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# Alternative formulations of the theory of evidence based on basic plausibility and commonality assignments

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**Abstract.** In this paper we introduce indeed two alternative formulations of the theory of evidence by proving that both plausibility and commonality functions share the same combinatorial structure of sum function of belief functions, and computing their Moebius inverses called basic plausibility and commonality assignments. The equivalence of the associated formulations of the ToE is mirrored by the geometric congruence of the related simplices. Applications to the probabilistic approximation problem are briefly presented.

## 1 Introduction

The *theory of evidence* (ToE) is one of the most popular uncertainty theory [1, 2], in which subjective probability is represented by *belief function* (b.f.) rather than a Bayesian mass distribution, assigning probability values to *sets* of possibilities rather than single events. Variants or continuous extensions of the ToE in terms of hints [3] or allocations of probability [4] have since been proposed. From a combinatorial point of view, in their finite incarnation, b.f.s are *sum functions*, i.e. functions on the power set $2^\Theta = \{A \subseteq \Theta\}$ of a finite domain $\Theta$ $b(A) = \sum_{B \subseteq A} m_b(B)$ induced by a *basic probability assignment* (b.p.a.) $m_b : 2^\Theta \rightarrow [0, 1]$ which is combinatorially the Moebius inverse [5] of $b$. The same evidence associated with a b.f. is carried by the related plausibility (pl.f.) $pl_b(A) = 1 - b(A^c)$ and commonality $Q_b(A) = \sum_{B \supseteq A} m_b(B)$ (comm.f.) functions, which lack though a similar coherent mathematical characterization.

In this paper we introduce indeed two alternative formulations of the theory of evidence by proving that both pl.f.s and comm.f.s share the same combinatorial structure of sum function, and computing their Moebius inverses which is natural to call *basic plausibility* and *commonality assignments*. We achieve this by resorting to a recent geometric approach to the theory of evidence [6] in which belief functions are represented by points of a Cartesian space. Besides giving the overall mathematical structure of the theory of evidence a more elegant symmetry, the notions of b.pl.a.s and b.comm.a.s turn out to be useful when solving problems like finding probabilistic approximations [7–9] of belief functions, or computing the canonical decomposition of support functions. Moreover, as they are discovered through geometric methods, basic plausibility and commonality

assignments inherit the same simplicial geometry as that of b.f.s.
The novel contributions of this paper are then the proofs that:

- commonality functions have a Moebius inverse that we call basic commonality assignment (Theorem 1), the study of its properties and geometries (Theorems 2 and 3);
- the equivalence of the alternative formulations of the ToE is geometrically mirrored by the congruence of the corresponding simplices (Theorem 4);

To support the usefulness of these alternative formulations, some applications of basic plausibility assignments to the approximation problem are discussed. We first recall the basic notions of the ToE and its geometric approach.

## 2 Belief, plausibility, and commonality functions

Even though belief functions can be given several alternative but equivalent definitions in terms of multi-valued mappings, random sets [10, 11], inner measures [12], in Shafer's formulation [1] a central role is played by the notion of "basic probability assignment". A *basic probability assignment* (b.p.a.) over a finite set (*frame of discernment* [1]) $\Theta$ is a function $m : 2^\Theta \to [0,1]$ on its power set $2^\Theta = \{A \subset \Theta\}$ such that $m(\emptyset) = 0$, $\sum_{A\subseteq\Theta} m(A) = 1$, $m(A) \geq 0 \ \forall A \subset \Theta$. Subsets of $\Theta$ associated with non-zero values of $m$ are called *focal elements*.
The *belief function* (b.f.) $b : 2^\Theta \to [0,1]$ associated with a b.p.a. $m_b$ is

$$b(A) = \sum_{B\subseteq A} m_b(B). \tag{1}$$

A finite probability or *Bayesian* belief function is a special b.f. assigning non-zero masses only to singletons : $m_b(A) = 0$, $|A| > 1$.
Functions of the form (1) on a partially ordered set are called *sum functions* [5]. A belief function $b$ is then the sum function associated with a basic probability assignment $m_b$ on the partially ordered set $(2^\Theta, \subseteq)$.
Conversely, the unique basic probability assignment $m_b$ associated with a given belief function $b$ can be recovered by means of the *Moebius inversion formula*

$$m_b(A) = \sum_{B\subseteq A} (-1)^{|A-B|} b(B). \tag{2}$$

A sum function can be seen as the discrete counterpart of the indefinite integral in calculus, and Moebius inversion as the discrete counterpart of the derivative.

