thus we obtain the following slightly stronger version of atomic completeness:

Corollary 3.20 *Let $\overrightarrow{\mathcal{D}}$ be a sequence $\mathcal{D}_1, \ldots, \mathcal{D}_n$ of* EUL-*diagrams which is semantically consistent. Let β be a minimal diagram. Then we have:*

If $\overrightarrow{\mathcal{D}} \models \beta$, then $\overrightarrow{\mathcal{D}} \vdash \beta$ in GDS *with* U3–U7 *and* Deletion *rules.*

Thus soundness (Theorem 3.6) and Corollary 3.20 imply that any minimal diagram is provable by using only U3–U7 and Deletion rules:

Proposition 3.21 (U3–U7 rules) *Let $\overrightarrow{\mathcal{D}}$ be a sequence $\mathcal{D}_1, \ldots, \mathcal{D}_n$ of* EUL-*diagrams which is semantically consistent. Let β be a minimal diagram.*

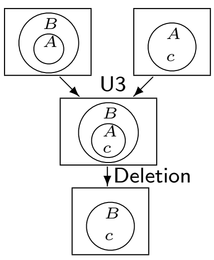
If $\overrightarrow{\mathcal{D}} \vdash \beta$ in GDS, *then $\overrightarrow{\mathcal{D}} \vdash \beta$ in* GDS *with* U3–U7 *and* Deletion *rules.*

Completeness (Theorem 3.15), the $\pm$-normal form theorem (Theorem 3.19), and the above Proposition 3.21 give a more precise classification of inference rules of GDS in terms of proof-construction as follows:

- U3–U7 and Deletion rules for derivation of a minimal diagram.
- U1, U2 (U8) rules for construction of a Venn-like diagram consisting of a single point (resp. no point).
- Point Insertion rule for construction of a Venn-like diagram consisting of multiple points.
- U9, U10 rules for construction of the conclusion.

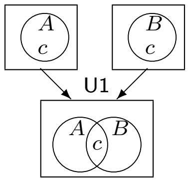
See also the d-proof given in Example 3.16.

The classification of inference rules and our canonical construction of d-proofs show that the structure of each canonical d-proof essentially corresponds to the structure of a normal (linguistic) proof of Gentzen's natural deduction system. (See, e.g., [Gentzen 1969, Prawitz 1965, Troelstra-Schwichtenberg 2000] for natural deduction.) Observe that each pair of U3–U7 rules and a Deletion rule essentially corresponds to a combination of *elimination rules* of natural deduction. For example, the pair of U3 and Deletion rules corresponds to $\to$-elimination accompanied by $\forall$-elimination as follows:



$$\dfrac{\dfrac{\forall x(A(x) \to B(x))}{A(c) \to B(c)}\ \forall E \qquad A(c)}{B(c)} \to E$$

Further note that U1,U2, Point Insertion, U9, U10 rules essentially correspond to $\wedge$-introduction rule of natural deduction. For example, there is a correspondence between the following application of U1 rule and $\wedge$-introduction rule:



$$\frac{A(c) \quad B(c)}{A(c) \wedge B(c)} \wedge I$$

In this way, in terms of natural deduction, it is shown that our canonical d-proof consists of, from top down, successive applications of elimination rules and successive applications of introduction rules. See, for example, [Troelstra-Schwichtenberg 2000] for the structure of a normal natural deduction proofs. A precise formulation of this discussion will appear in a forthcoming paper.

4 Some extensions

In this section, we discuss some possible extensions of our diagrammatic system.

We defined a formal Euler diagrammatic system in terms of the topological relations, called EUL-relations, between two diagrammatic objects. Although, in this paper, we considered only circles and points as diagrammatic objects and excluded any intersection, union, and complement regions and linking of points, we can naturally extend our framework by introducing these diagrammatic objects in the following way.

1. Our framework can be naturally extended by regarding intersection regions of circles as diagrammatic objects. As suggested in Section 3.4, the set of relations $\mathsf{rel}(\mathcal{D})$ of an EUL-diagram $\mathcal{D}$ can be seen as a partially ordered structure, which we call EUL-structure. The greatest lower bounds in the structure correspond to intersection regions of circles. With this extension, the diagrams $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ of Fig.9 in Section 2.1 can be distinguished.

2. We sometimes regarded a named point of EUL as a special named circle which does not contain, nor cross, any other object. Let us extend the notion of point by allowing it to make crosses with circles as illustrated in the following diagram $\mathcal{D}$.

   $\mathcal{D}$ $\mathcal{D}'$

   According to our semantic interpretation of $\bowtie$-relation (cf. Remark 2.9 of Section 2.2), $\mathcal{D}$ can be considered to represent "the point b is inside A or outside A," which is equivalently expressed by using Peirce's linking as in $\mathcal{D}'$. Although this linking does not fully correspond to Peirce's linking (cf. [Peirce 1897, Shin 1994]), we can naturally extend our framework to represent some disjunctive information on objects in this way.

Our soundness, completeness, and normal form theorems are naturally retained with these extensions of EUL.

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