A dual mathematical representation of the evidence encoded by a belief function $b$ is the *plausibility function* (pl.f.) $pl_b : 2^\Theta \to [0,1]$, $A \mapsto pl_b(A)$, where

$$pl_b(A) \doteq 1 - b(A^c) = 1 - \sum_{B\subseteq A^c} m_b(B) = \sum_{B\cap A\neq\emptyset} m_b(B)$$

expresses the amount of evidence *not against* $A$.

A third mathematical model of the evidence carried by a b.f. is represented by the *commonality function* (comm.f.) $Q_b : 2^\Theta \rightarrow [0,1]$, $A \mapsto Q_b(A)$, where the *commonality number* $Q_b(A)$ can be interpreted as the amount of mass which can move freely through the entire event $A$,

$$Q_b(A) \doteq \sum_{B \supseteq A} m_b(B). \tag{3}$$

**Example.** Let us consider a b.f. $b$ on a frame of size 3, $\Theta = \{x, y, z\}$ with b.p.a. (see Figure 1) $m_b(x) = 1/3$, $m_b(\Theta) = 2/3$. The belief values of $b$ on all possible

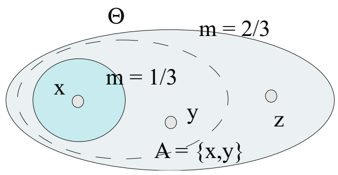


**Fig. 1.** The belief function of the example has two focal elements, $\{x\}$ and $\Theta$.

events of $\Theta$ are (Eq. 1): $b(x) = m_b(x) = 1/3$, $b(y) = b(z) = 0$, $b(\Theta) = m_b(x) + m_b(\Theta) = 1$, $b(\{x,y\}) = m_b(x) = 1/3$, $b(\{x,z\}) = m_b(x) = 1/3$, $b(\{y,z\}) = 0$. To appreciate the difference between belief, plausibility, and commonality let us consider in particular the event $A = \{x, y\}$. Its belief value $b(\{x,y\}) = \sum_{A \subseteq \{x,y\}} m_b(A) = m_b(x) = 1/3$ represents the amount of evidence which *surely support* $\{x, y\}$ as it counts all the events which imply $\{x, y\}$ On the other side, $pl_b(\{x,y\}) = 1 - b(\{x,y\}^c) = 1 - b(z) = 1$ measures the evidence *not surely against* it, as it counts all the events which no not imply its complement $\{x, y\}^c$. Finally, the commonality number $Q_b(\{x,y\}) = \sum_{A \supseteq \{x,y\}} m_b(A) = m_b(\Theta) = 2/3$ tells us which is the amount of evidence which can (possibly) *equally support* each of the outcomes in $\{x, y\}$ (i.e. $x$ and $y$), as the evidence represented by events $A \supseteq \{x, y\}$ can focus on both elements.

## 3 Two alternative formulations of the ToE

As plausibility and commonality functions are both equivalent representations of the evidence carried by a belief function, it is natural to guess that they should share the form of sum function on the power set $2^\Theta$.

We can indeed use results and tools provided by the geometric interpretation of the ToE to develop alternative models of uncertainty which are parallel to the standard formulation of the ToE. Evidence is there represented by cumulating basic probabilities on intervals of events $\{B \subseteq A\}$ (yielding a belief value $b(A) = \sum_{B \subseteq A} m(B)$). Equivalently we can represent pieces of evidence as basic plausibility (commonality) assignments on the power set, and compute the related plausibility (commonality) set function by adding basic assignments over intervals. Let us first recall the geometry of belief measures.

**Belief space**. A b.f. $b : 2^{\Theta} \rightarrow [0,1]$ on a frame of discernment $\Theta$ is completely specified by its $N-2$ belief values $\{b(A), \emptyset \subsetneq A \subsetneq \Theta\}$, $N \doteq 2^{|\Theta|}$ (since $b(\emptyset) = 0$, $b(\Theta) = 1$ always). It can then be represented as a point of $\mathbb{R}^{N-2}$ like

$$b = \sum_{\emptyset \subsetneq A \subsetneq \Theta} b(A) \mathbf{v}_A$$

where $\{\mathbf{v}_A : \emptyset \subsetneq A \subseteq \Theta\}$ is a reference frame in $\mathbb{R}^{N-2}$. The set of points $\mathcal{B}$ of $\mathbb{R}^{N-1}$ which correspond to a b.f. is called "belief space" [6], i.e. the *simplex*

$$\mathcal{B} = Cl(b_A, \ \emptyset \subsetneq A \subseteq \Theta),$$

where $b_A$ is the unique belief function assigning all the mass to a single subset $A$ of $\Theta$ ($A$-th *dogmatic belief function*), and $Cl$ denotes the convex closure operator: $Cl(b_1, ..., b_k) = \{b \in \mathcal{B} : b = \alpha_1 b_1 + \cdots + \alpha_k b_k, \sum_i \alpha_i = 1, \ \alpha_i \geq 0 \ \forall i\}$. The *faces* of a simplex $Cl(b_1, ..., b_k)$ are all possible simplices generated by a subset of its vertices. Each b.f. $b \in \mathcal{B}$ can be written as a convex sum as follows:

$$b = \sum_{\emptyset \subsetneq A \subseteq \Theta} m_b(A) b_A. \tag{4}$$

A b.p.a. (the Moebius inverse of a belief function) is then the set of simplicial coordinates of $b$ in $\mathcal{B}$: The simplicial form of $\mathcal{B}$ is the geometric counterpart of the nature of b.f.s as sum functions. The set $\mathcal{P}$ of all Bayesian b.f.s is the simplex formed by all dogmatic b.f.s associated with singletons: $\mathcal{P} = Cl(b_{\{x\}}, x \in \Theta)$.

**Binary case**. Consider as an example a frame of discernment with just two elements $\Theta_2 = \{x, y\}$. Each b.f. $b : 2^{\Theta_2} \rightarrow [0,1]$ is completely determined by its belief values $b(x), b(y)$ (since $b(\emptyset) = 0$, $b(\Theta) = 1$ for all $b$). This means that we can represent $b$ as the vector

$$b(x)\mathbf{v}_x + b(x)\mathbf{v}_y = [b(x), b(y)]' = [m_b(x), m_b(y)]' \in \mathbb{R}^2. \tag{5}$$

where $\mathbf{v}_x = [1,0]'$ is the versor of the $x$ axis, and $\mathbf{v}_y = [0,1]'$ that of the $y$ axis. Since $m_b(x) \geq 0$, $m_b(y) \geq 0$, and $m_b(x) + m_b(y) \leq 1$ the set $\mathcal{B}_2$ of all the possible belief functions on $\Theta_2$ is the triangle in the Cartesian plane of Figure 2, whose vertices are the vacuous belief function $b_\Theta = [0,0]'$ with $m_{b_\Theta}(\Theta) = 1$, the Bayesian b.f. $b_x = [1,0]'$ with $m_{b_x}(x) = 1$, and the Bayesian b.f. $b_y = [0,1]'$ with $m_{b_y}(y) = 1$. Bayesian b.f.s on $\Theta_2$ obey the constraint $m_b(x) + m_b(y) = 1$ and form then the points of the segment $\mathcal{P}_2$ joining $b_x = [1,0]'$ and $b_y = [0,1]'$. In the binary case each b.f. $b$ decomposes according to Equation (4) as

$$b = m_b(x) b_x + m_b(y) b_y.$$

**Change of reference frame**. In the case of a general domain $\Theta$, the dogmatic belief functions $\{b_A : \emptyset \subsetneq A \subsetneq \Theta\}$ form a set of independent vectors in $\mathbb{R}^{N-2}$, so that the collections $\{\mathbf{v}_A\}$ and $\{b_A\}$ represent two distinct coordinate frames in $\mathcal{B}$. We can then compute the transformation linking them [13].

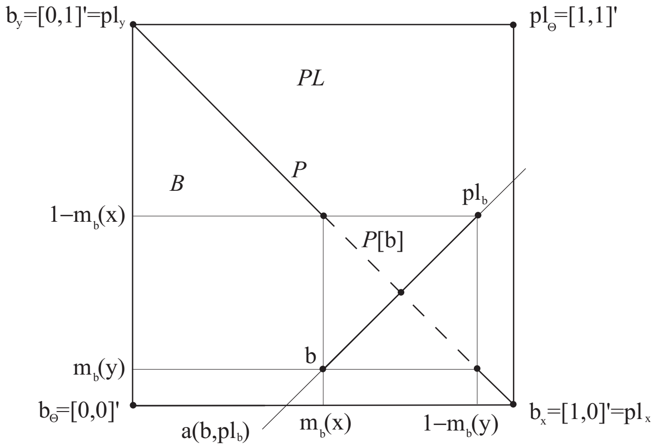


**Fig. 2.** The belief space $\mathcal{B}$ for a binary frame is a triangle in $\mathbb{R}^2$ whose vertices are the dogmatic b.f.s focused on $\{x\}$, $\{y\}$ and $\Theta$, $b_x, b_y, b_\Theta$ respectively. The probability region is the segment $\mathcal{P} = Cl(b_x, b_y)$. Belief and plausibility functions lie on opposite locations with respect to $\mathcal{P}$. The line $a(b, pl_b)$ joining them intersect $\mathcal{P}$ in the intersection probability $p[b]$ (Section 5).

**Lemma 1.** *The two coordinate frames* $\{\mathbf{v}_A : \emptyset \subsetneq A \subsetneq \Theta\}$ *and* $\{b_A : \emptyset \subsetneq A \subsetneq \Theta\}$ *are linked by the relation* $\mathbf{v}_A = \sum_{B \supseteq A} (-1)^{|B \setminus A|} b_B$.

### 3.1 Basic plausibility assignment

The geometry of belief measures can be exploited to prove the structure of sum function of both plausibility and commonality functions, establishing this way two equivalent formulations of the ToE in terms of basic plausibilities and commonalities. To get there we need to compute the Moebius inverse of pl.f.s and comm.f.s respectively.

**Plausibility space.** Plausibility functions are indeed also completely specified by their $N - 2$ plausibility values $\{pl_b(A), \emptyset \subsetneq A \subsetneq \Theta\}$ and can then be represented in the same reference frames as before as

$$pl_b = \sum_{\emptyset \subsetneq A \subsetneq \Theta} pl_b(A) \mathbf{v}_A \in \mathbb{R}^{N-2}. \tag{6}$$

It can be proved that [13]

**Proposition 1.** *The region* $\mathcal{PL}$ *of* $\mathbb{R}^{N-2}$ *whose points correspond to admissible pl.f.s is a simplex* $\mathcal{PL} = Cl(pl_A, \emptyset \subsetneq A \subseteq \Theta)$ *whose vertices are given by* $pl_A = -\sum_{\emptyset \subsetneq B \subseteq A} (-1)^{|B|} b_B$, *and represent the plausibility functions associated with all dogmatic belief functions* $b_A$*:* $pl_A = pl_{b_A}$.

Figure 2 shows the geometry of belief and plausibility spaces for a binary frame $\Theta_2 = \{x, y\}$, where pl.f.s are also vectors of $\mathbb{R}^2$: $pl_b = [pl_b(x) = 1 - m_b(y), pl_b(y) = 1 - m_b(x)]'$. The two simplices $\mathcal{B} = Cl(b_\Theta = \mathbf{0}, b_x, b_y)$, $\mathcal{PL} = Cl(pl_\Theta = \mathbf{1}, pl_x = b_x, pl_y = b_y)$ are symmetric with respect to the segment of all probability measures $\mathcal{P}$ and congruent, so that they can be moved onto each other by means of a rigid transformation.

**Plausibility assignment.** We can use Lemma 1 to compute the Moebius inverse of a pl.f., by putting (6) in the same form as Equation (4). We get that $pl_b = \sum_{\emptyset \subsetneq A \subseteq \Theta} \mu_b(A) b_A$, where

$$\mu_b(A) \doteq \sum_{B \subseteq A} (-1)^{|A-B|} pl_b(B). \tag{7}$$

It is natural to call the function $\mu_b : 2^\Theta \to \mathbb{R}$ defined by expression (7) *basic plausibility assignment* (b.pl.a.). By comparing (7) with the Moebius formula for b.f.s (2) it is easy to recognize the Moebius equation for plausibilities: hence

$$pl_b(A) = \sum_{B \subseteq A} \mu_b(B). \tag{8}$$

PL.F.s are then sum functions on $2^\Theta$ of the form (8), whose Moebius inverse is the b.pl.a. (7). Basic probabilities and plausibilities are obviously related.

**Proposition 2.** $\mu_b(A) = (-1)^{|A|+1} \sum_{C \supseteq A} m_b(C)$ *for* $A \neq \emptyset$, $\mu_b(\emptyset) = 0$.

As b.p.a.s do, basic plausibility assignments meet the normalization constraint. In other words, pl.f.s are *normalized sum functions* [5].
However, $\mu_b(A)$ is not always positive on all events $A \subseteq \Theta$.

**Example.** Let us consider as an example a b.f. $b$ on the binary frame $\Theta_2 = \{x, y\}$ with b.p.a. $m_b(x) = \frac{1}{3}$, $m_b(\Theta) = \frac{2}{3}$. The corresponding pl. vector is

$$pl_b = [pl_b(x), pl_b(y)]' = [1 - b(\{x\}^c), 1 - b(\{y\}^c)]' = [1,\ 2/3]'.$$

Using Equation (7) we can compute its b.pl.a. as

$$\begin{aligned}
\mu_b(x) &= (-1)^{|x|+1} \sum_{C \supseteq x} m_b(C) = (-1)^2 (m_b(x) + m_b(\Theta)) = 1, \\
\mu_b(y) &= \qquad (-1)^{|y|+1} \sum_{C \supseteq y} m_b(C) = (-1)^2 m_b(\Theta) = 2/3, \\
\mu_b(\Theta) &= \quad (-1)^{|\Theta|+1} \sum_{C \supseteq \Theta} m_b(C) = (-1) m_b(\Theta = -2/3 < 0
\end{aligned}$$

confirming that b.pl.a. meet the normalization but not the positivity constraint.

### 3.2 Basic commonality assignment

It is straightforward to prove that commonality functions are also sum functions and possess some interesting similarities with pl.f.s. They present though some peculiarities we need to take care of. We know that b.f.s and pl.f.s are such that

$$b(\emptyset) = pl_b(\emptyset) = 0, \qquad b(\Theta) = pl_b(\Theta) = 1;$$

in other words, both $b$ and $pl_b$ can be represented by vectors with $N-2$ coordinates as we have previously seen. On the other side

$$Q_b(\emptyset) = \sum_{A \supseteq \emptyset} m_b(A) = \sum_{A \subseteq \Theta} m_b(A) = 1, \quad Q_b(\Theta) = \sum_{A \supseteq \Theta} m_b(A) = m_b(\Theta)$$

so that $Q_b$ needs $N$ coordinates to be represented (even though the dimension of $\mathcal{Q}$ is still $N-2$). A comm.f. corresponds then to a vector of $\mathbb{R}^N = \mathbb{R}^{2^{|\Theta|}}$

$$Q_b = \sum_{\emptyset \subseteq A \subseteq \Theta} Q_b(A) \mathbf{v}_A$$

where $\{\mathbf{v}_A : \emptyset \subseteq A \subseteq \Theta\}$ is an extended reference frame in $\mathbb{R}^N$ ($A = \Theta, \emptyset$ this time included).

**Commonality assignment.** We can as before express $Q_b$ as a sum function by computing its Moebius inverse. We can use Lemma 1 to change the coordinate base and get the coordinates of $Q_b$ with respect to the base $\{b_A, \emptyset \subseteq A \subseteq \Theta\}$:

$$\begin{aligned} Q_b &= \sum_{\emptyset \subseteq A \subseteq \Theta} Q_b(A) \big( \sum_{B \supseteq A} b_B (-1)^{|B \setminus A|} \big) \\ &= \sum_{\emptyset \subseteq B \subseteq \Theta} b_B \big( \sum_{A \subseteq B} (-1)^{|B \setminus A|} Q_b(A) \big) = \sum_{\emptyset \subseteq B \subseteq \Theta} q_b(B) b_B \end{aligned}$$

i.e. $Q_b$ is a sum function with Moebius inverse $q_b : 2^\Theta \to [0,1]$, $B \mapsto q_b(B)$ with

$$q_b(B) = \sum_{\emptyset \subseteq A \subseteq B} (-1)^{|B \setminus A|} Q_b(A)$$

which we can call *basic commonality assignment* (b.comm.a.).
$q_b$ has an interesting interpretation in terms of belief values.

**Theorem 1.** $q_b(B) = (-1)^{|B|} b(B^c)$.

*Proof.*

$$\begin{aligned} q_b(B) &= \sum_{\emptyset \subseteq A \subseteq B} (-1)^{|B \setminus A|} \big( \sum_{C \supseteq A} m_b(C) \big) = \sum_{\emptyset \subsetneq A \subseteq B} (-1)^{|B \setminus A|} \big( \sum_{C \supseteq A} m_b(C) \big) + \\ &+ (-1)^{|B|-|\emptyset|} \sum_{C \supseteq \emptyset} m_b(C) = \sum_{B \cap C \neq \emptyset} m_b(C) \big( \sum_{\emptyset \subsetneq A \subseteq B \cap C} (-1)^{|B \setminus A|} \big) + (-1)^{|B|}. \end{aligned}$$

But now, since $B \setminus A = B \setminus C + B \cap C \setminus A$, we have that

$$\begin{aligned} \sum_{\emptyset \subsetneq A \subseteq B \cap C} (-1)^{|B \setminus A|} &= (-1)^{|B \setminus C|} \sum_{\emptyset \subsetneq A \subseteq B \cap C} (-1)^{|B \cap C| - |A|} \\ &= (-1)^{|B \setminus C|} [(1-1)^{|B \cap C|} - (-1)^{|B \cap C| - |\emptyset|}] = (-1)^{|B|+1} \end{aligned}$$

so that the b.comm.a. $q_b(B)$ can be expressed as

$$q_b(B) = (-1)^{|B|+1} \sum_{B \cap C \neq \emptyset} m_b(C) + (-1)^{|B|} = (-1)^{|B|} (1 - \sum_{B \cap C \neq \emptyset} m_b(C)) = \quad (9)$$

$= (-1)^{|B|}(1 - pl_b(B))$ i.e. we have as desired. Note that $q_b(\emptyset) = (-1)^{|\emptyset|} b(\emptyset) = 1$.

**Properties of basic commonality assignments.** Basic commonality assignments are not normalized, as

$$\sum_{\emptyset \subseteq B \subseteq \Theta} q_b(B) = Q_b(\Theta) = m_b(\Theta).$$

In other words, whereas belief functions are normalized sum functions (n.s.f.) with non-negative Moebius inverse, and plausibility functions are normalized sum functions, commonality functions are *unnormalized* sum functions.
Going back to the above example, the b.comm.a. associated with $m_b(x) = 1/3$, $m_b(\Theta) = 2/3$ is (by Equation (9))

$$\begin{array}{l} q_b(\emptyset) = (-1)^{|\emptyset|} b(\Theta) = 1, \quad q_b(x) = (-1)^{|x|} b(y) = -m_b(y) = 0, \\ q_b(y) = (-1)^{|y|} b(x) = -m_b(x) = -1/3, \quad q_b(\Theta) = (-1)^{|\Theta|} b(\emptyset) = 0 \end{array}$$

so that $\sum_{\emptyset \subseteq B \subseteq \Theta} q_b(B) = 2/3 = m_b(\Theta) = Q_b(\Theta)$.

**Commonality space.** Analogously to the case of belief and plausibility functions, we can use here the notion of basic commonality assignment (Theorem 1) to recover the shape of the space $\mathcal{Q} \subset \mathbb{R}^N$ of all commonality functions, or "commonality space".

**Theorem 2.** *The commonality space $\mathcal{Q}$ is a simplex*

$$\mathcal{Q} = Cl(Q_A, \emptyset \subsetneq A \subseteq \Theta)$$

*whose vertices are*

$$Q_A \doteq \sum_{\emptyset \subseteq B \subseteq A^c} (-1)^{|B|} b_B. \tag{10}$$

*Proof.*

$$\begin{aligned} Q_b &= \sum_{\emptyset \subseteq B \subseteq \Theta} (-1)^{|B|} b_B \Big( \sum_{\emptyset \subseteq A \subseteq B^c} m_b(A) \Big) = \sum_{\emptyset \subseteq A \subseteq \Theta} m_b(A) \Big( \sum_{\emptyset \subseteq B \subseteq A^c} (-1)^{|B|} b_B \Big) \\ &= \sum_{\emptyset \subseteq A \subseteq \Theta} m_b(A) Q_A \end{aligned}$$

with $Q_A$ given by Equation (10). □

**Theorem 3.** *$Q_A$ is the commonality function associated with the dogmatic belief function $b_A$, i.e.*

$$Q_{b_A} = \sum_{\emptyset \subseteq B \subseteq \Theta} q_{b_A}(B) b_B.$$

*Proof.* Indeed $q_{b_A}(B) = (-1)^{|B|}$ if $B^c \supseteq A$ i.e. $B \subseteq A^c$, while $q_{b_A}(B) = 0$ otherwise, so that $Q_{b_A} = \sum_{\emptyset \subseteq B \subseteq A^c} (-1)^{|B|} b_B = Q_A$ and the two quantities coincide. □

**Binary case**. In the binary case $\mathcal{Q}_2$ needs $N - 1 = 3$ coordinates to be represented. We have indeed $Q_b(\emptyset) = 1$, $Q_b(x) = \sum_{A \supseteq \{x\}} m_b(A) = pl_b(x)$,

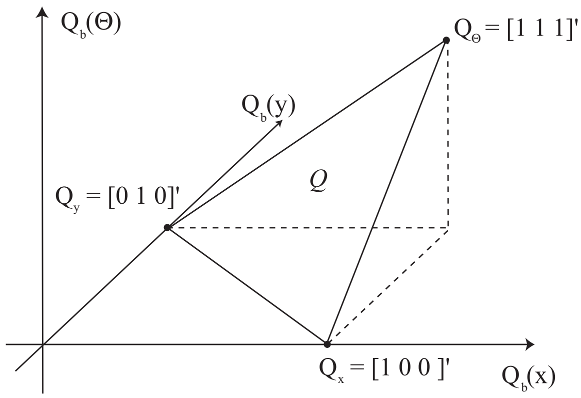


**Fig. 3.** Commonality space in the binary case.

$Q_b(y) = \sum_{A \supseteq \{y\}} m_b(A) = pl_b(y)$, and $Q_b(\Theta) = m_b(\Theta)$.
If we neglect the coordinate $Q_b(\emptyset)$ which is constant $\forall b$, the commonality space $\mathcal{Q}_2$ can then be drawn as in Figure 3. The vertices of $\mathcal{Q}_2$ are, according to Equation (10) and using all $N$ coordinates, $Q_\Theta = b_\emptyset = [1111]'$,

$$Q_x = \sum_{\emptyset \subseteq B \subseteq \{y\}} (-1)^{|B|} b_B = b_\emptyset - b_y = [1111]' - [0011]' = [1100]' = Q_{b_x}$$
$$Q_y = \sum_{\emptyset \subseteq B \subseteq \{x\}} (-1)^{|B|} b_B = b_\emptyset - b_x = [1111]' - [0101]' = [1010]' = Q_{b_y}.$$

## 4 Congruence of equivalent models

The *equivalence* of the three models based on basic probability, plausibility, and commonality assignments as descriptions of uncertainty geometrically translates as *congruence* of the associated simplices.
We saw that for binary frames, $\mathcal{B}$ and $\mathcal{PL}$ are congruent, i.e. they can be superposed by means of a rigid transformation. This is indeed a general property.

**Lemma 2.** *The corresponding 1-dimensional sides $Cl(b_A, b_B)$ and $Cl(pl_A, pl_B)$ of belief and plausibility spaces are congruent, namely*

$$\|pl_B - pl_A\|_p = \|b_A - b_B\|_p$$

*where $\|\|_p$ denotes the classical norm $\|\mathbf{v}\|_p \doteq \sqrt{\sum_{i=1}^N |v_i|^p}$, for all $p = 1, 2, ..., +\infty$.*

*Proof.* This a direct consequence of the definition of plausibility function. Let us denote with $C, D$ two generic subsets of $\Theta$. As $pl_A(C) = 1 - b_A(C^c)$ we have $b_A(C^c) = 1 - pl_A(C)$, which implies

$$b_A(C^c) - b_B(C^c) = 1 - pl_A(C) - 1 + pl_B(C) = pl_B(C) - pl_A(C).$$

This in turn means that

$$\sum_{C\subset\Theta} |pl_B(C) - pl_A(C)|^p = \sum_{C\subset\Theta} |b_A(C^c) - b_B(C^c)|^p = \sum_{D\subset\Theta} |b_A(D) - b_B(D)|^p \ \ \forall p.$$

A straightforward implication is then that

**Theorem 4.** *$\mathcal{B}$ and $\mathcal{PL}$ are congruent.*

as their corresponding 1-dimensional faces have the same length. This is due to the generalization of a well-known Euclid's theorem stating that triangles with sides of the same length are congruent.[2]

The situation is a bit more complicated for plausibility and commonality spaces, but we can still prove that $\mathcal{Q}$ and $\mathcal{PL}$ are congruent in the case of unnormalized belief functions [14].

## 5 Applications of basic plausibility assignments

Besides being a natural complement to the mathematical apparatus of the theory of evidence, these alternative models of the ToE and the related basic assignments can actually be useful in the solution of practical problems. This is true when dealing with plausibility functions as we can recur to their equivalent basic assignments and operate on them. In particular, it becomes necessary when we need to apply combination rules for the aggregation of evidence to those plausibility functions.

**Relative belief of singletons.** The problem of approximating a given belief function with a probability, for instance, has been studied by many researchers [7, 8, 15]. The "relative plausibility of singletons"

$$\tilde{pl}_b(x) = \frac{pl_b(x)}{\sum_{y\in\Theta} pl_b(y)},$$

in particular, is an interesting candidate as it can be proven that it commutes with Dempster's combination $\oplus$ [2, 15] and it perfectly represents a belief function when combined with a probability: $\tilde{pl}_b \oplus p = b \oplus p$ for all $p \in \mathcal{P}$.

**Definition 1.** *The* Dempster's sum *of two belief functions $b_1, b_2$ on the same frame $\Theta$ is a new belief function $b_1 \oplus b_2$ on $\Theta$ with b.p.a.*

$$m_{b_1\oplus b_2}(A) = \frac{\sum_{B\cap C=A} m_{b_1}(B)\, m_{b_2}(C)}{\sum_{B\cap C\neq\emptyset} m_{b_1}(B)\, m_{b_2}(C)} \tag{11}$$

*where $m_{b_i}$ denotes the b.p.a. associated with $b_i$.*

[2] Note that this holds for *simplices* but not for *polytopes* in general, think of a square and a rhombus with sides of length 1.

However, as belief and plausibilities are dual representations of the same evidence, a dual probability can be defined as the *relative belief of singletons*

$$\tilde{b}(x) \doteq \frac{b(x)}{\sum_{y \in \Theta} b(y)}. \tag{12}$$

We can prove that $\tilde{b}$ meets a set of dual properties with respect to $\oplus$, which are the dual of those of $\tilde{pl}_b$ [8, 15]. These dual properties involve the Dempster's sum *of plausibility functions* (instead of belief functions).
This should not surprise at this point. We have proven in Section 3.1 that plausibility functions are themselves sum functions, which admit a Moebius inverse: the basic plausibility assignment. But then nothing prevents from applying Equation (11) to the b.pl.a.s of a pair of plausibility functions, instead of belief functions. We can then easily prove that

**Proposition 3.** *The relative belief of singletons $\tilde{b}$ represents perfectly the corresponding* plausibility *function $pl_b$ when combined with any probability through (extended) Dempster's rule: $\tilde{b} \oplus p = pl_b \oplus p \ \forall p \in \mathcal{P}$.*

**Intersection probability.** From a different point of view, each belief function determines an "interval probability", i.e. a set of probability measures $p : \Theta \to [0, 1]$ on the same domain $\Theta$ which meet a lower bound associated with the belief values on all outcomes $x \in \Theta$, and an upper bound determined by the corresponding plausibility values:

$$(b, pl_b) \doteq \big\{ b(x) \leq p(x) \leq pl_b(x), \forall x \in \Theta \big\}. \tag{13}$$

Now, there are clearly many ways of selecting one of those measures as representative of the above interval probability. However, as each interval $[b(x), pl_b(x)]$ has the same weight in the interval probability, there is no reason for the different singletons $x$ to be treated differently.
Mathematically this translates into seeking the unique probability $p[b]$ such that

$$p[b](x) = b(x) + \alpha(pl_b(x) - b(x)), \qquad \alpha \in [0, 1].$$

This function has been called *intersection probability* [16], as it is geometrically located on the segment joining a pair of belief-plausibility functions. The situation is clearly visible in the binary case of Figure 2, where the line $a(b, pl_b)$ joining such a pair is drawn: $p[b]$ lies at the intersection of this line with the region $\mathcal{P}$ of all probability functions.
Again, as linear combination of a b.f. and a pl.f., its analysis requires the Moebius inversion of $pl_b$ [16].

## 6 Conclusions

In this paper we introduced two alternative formulations of the theory of evidence by proving that both pl.f.s and comm.f.s share with belief functions the combinatorial structure of sum function, and computing their Moebius inverses which we called basic plausibility and commonality assignments. From a combinatorial point of view, b.f.s, pl.f.s and comm.f.s form a hierarchy of sum functions whose Moebius inverse meets both normalization and positivity axiom (b.p.a.), only the normalization constraint (b.pl.a.), and none of them (b.comm.a.) respectively. The related spaces possess a similar convex geometry. Their congruence is the geometric reflection of the equivalence of those alternative formulations, which can be successfully applied to problems like the probabilistic approximation of a belief function.

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